现代数学基础丛书

高斯过程的样本轨道性质

林正炎 陆传荣 张立新 著

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内 容 简 介

本书论述Gauss过程的样本轨道性质，内容包括：Gauss变量和Gauss过程的一些基本性质，Gauss过程的连续性、Gauss过程的连续模与大增量的极限性质，无穷维Gauss过程的连续模与大增量的极限性质，Gauss过程的重对数律和增量的下极限性质，以及Gauss过程的 p 变差和一些分形性质.

本书大部分内容是作者们的研究成果，具有较高的学术水平.

本书适合高等学校概率论专业的大学生、研究生与数学研究工作者阅读与参考.

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序

我十分高兴介绍这部关于 Gauss 过程及其相关过程的样本轨道性质的精细分析的优秀著作. 在 20 世纪 20 年代的一系列论文中, N. Wiener 从事 Brown 运动的数学分析的研究. 他指出, 除去一个概率 (关于 Wiener 测度) 为零的集合外, 所有 Brown 运动的样本轨道是连续而不可微的曲线. 在 20 世纪 40 年代左右, P. Lévy 证明了著名的连续模定理, 即对 Brown 运动 (Wiener 过程) 的几乎所有样本轨道建立了精确的连续性速度. 自此以后, 在关于一般 Gauss 过程和许多其他相关的随机过程样本轨道性质的研究文献中, 这些基本贡献已成为首要的指导性结果.

在 M. Csörgő 和 P. Révész 的 1981 年的专著的第一章中, 我们给出了 Wiener 过程的构造性证明并论证了它的样本轨道的精确细致的分析性质, 受此启发, 我们在 1987 年和林正炎一起, 沿着类似的思路开始了无穷维 Ornstein-Uhlenbeck 过程样本轨道性质的研究. 后来, 这一研究思路已发展到对较一般的 Gauss 过程这样一个广泛类型的过程以及其他随机过程的类似研究. 结合世界范围的"法国学派"的基本方法和成就, 这个课题方面的研究文献是十分浩瀚的. 由林正炎、陆传荣和张立新撰写的这一著作对研究随机过程的内在样本轨道性质这一相当漂亮的、发展中的复杂领域, 在全面的基础知识和若干最近的进展方面作了极为及时的基本阐述.

Miklos Csörgő

1997 年 9 月

目　录

引　论

Gauss 过程的样本轨道性质是研究 Gauss 过程的基本性质的重要方面. 对 Gauss 过程样本轨道性质的研究最早是关于它的连续性、有界性及连续模定理, 首先从具有优良性质的 Wiener 过程开始. Lévy 于 1937,1948 年就给出了 Wiener 过程 $\{W(t); t \geq 0\}$ 的精确连续模定理.

定理 0.1 我们有

$$\lim_{h\to 0} \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \frac{|W(t+s)-W(t)|}{(2h\log(1/h))^{1/2}} = 1 \quad \text{a.s.},$$

$$\lim_{h\to 0} \sup_{0\leq t\leq 1-h} \frac{|W(t+h)-W(t)|}{(2h\log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

对于一般的 Gauss 过程的连续性和有界性的研究, Adler (1990) 的专著综述了这一方面的成果. 本书将在 §2.1 中介绍部分主要结果.

1964 年 Strassen 对 Wiener 过程的泛函重对数律是一个关于 Wiener 过程样本轨道性质的重要成果. 20 世纪 70 年代, M. Csörgő 和 P. Révész 等对 Wiener 过程的大增量作了系统的开创性研究. 他们 1981 年的专著 "Strong Approximations in Probability and Statistics" 综述了当时的主要成果, 在 Wiener 过程增量理论方面有:

定理 0.2 设 a_T 是 T 的单调非降函数, 满足

(i) $0 < a_T \leq T$,

(ii) T/a_T 非降.

那么有

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|W(t+s)-W(t)|=1\quad \text{a.s.},$$

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\beta_T|W(t+a_T)-W(t)|=1\quad \text{a.s.},$$

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{0\le s\le a_T}\beta_T|W(T+s)-W(T)|\\&\quad=\limsup_{T\to\infty}\beta_T|W(T+a_T)-W(T)|=1\quad \text{a.s.},\end{aligned}$$

其中 $$\beta_T=\{2a_T(\log(T/a_T)+\log\log T)\}^{-1/2}.$$

若还满足

(iii) $$\lim_{T\to\infty}\left(\log(T/a_T)\right)/\log\log T=\infty,$$

那么

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|W(t+s)-W(t)|=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\beta_T|W(t+a_T)-W(t)|=1\quad \text{a.s.}$$

若 (iii) 不真，那么

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|W(t+s)-W(t)|<1\quad \text{a.s.}$$

以后就 a_T 更靠近 T 的情形有若干作者作了进一步讨论．例如，若将 (iii) 改为

(iv) $\lim_{T\to\infty}\left(\log(T/a_T)\right)/\log\log T=r$, $0<r\le\infty$, 则有

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\beta_T|W(t+a_T)-W(t)|=\left(\frac{r}{1+r}\right)^{1/2}\quad \text{a.s.};$$

若将 (iii) 改为

(v) $\lim_{T\to\infty}\left(\log(T/a_T)\right)/\log\log\log T=\infty$, 则有

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma_1(T)|W(t+s)-W(t)|=1\quad \text{a.s.},$$

其中

$$\gamma_1(T)=\left\{2a_T\log\left(1+\frac{\pi^2}{16}\frac{T}{a_T\log\log T}\right)\right\}^{-1/2};$$

而若

(vi) $\lim_{T\to\infty}(T/a_T)/\log\log T=\infty$ 成立，则有

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma_0(T)|W(t+s)-W(t)|=1\quad \text{a.s.},$$

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\gamma_0(T)|W(t+a_T)-W(t)|=1\quad \text{a.s.},$$

其中 $\gamma_0(T)=\left\{2a_T(\log(T/a_T)-\log\log\log T)\right\}^{-1/2}$.

显然，定理 0.1 和定理 0.2 与下述 Lévy 重对数律相关:

定理 0.3 (Lévy 1937, 1948). 我们有

$$\limsup_{T\to\infty}\frac{|W(T)|}{\sqrt{2T\log\log T}}=1\quad \text{a.s.},$$

$$\limsup_{h\to 0}\frac{|W(h)|}{\sqrt{2h\log\log 1/h}}=1\quad \text{a.s.}$$

在 Csörgő 和 Révész 的上述专著中还研究了 Wiener 过程的另一类样本轨道性质. 他们给出了 Wiener 过程的不可微模，并研究了 Wiener 过程增量有多小.

定理 0.4 我们有

$$\lim_{h\to 0}\inf_{0\le t\le 1-h}\sup_{0\le s\le h}\sqrt{\frac{8\log h^{-1}}{\pi^2 h}}|W(t+s)-W(t)|=1\quad \text{a.s.}$$

定理 0.5 设 a_T 如定理 0.2, 那么

$$\liminf_{T\to\infty}\inf_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma(T)|W(t+s)-W(t)|=1\quad \text{a.s.}$$

其中

$$\gamma(T)=\left(\frac{8}{\pi^2 a_T}(\log(T/a_T)+\log\log T)\right)^{1/2}.$$

进一步若 (iii) 被满足，那么 lim inf 可换成 lim.

由定理 0.4 易知，Wiener 过程的几乎所有样本轨道是无处可微的. 容易看出定理 0.4 和定理 0.5 与下述 Chung 重对数律相关：

定理 0.6 我们有

$$\liminf_{T\to\infty}\left(\frac{8\log\log T}{\pi^2 T}\right)^{1/2}\sup_{0\le t\le T}|W(t)|=1 \quad \text{a.s.},$$

$$\liminf_{h\to 0}\left(\frac{8\log\log 1/h}{\pi^2 h}\right)^{1/2}\sup_{0\le t\le h}|W(t)|=1 \quad \text{a.s.}$$

在上述专著中还讨论了两参数 Wiener 过程增量有多大.

自 Csörgő 和 Révész 的 1981 年专著发表以后，强逼近理论发展十分迅速，对 Wiener 过程样本轨道性质，进一步在对许多特殊的 Gauss 过程的轨道性质的研究有着一系列重大进展. 在 1987 年，应 M. Csörgő 院士之邀，林正炎访问了加拿大的 Carleton 大学. M. Csörgő 提议对 Ornstein-Uhlenbeck 过程的精确样本轨道性质加以研究. 从那时开始，M. Csörgő, 林正炎，邵启满和许多匈牙利及中国学者在这一课题上，不仅对某些特殊的 Gauss 过程而且对较一般的 Gauss 过程进行了深入的研究. 林正炎、陆传荣 1992 年的专著《强极限定理》综述了当时国内外有关成果，在 Wiener 过程与 Gauss 过程的样本轨道性质方面有：

1. 完善了对 Wiener 过程增量理论的研究，如详细介绍了在条件 (vi) 下邵启满证明的关于 Wiener 过程增量的下极限结果，讨论了 Wiener 过程滞后增量及增量的一般形式，也讨论了滞后形式的下极限结果及精确收敛速度.

2. 将 Wiener 过程增量在条件 (v) 下的结果拓广于两参数 Wiener 过程情形，讨论了两参数 Wiener 过程滞后增量及增量的一般形式.

3. 介绍了 Ortega (1984) 关于分数 Wiener 过程增量的大小与连续模结果.

4. 开创性地研究了无穷维 Ornstein-Uhlenbeck 过程导出的过程，如部分和过程、无穷级数及 l^2 模平方过程等的样本轨道性质.

在 20 世纪 90 年代中，除去进一步对 Wiener 过程增量作深入讨论外，对各种有实际背景的 Gauss 过程增量的上极限与下极限的研究取得了一系列完美的成果. 本书汇集了样本轨道性质方面的主要结果，包括一般 Gauss 过程的连续性、不可微性、连续模和大增量性质等等. 对某些特殊的 Gauss 过程 (例如 Ornstein-Uhlenbeck 过程)，进行了深入细致讨论. 本书拓展和深化了上述两专著的内容，可以看作是它们的续编.

在第一章中，介绍了 Gauss 过程尾概率估计，如 Borell 不等式 (定理 1.1.1), Fernique 不等式 (定理 1.1.3); 用于研究一般 Gauss 过程的 Slepian 不等式 (定理 1.2.1), Anderson 不等式 (定理 1.2.2) 和 Khatri-Šidák 不等式 (定理 1.2.4) 等基本不等式. 关于 Borel-Cantelli 引理的一个拓广形式也于引理 2.1.1 中给出.

在第二章中，先将 Gauss 过程连续性、有界性等基本结果作一综合论证于 §2.1 中. 在 §2.2 中将对 Wiener 过程成立的许多结果拓展到分数 Wiener 过程上, 介绍分数 Wiener 过程连续模、大增量及其下极限结果、增量的一般形式等, 最后还给出增量负相关的 Gauss 过程的连续模及大增量等有关结果. 在 §2.3 中, 讨论了两参数 Wiener 过程在类似于条件 (vi) 下的下极限结果. 在 §2.4—2.6 中讨论了几类两参数 Gauss 过程如两参数 Lévy-Wiener 过程、两参数 Ornstein-Uhlenbeck 过程及由此拓广的带核两参数 Gauss 过程的连续模及大增量结果. 最后在 §2.7 中还讨论了 Gauss 过程局部时的有关结果.

在第三章中详细介绍了无穷维 Gauss 过程的连续模和大增

量. 首先介绍了 l^p 值 Gauss 过程连续性条件, 特别给出了 l^2- 值 Ornstein-Uhlenbeck 过程连续性的充分必要条件. 为讨论 l^p- 值 Gauss 过程的连续模与大增量, 从对 B 值 Gauss 过程给出增量的上极限估计开始, 在增量具有某种负相关条件下给出了连续模与大增量结果. 最后还讨论了 l^∞ 值 Gauss 过程的连续模与大增量.

在第四章中, 首先在 §4.1 对分数 Wiener 过程及一类具有平稳增量的 Gauss 过程证明了 Strassen 型泛函重对数律, 讨论了具有指数 α ($\alpha > 0$) 的自相似 Gauss 过程的 Strassen 重对数律的收敛速度. 并简述了最近关于 Wiener 过程的 Strassen 型泛函连续模与大增量定理及其精确收敛速度. 在 §4.1 中还讨论了 Wiener 过程及一类平稳 Gauss 过程的 Erdös-Révész 重对数律, 由后者即可导出关于独立 Ornstein-Uhlenbeck 过程无穷级数及分数 Wiener 过程的 Erdös-Révész 重对数律.

第四章的 §4.2—§4.5 讨论 Gauss 过程和 Gauss 随机场的 Chung 重对数律、不可微模及增量有多小等下极限问题. 此时小球概率估计是证明有关结果的关键. 在 §4.2 中, 详细介绍了邵启满关于具有平稳增量 Gauss 过程, 特别是分数 Wiener 过程的小球概率估计, 由此即可导出分数 Wiener 过程的 Chung 重对数律, 该节中也给出了 Ornstein-Uhlenbeck 过程无穷级数的 Chung 重对数律. 在 §4.3 节讨论了 Gauss 场的小球概率估计、 Chung 重对数律. 在 §4.4 中介绍了 Gauss 过程增量的下极限, 特别给出了独立 Ornstein-Uhlenbeck 过程无穷级数的不可微模. 在 §4.5 中, 讨论了两参数 Wiener 过程的下极限, 给出了两参数 Ornstein-Uhlenbeck 过程的不可微模和 Chung 重对数律. 值得指出的是, 关于小球概率估计及有关结果至今仍是人们关注的热点问题, 近期仍有许多学者在研究这一问题.

在 §4.6 中介绍了 Gauss 过程的 p 变差和分形性质. 首先给出 Gauss 过程 p 变差的一个较一般的结果. 其次介绍了 Gauss 场的像与图的分形性质, 还讨论了 Gauss 过程增量的分形性质, 最后在 §5.4 中论述了 Ornstein-Uhlenbeck 过程的无穷级数的增量与

Chung 重对数律有关的分形性质.

本书的写作和出版得到中国科学院出版基金、中国国家自然科学基金、浙江省自然科学基金和浙江大学数学系的资助，加拿大皇家科学院院士 M. Csörgő 为本书写了序言，谨此一并致谢.

林正炎　陆传荣　张立新

2000 年 2 月于浙江大学

第一章 Gauss 变量和 Gauss 过程的若干基本结果

在这一章中，我们将介绍本书需要的若干准备结果，特别是给出一批重要不等式，包括 Borell 不等式、 Fernique 不等式、Slepian 不等式、 Anderson 不等式、 Khatri-Šidák 不等式等.

我们总用 c 表示正常数，其取值在不同的地方可以不同. $K, C, \cdots$ 通常表示绝对常数. $\mathcal{N}$ 为非负整数集， $\mathcal{Z}$ 为整数集， $\mathcal{R}$ 为实数集， $\mathcal{R}_+$ 为正实数集， $\mathcal{Z}_+$ 为正整数集. $A \approx B$ 表示对某个常数 C 成立 $C^{-1}A \le B \le CA$. 符号 $\sim$ 表示两序列等价. $(\cdot)^+, (\cdot)^-, [\cdot]$ 分别表示取正部，取负部，取整函数. Card(A) 表示 (有限) 集 A 中元素的个数， A^c 表示 A 的余集， I_A 表示集合 A 的示性函数. $(\Omega, \mathcal{F}, P)$ 为一完备概率空间. 具有期望 $\mu \in \mathcal{R}$ 和方差 $\sigma^2 \in \mathcal{R}_+$ 的随机变量 X 称为是 Gauss 的 (或正态的)，如果其 Fourier 变换满足

$$Ee^{itX} = e^{i\mu t - \frac{1}{2}\sigma^2 t^2},$$

或等价地， X 的分布具有密度函数 $\sigma^{-1}\phi((x-u)/\sigma)$, 其中

$$\phi(x) := (2\pi)^{-\frac{1}{2}} \exp(-x^2/2),$$

即

$$P\{(X-\mu)/\sigma \le x\} = \Phi(x) := \int_{-\infty}^{x} \phi(t)\,dt, \quad x \in (-\infty, \infty).$$

若 $\mu = 0$, 我们称 X 为零均值的 (中心化的), 若还有 $\sigma = 1$, 则称 X 是标准正态变量. $\mathcal{R}^N$ 中的随机向量 $X = (X_1, \cdots, X_N)$ 称为 (零均值)Gauss 向量，如果对任何实数 $\alpha_1, \cdots, \alpha_N$, $\sum_{i=1}^N \alpha_i X_i$ 为实值 (零均值)Gauss 随机变量. $\mathcal{R}^N$ 中的零均值 Gauss 随机向量 X 的分布完全由它的对称 (半) 正定协方差矩阵 $\Gamma = (EX_iX_j)_{1\le i,j\le N}$ 确

定. 事实上, 若 $\Gamma = AA'$, 则 X 与 Ag 同分布, 其中 $g=(g_1,\cdots,g_N)$ 具有 $\mathcal{R}^N$ 中的标准 Gauss 分布 γ_N, 其密度函数为

$$(2\pi)^{-N/2}\exp(-\|x\|^2/2),$$

其中 $\|\cdot\|$ 表示 Euclidean 范数.

设 $X=\{X_t;t\in T\}$ 为具有指标集 T 的一族随机变量, 若每一个线性组合 $\sum\alpha_t X_t$ 是 (零均值)Gauss 变量, 则 X 称为 (零均值)Gauss 过程. 本书中, T 通常是 $\mathcal{R}$ 的某个子集, 有时也是 $\mathcal{R}^k$ 或 $[0,1]^k$ 的某个子集 ($k>1$ 时, 我们称 X 是多参数的). 若 X 是一个零均值 Gauss 过程, 则它的协方差函数 $\Gamma(s,t)=EX_sX_t, s,t\in T$ 完全确定了它的分布.

若无特别说明, 我们通常设 T 是具有可数稠密子集的距离空间, 并设 X 是一个可分的随机过程, 即存在一个零测集 $\Omega_0\subset\Omega$ 和一个 T 的可数稠密子集 S 使得对任何 $\omega\notin\Omega_0$, $t\in T$ 和 $\varepsilon>0$ 有

$$X_t(\omega)\in\overline{\{X_s(\omega);s\in S,d(s,t)<\varepsilon\}},$$

其中闭包取自 $\mathcal{R}\cup\{\infty\}$, 且 $d(\cdot,\cdot)$ 是 T 中的距离. 若 X 可分, 则对每个 $\omega\notin\Omega_0$ 有 $\sup_{t\in T}|X_t(\omega)|=\sup_{s\in S}|X_s(\omega)|$, $\sup_{t\in T}X_t(\omega)=\sup_{s\in S}X_s(\omega)$, 并且 T 中每个可数稠密子集 S 都可取作为可分集. 一个随机过程 $\{X_t;t\in\mathcal{R}^k\}$ 称为是 (强) 平稳的, 如果对任何 $s\in\mathcal{R}^k$, 它与过程 $\{X_{t+s};t\in\mathcal{R}^k\}$ 同分布. 一个随机过程 $X=\{X_t;t\in T\}$ 称为几乎处处有界 (连续), 或者称它的样本轨道几乎处处有界 (连续), 是指对几乎所有的 ω, 轨道 $t\to X_t(\omega)$ 是有界 (连续) 的.

给定一个 Banach 空间 B, 其对偶空间记为 B', 设存在 B' 中单位球的可数子集 D 使得 $\|x\|=\sup_{f\in D}f(x)$, $x\in B$. 一个 B 中的随机变量 X 称为是 (零均值)Gauss 的, 如果对每个 $f\in D$, $f(X)$ 可测, 并且每个线性组合 $\sum_i\alpha_i f_i(X)$, $\alpha_i\in\mathcal{R}$, $f_i\in D$ 是 (零均值) Gauss 的. 易知, X 可以看作一个以 D 为指标集的 Gauss 过程 $\{f(X);\ f\in D\}$.

有许多关于 Gauss 向量的现代概率不等式，我们这里列出两个，其中一个称为等周不等式：对任何 $\mathcal{R}^N$ 中的 Borel 集 A 有

$$\mathrm{inv}\Phi\big(\gamma_N(A_r)\big) \geq \mathrm{inv}\Phi\big(\gamma_N(A)\big) + r, \tag{1.0.1}$$

其中 A_r 为 A 的 r 阶 Euclidean 邻域；特别地若 $\gamma_N(A) \geq 1/2$，则有

$$1 - \gamma_N(A_r) \leq 1 - \Phi(r) \leq \frac{1}{2}e^{-r^2/2}.$$

另一个不等式是 Brunn-Minkowski 型不等式：对 $\mathcal{R}^N$ 中的任何凸子集 A, B 和数 $\lambda \in [0,1]$ 有，

$$\begin{aligned}&\mathrm{inv}\Phi\big(\gamma_N(\lambda A + (1-\lambda)B)\big)\\ \geq &\lambda\, \mathrm{inv}\Phi\big(\gamma_N(A)\big) + (1-\lambda)\mathrm{inv}\Phi\big(\gamma_N(B)\big),\end{aligned} \tag{1.0.2}$$

其中 $\lambda A + (1-\lambda)B = \{x \in R^N : x = \lambda a + (1-\lambda)b, a \in A, b \in B\}$.

当 A 是凸集时，(1.0.1) 可由 (1.0.2) 得到，这只要在 (1.0.2) 中取 B 为以原点为中心、半径为 $r/(1-\lambda)$ 的 Euclidean 球，并令 $\lambda \to 1$ 即得.

(1.0.1) 和 (1.0.2) 的证明将不在这里给出，读者可分别参看 Ledoux 和 Talagrand (1991), Ehrhard (1983, 1984, 1986).

在这一章中，我们给出关于 Gauss 过程的两类基本结果. 其中一类是 Gauss 过程最大值的尾概率估计，另一类是比较原理.

§1.1 Gauss 过程最大值的尾概率估计

1.1.1 Borell 不等式

设 X 为一个零均值 Gauss 变量，具有方差 σ^2. 令

$$\Psi(x) := 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\,dt.$$

则对任何 $x > 0$ 有

$$\begin{aligned}&(1-\sigma^2 x^{-2})\big(\sigma/\sqrt{2\pi}\big)x^{-1}e^{-\frac{1}{2}x^2/\sigma^2}\\ &\leq P\{X > x\} = \Psi(x/\sigma) \leq \big(\sigma/\sqrt{2\pi}\big)x^{-1}e^{-\frac{1}{2}x^2/\sigma^2}.\end{aligned} \tag{1.1.1}$$

假设 $\{X_t; t\in T\}$ 为一零均值 Gauss 过程，其样本轨道以概率 1 有界．下述 Borell 不等式告诉我们，只要用 $\sigma_T^2 := \sup_{t\in T} EX_t^2$ 代替 σ^2，则 $\sup_{t\in T} X_t$ 有与 X 类似的尾概率估计．

定理 1.1.1 设 $\{X_t; t\in T\}$ 为零均值可分 Gauss 过程，样本轨道几乎处处有界．记 $\|X\| = \sup_{t\in T} X_t$．则对任何 $\lambda > 0$ 有

$$P\{|\|X\| - E\|X\|| > \lambda\} \le 2\exp\left(-\lambda^2/(2\sigma_T^2)\right), \tag{1.1.2}$$

其中 $\sigma_T^2 := \sup_{t\in T} EX_t^2$．

由于一个取值于 Banach 空间的 Gauss 变量可以看作一个 Gauss 过程，定理 1.1.1 中的过程可以用取值于 Banach 空间的 Gauss 变量代替．另外，(1.1.2) 中的 $E\|X\|$ 可以用 $\|X\|$ 的中位数代替．若用 $\|X\|$ 的中位数代替其期望，则 (1.1.2) 可由等周不等式 (1.0.1) 得到 (参看 Ledoux 和 Talagrand 1991)．我们这里列出一个纯概率的证明，这是由 Maurey 和 Pisier (Pisier 1986) 给出的．

注意到 $\{X_t; t\in T\}$ 是可分的，我们可设存在 T 的一个可数子集 D 使得

$$\sup_{t\in D} X_t = \sup_{t\in T} X_t.$$

从而，证明 Borell 不等式，只要证明用 $\sup_{t\in D} X_t$ 代替 $\|X\|$ 时 (1.1.2) 成立，进而，只要证明 (1.1.2) 对有限的 D 成立．取 $D = \{t_1, \cdots, t_k\}$, $t_i \in T$, $k < \infty$, 下述引理是证明的关键．

引理 1.1.1 设 $f: \mathcal{R}^k \to \mathcal{R}$ 具有一阶，二阶偏导数，并且 f 的所有导数都由 $Ae^{B\|x\|}$ 控制，其中 $A, B < \infty$ 为某两个常数，$\|\cdot\|$ 为通常的 Euclidean 范数．设 $X = (X_{t_1}, \cdots, X_{t_k})$ 为一 k 维的零均值 Gauss 变量，其协方差矩阵为 $V_D = (EX_{t_i}X_{t_j})_{1\le i,j\le k}$．若对任何 $x, y \in \mathcal{R}^k$ 有 $|f(x) - f(y)| \le \|x - y\|$，则对任何 $\lambda > 0$ 有

$$P\{|f(X) - Ef(X)| > \lambda\} \le 2\exp(-\lambda^2/(2\sigma^2)), \tag{1.1.3}$$

其中

$$\sigma^2 = \sup_{1\le i\le k} V_D(i,i) = \sup_{1\le i\le k} EX_{t_i}^2.$$

证明 设 $\{B_s; s\ge 0\} = \{(B_s^1,\cdots,B_s^k); s\ge 0\}$ 为 k 维 Wiener 过程, 即 B^i, $i=1,\cdots,k$, 为 i.i.d. 标准实值 Wiener 过程. 取 $0\le s_0<s_1<\cdots<s_n\le 1$, 令 $\mathcal{F}_j$ 为由 $\{B_{s_0},\cdots,B_{s_j}\}$ 生成的 σ-域, 令 $\{V_j; 0\le j\le n\}$ 为一 $\mathcal{R}^k$ 值随机变量序列, 对每个 j, V_j 是 $\mathcal{F}_{j-1}$ 可测的. 假设对每个 j, $\|V_j\|<\sigma$ a.s., 记

$$S_m = \sum_{j=1}^m \langle V_j, B_{s_j} - B_{s_{j-1}}\rangle. \tag{1.1.4}$$

由 V_j 的可测性和 B_t 的独立增量性, 对任何实的 θ 我们有

$$Ee^{\theta S_n} = E\left(e^{\theta S_{n-1}} e^{\theta\langle V_n, B_{s_n} - B_{s_{n-1}}\rangle}\right) \le E\left(e^{\theta S_{n-1}} e^{\frac12\theta^2(s_n-s_{n-1})\sigma^2}\right).$$

从而

$$Ee^{\theta S_n} \le e^{\frac12\theta^2\sigma^2}.$$

取 $\theta=\lambda/\sigma^2$, 由 Chebycheff 不等式得

$$\begin{aligned} P\{|S_n|>\lambda\} =& 2P\{S_n>\lambda\} \le 2e^{-\theta\lambda}Ee^{\theta S_n} \\ \le& 2e^{-\theta\lambda}e^{\frac12\theta^2\sigma^2} = 2e^{-\frac12\lambda^2/\sigma^2}. \end{aligned} \tag{1.1.5}$$

由 Itô 公式, 对充分光滑的函数 $F=F(x,t):\mathcal{R}^k\times\mathcal{R}_+\to\mathcal{R}$ 有

$$\begin{aligned} F(B_t,t) - F(B_s,s) =& \int_s^t \langle\nabla_x F(B_u,u), dB_u\rangle \\ &+ \int_s^t \left(\frac12\triangle_{xx}F(B_u,u) + F_t(B_u,u)\right)du, \end{aligned}$$

((1.1.6))

其中 $\nabla_x F(x,t)$ 表示 $F(x,t)$ 关于 x 的偏导数向量 $(\partial F(x,t)/\partial x_1,\cdots,\partial F(x,t)/\partial x_k)$, 而 $\triangle_{xx}=\sum_{i,j=1}^k \partial^2/\partial x_i\partial x_j$, $F_t(x,t)=\partial F(x,t)/\partial t$.

令 $(P_t)_{t\geq 0}$ 为与 B 相关的 Markov 半群，从而对光滑函数 $g:\mathcal{R}^k\to\mathcal{R}$ 有

$$\begin{aligned}(P_tg)(x)=&E^xg(B_t)\\ =&(2\pi t)^{-k/2}\int_{\mathcal{R}^k}g(y)e^{-\frac{1}{2}\|x-y\|^2/t}\,dy,\end{aligned}$$

其中 E^x 表示关于在零时刻从 $x\in\mathcal{R}^k$ 出发的 Wiener 过程 B 的期望. 令 $\widehat{f}:\mathcal{R}^k\to\mathcal{R}$ 满足引理中对 f 的可导要求，并假设 $|\widehat{f}(x)-\widehat{f}(y)|\leq\sigma\|x-y\|$. 取 $F(x,t)=(P_{1-t}\widehat{f})(x)$, 由引理中的条件可知 F 是充分光滑的，使得 (1.1.6) 成立. 令 $t=1,s=0$, 通过一些必要的计算得

$$\widehat{f}(B_1)-E\widehat{f}(B_1)=\int_0^1\langle\nabla_x(P_{1-u}\widehat{f})(B_u),dB_u\rangle.$$

由 $|\widehat{f}(x)-\widehat{f}(y)|\leq\sigma\|x-y\|$ 易知 $P_{1-u}\widehat{f}$ 满足同样的不等式，从而 $\|\nabla_xP_{1-u}\widehat{f}\|\leq\sigma$ a.s. 然后由 (1.1.5) 易得

$$P\{|\widehat{f}(B_1)-E\widehat{f}(B_1)|>\lambda\}\leq 2e^{-\frac{1}{2}\lambda^2/\sigma^2}. \tag{1.1.7}$$

最后注意到 $f(X)\overset{L}{=}f(V_D^{\frac{1}{2}}B_1)$, 其中 $V_D^{\frac{1}{2}}$ 满足 $V_D=V_D^{\frac{1}{2}}(V_D^{\frac{1}{2}})'$. 令 $\widehat{f}=f(V_D^{\frac{1}{2}}x)$, 则函数 $\widehat{f}$ 满足所需条件，由 (1.1.7) 即得 (1.1.3).

定理 1.1.1 的证明

如果函数 $\sup(\cdot)$ 是充分光滑的，则定理 1.1.1 由引理 1.1.1 即得. 但是，众所周知，函数 $\sup(\cdot)$ 在对角线上不可导. 然而幸运的是， $\sup(\cdot)$ 可由光滑函数逼近. 由标准的函数逼近方法即可完成证明.

显然由 (1.1.2) 有

$$\lim_{\lambda\to\infty}\lambda^{-2}\log P\{\sup_{t\in T}X_t>\lambda\}=-(2\sigma^2)^{-1}.$$

有许多比这个不等式更精细的不等式. 在 $T=[0,h]$ 的情形，下述结论是最佳的.

定理 1.1.2 令 X 为 $\mathcal{R}$ 上的零均值可分 Gauss 过程，其方差为 1，协方差函数 $\Gamma(s,t)$ 满足

$$\Gamma(s,t) = 1 - C_0|s-t|^\alpha + o\big(|s-t|^\alpha\big) \quad \text{当}|s-t| \to 0\text{时},$$

其中 $0 < \alpha \le 2$, $C_0 > 0$. 则对任何 $h > 0$ 和 $\theta > 0$ 有

$$\lim_{\lambda\to\infty} \frac{P\{\max_{t\in[0,h]} X(t) > \lambda\}}{\lambda^{2/\alpha}\Psi(\lambda)} = hC_0^{1/\alpha} H_\alpha,$$

$$\lim_{\lambda\to\infty} \frac{P\big\{\max_{0\le j\le[h\lambda^{2/\alpha}/\theta]} X(j\theta\lambda^{-2/\alpha}) > \lambda\big\}}{\lambda^{2/\alpha}\Psi(\lambda)} = hC_0^{1/\alpha}\frac{H_\alpha(\theta)}{\theta},$$

其中 $H_\alpha(\theta)$ 是只依赖于 θ 和 α 的正常数，满足 $\lim_{\theta\to0} H_\alpha(\theta)/\theta = H_\alpha$, $0 < H_\alpha := \lim_{T\to\infty} T^{-1}\int_0^\infty e^s P\big\{\sup\limits_{0\le t\le T} Y(t) > s\big\}\, ds < \infty$, $Y(t)$ 是一个非平稳 Gauss 过程，具有均值 $EY(t) = -|t|^\alpha$ 和协方差函数 $Cov(Y(s),Y(t)) = -|t-s|^\alpha + |s|^\alpha + |t|^\alpha$.

这一结果由 Pickands (1969a,b) 得到，详细而完整的证明可在 Leadbetter, Lindgrenn 和 Rootzén (1983) 中找到，这里不再列出. 这一结果在 $\mathcal{R}^k$ 中的推广，可参看 Qualls 和 Watanabe(1973). 我们知道 $H_1 = 1$ 和 $H_2 = 1/\sqrt{\pi}$, 除了这两特殊情形，人们还不知道 H_α 的精确值. 然而 Shao (1996a) 给出了 H_α 的上下界估计和两个统计估计:

$$\begin{aligned}
&5.2^{-1/\alpha}0.625 \le H_\alpha \le (\alpha e/\sqrt{\pi})^{2/\alpha}, \quad \text{若 } 1 \le \alpha \le 2;\\
&(\alpha/4)^{1/\alpha}\big(1 - e^{-1/\alpha}(1+1/\alpha)\big) \le H_\alpha\\
&\quad \le \big(\sqrt{\alpha}(0.77\sqrt{\alpha} + 2.41(8.8 - \alpha\log(0.4 + 2.5/\alpha))^{1/2})\big)^{2/\alpha},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{若 } 0 < \alpha < 1.
\end{aligned}$$

特别地，我们有

$$0.12 \le H_\alpha \le 3.1, \qquad \text{若 } 1 \le \alpha \le 2,$$

$$\lim_{\alpha\to0} \alpha\log H_\alpha/\log\alpha = 1.$$

1.1.2 Fernique 不等式

现在，我们考察 $T=[0,1]^k$ 的情形，并设 X 为零均值可分 Gauss 过程，具有协方差函数 Γ. 对任何 $(s,t)\in T\times T$, 令 $d(s,t)=|s-t|=\sup_{1\le i\le k}|s_i-t_i|$. 记 $\varphi:[0,1]\to\mathcal{R}_+$ 为一个实函数

$$\begin{aligned}\varphi(h)&=\sup_{\substack{(s,t)\in T\times T\\|s-t|\le h}}\sqrt{E(X(s)-X(t))^2}\\&=\sup_{\substack{(s,t)\in T\times T\\|s-t|\le h}}\sqrt{\Gamma(s,s)-2\Gamma(s,t)+\Gamma(t,t)}.\end{aligned}\tag{1.1.8}$$

定理 1.1.3 假设 $\int_1^\infty\varphi(e^{-x^2})\,dx<\infty$, 并设对某个 A, $EX^2(t)\le A^2$ 对任何 $t\in T$ 成立. 则对 $x\ge\sqrt{1+4k\log p}$ 我们有

$$\begin{aligned}&P\left\{\sup_{t\in T}|X(t)|\ge x\left\{A+(2+\sqrt{2})\int_1^\infty\varphi(\frac{1}{2}p^{-u^2})du\right\}\right\}\\&\qquad\le\frac{5}{2}p^{2k}\int_x^\infty e^{-u^2/2}du,\end{aligned}\tag{1.1.9}$$

其中 $p\ge2$ 为整数.

证明 为了叙述简单，对任何定义在 $S=T$ 或 $T\times T$ 上的 f, 我们定义范数: $\|f\|=\sup_{s\in S}|f(s)|$. 对整数 $m>0$, 令 $I_m=\{i=(i_j);1\le j\le k,0\le i_j<m\}$, 对任何 $i\in I_m$ 定义

$$\begin{aligned}A_i^m&=\{x\in[0,1)^k:\forall j\in[1,k],i_j\le mx_j<i_j+1\},\\a_i^m&=\left(\frac{2i_j+1}{2m},1\le j\le k\right).\end{aligned}$$

对每一个 m, 定义 X 在 $[0,1)^k$ 上的一个 (惟一的) 逼近 X_m :

$$\forall i\in I_m,\quad\forall x\in A_i^m,\quad X_m(x)=X(a_i^m).$$

则 $\|X_m\|$ 为 m^k 个方差不超过 A 的零均值 Guass 变量的绝对值的最大值，从而

$$\forall y \in \mathcal{R}_+, \quad P\{\|X_m\| \geq yA\} \leq m^k \sqrt{\frac{2}{\pi}} \int_y^\infty e^{-u^2/2}\, du. \tag{1.1.10}$$

给定整数 m_1，取整数 $m_2 > m_1$ 使得 m_2/m_1 仍然是整数．则 $\{A_i^{m_2} : i \in I_{m_2}\}$ 是 $\{A_i^{m_1} : i \in I_{m_1}\}$ 的一个分划，且 $\|X_{m_1} - X_{m_2}\|$ 是 m_2^k 个方差不超过 $\varphi\left(\frac{1}{2m_1}\right)$ 的零均值 Gauss 变量的绝对值的最大值．因此 $\forall y \in \mathcal{R}_+$，

$$\begin{aligned} & P\left\{\|X_{m_1} - X_{m_2}\| \geq y\varphi\left(\frac{1}{2m_1}\right)\right\} \\ \leq & m_2^k \sqrt{\frac{2}{\pi}} \int_y^\infty e^{-u^2/2}\, du. \end{aligned} \tag{1.1.11}$$

假设 $\{y_n; n \geq 0\}$ 是一列正实数，$\{m_n; n \geq 1\}$ 是一列正整数，使得对每个 n, m_{n+1}/m_n 是整数．由 (1.1.10) 和 (1.1.11) 得

$$\begin{aligned} & P\Big\{\|X_{m_1}\| + \sum_{n=1}^\infty \|X_{m_n} - X_{m_{n+1}}\| \geq y_0 A + \sum_{n=1}^\infty y_n \varphi\Big(\frac{1}{2m_n}\Big)\Big\} \\ & \quad \leq \sqrt{\frac{2}{\pi}} \sum_{n=0}^\infty (m_{n+1})^k \int_{y_n}^\infty e^{-u^2/2}\, du. \end{aligned} \tag{1.1.12}$$

令 $A = \bigcup_n \{a_i^{m_n} : i \in I_{m_n}\}$，则 A 是 $[0,1]^k$ 中的可数稠密子集．由于 X 可分，$\|X\|$ 与 $\sup_{t\in A} |X(t)|$ 几乎处处相等，而后者不超过 $\|X_{m_1}\| + \sum_{n=1}^\infty \|X_{m_n} - X_{m_{n+1}}\|$. 因而

$$\begin{aligned} & P\left\{\|X\| \geq y_0 A + \sum_{n=1}^\infty y_n \varphi\left(\frac{1}{2m_n}\right)\right\} \\ & \quad \leq \sqrt{\frac{2}{\pi}} \sum_{n=0}^\infty (m_{n+1})^k \int_{y_n}^\infty e^{-u^2/2}\, du. \end{aligned} \tag{1.1.13}$$

对整数 $p \geq 2$, 令

$$m_n = p^{2^n},\ y_n = x2^{n/2},\ x \geq \sqrt{1 + 4k\log p},\ x_n = 2^{n/2}, \forall n \geq 0.$$

则对任何 $n \geq 1$ 有

$$y_n \varphi\left(\frac{1}{2m_n}\right) \leq x(2+\sqrt{2})(x_n - x_{n-1})\varphi\left(\frac{1}{2}p^{-x_n^2}\right)$$
$$\leq x(2+\sqrt{2})\int_{x_{n-1}}^{x_n} \varphi\left(\frac{1}{2}p^{-u^2}\right)\,du,$$

因此

$$\sum_{n=1}^{\infty} y_n \varphi\left(\frac{1}{2m_n}\right) \leq x(2+\sqrt{2})\int_1^{\infty} \varphi\left(\frac{1}{2}p^{-u^2}\right)\,du.$$

另一方面，对任何 $n \geq 0$ 有

$$(m_{n+1})^k \int_{y_n}^{\infty} e^{-u^2/2}\,du$$
$$= \int_x^{\infty} \exp\left\{k2^{n+1}\log p + \frac{n}{2}\log 2 - \frac{v^2}{2}2^n\right\}\,dv$$
$$\leq \int_x^{\infty} \exp\left\{-\frac{v^2}{2} + 2k\log p + \frac{1}{2}(n\log 2 + 1 - 2^n)\right\}\,dv,$$

由此得

$$\sum_{n=0}^{\infty}(m_{n+1})^k \int_{y_n}^{\infty} e^{-u^2/2}\,du$$
$$\leq p^{2k}\sum_{n=0}^{\infty} 2^{n/2} e^{-\frac{2^n-1}{2}} \int_x^{\infty} e^{-u^2/2}\,du \leq \frac{5}{2}p^{2k}\int_x^{\infty} e^{-u^2/2}\,du.$$

定理得证.

推论 1.1.1 设 X 为 $T=[a,b]^k$ 上的零均值可分 Gauss 过程，具有协方差函数 Γ，并且 $EX^2(t) \leq A$ 对任何 $t \in T$ 成立. 令 $\varphi(h)$ 为 (1.1.8) 所定义. 则对任何 $x \geq \sqrt{1+4k\log p}$ 我们有

$$P\left\{\sup_{t\in T}|X(t)| \geq x\left(A + (2+\sqrt{2})\int_1^{\infty}\varphi\left(\frac{b-a}{2}p^{-u^2}\right)du\right)\right\}$$
$$\leq \frac{5}{2}p^{2k}\int_x^{\infty} e^{-u^2/2}\,du,$$

其中 $p \geq 2$ 为整数.

注 1.1.1 设 X 为 $T=[a,b]^k$ 上的可分过程. 假设存在 $T\times T$ 上的一个函数 $d(\cdot,\cdot)$ 和常数 $C_0,\gamma,\beta>0$ 使得

$$P\big\{|X(s)-X(t)|\geq xd(s,t)\big\}\leq C_0\exp\big(-\gamma d^{\beta}(s,t)\big),\quad \forall x\geq 0. \tag{1.1.14}$$

定义函数 $\varphi: T\to\mathcal{R}^+$ 为

$$\varphi(h)=\sup_{\substack{s,t\in T\\ |s-t|\leq h}} d(s,t).$$

由定理 1.1.3 的证明, 对任何 $x\geq\big((1+4k\log p)/\gamma\big)^{1/\beta}$ 有

$$P\Bigg\{\sup_{t\in T}|X(t)|\geq x\Bigg(\sup_{(s,t)\in T\times T} d(s,t)+(1-2^{-1/\beta})^{-1}\int_1^{\infty}\varphi\Big(\frac{b-a}{2}p^{-u^2}\Big)\,du\Bigg)\Bigg\}\leq C_0\frac{5}{2}p^{2k}\exp(-\gamma x^{\beta}),$$

其中 $p\geq 2$ 为整数.

§1.2 比较原理

这一节, 我们介绍 Gauss 过程的比较原理, 它们和尾概率估计是研究 Gauss 过程理论的非常重要而又有效的工具. 首先我们介绍著名的 Slepian 引理. Gauss 过程理论中的许多重要的基本结果都是通过 Slepian 引理证明的.

1.2.1 Slepian 不等式

有许多叙述 Slepian 型不等式的方式. 我们采用如下 Kahane (1986) 的公式, 它包涵了许多有兴趣的结果.

定理 1.2.1 设 $X=(X_1,\cdots,X_N)$ 和 $Y=(Y_1,\cdots,Y_N)$ 为 $\mathcal{R}^N$ 中的零均值 Gauss 向量. 假设

$$EX_iX_j\leq EY_iY_j \qquad 若 (i,j)\in A,$$

$$EX_iX_j \geq EY_iY_j \qquad 若 (i,j) \in B,$$

$$EX_iX_j = EY_iY_j \qquad 若 (i,j) \notin A \cup B,$$

其中 A 和 B 为 $\{1,\cdots,N\} \times \{1,\cdots,N\}$ 的子集. 若 h 为 $\mathcal{R}^N$ 上的函数, 其在分布意义下的二阶导数满足

$$D_{ij}h \geq 0 \qquad 若 (i,j) \in A,$$

$$D_{ij}h \leq 0 \qquad 若 (i,j) \in B,$$

其中 $D_i = \frac{\partial}{\partial x_i}$, $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, 则

$$Eh(X) \leq Eh(Y).$$

证明 我们不妨设 X 与 Y 独立. 对每个 $t \in [0,1]$, 令 $Z(t) = (1-t)^{1/2}X + t^{1/2}Y$ 和 $\psi(t) = Eh(Z(t))$. 我们要证 $\psi(0) \leq \psi(1)$, 这只要证 $\psi'(t) \geq 0$ 对任何 $t \in [0,1]$ 成立. 我们有

$$\psi'(t) = \sum_{i=1}^{N} E\{D_ih(Z(t))Z_i'(t)\}.$$

对固定的 t 和 i, 易知对每个 j 有

$$EZ_j(t)Z_i'(t) = E(Y_jY_i - X_jX_i)/2.$$

由定理中的假设我们可将 Z_j 表示为

$$Z_j(t) = \alpha_j Z_i'(t) + W_j,$$

其中 $(W_1,\cdots,W_N)$ 为一个与 X 和 Y 都独立的新的零均值 Gauss 向量序列, 并且当 $(i,j) \in A$ 时 $\alpha_j \geq 0$, 当 $(i,j) \in B$ 时 $\alpha_j \leq 0$, 当 $(i,j) \notin A \cup B$ 时 $\alpha_j = 0$. 如果我们把 $E\{D_ih(Z(t))Z_i'(t)\}$ 看作是 α_j ($(i,j) \in A \cup B$) 的函数, 对它求关于 α_j 的偏导数并注意到关于 h 的假设条件我们知, 这些偏导数值当 $(i,j) \in A$ 时是正的, 而当 $(i,j) \in B$ 时是负的. 这意味着 $E\{D_ih(Z(t))Z_i'(t)\}$ 对满

足 $(i,j)\in A$ 的 α_j 是单调增加的，而对满足 $(i,j)\in B$ 的 α_j 是单调减少的. 另一方面，当所有 α_j 为 0 时，这一函数取值为 0, 这是因为

$$\begin{aligned}E\{D_ih(Z(t))Z_i'(t)\} &= E\{D_ih(W)Z_i'(t)\}\\ &= E\{D_ih(W)\}E\{Z_i'(t)\}=0.\end{aligned}$$

因此， $E\{D_ih(Z(t))Z_i'(t)\}\geq 0$. 从而 $\psi'(t)\geq 0$, 即我们要证的.

由此定理立即可得下述 Slepian 不等式，这只要在定理中取 $A=\{(i,j),i\neq j\}$, $B=\emptyset$ 和 $h=I_G$, 其中 G 为区间 $(-\infty,\lambda_j]$ 的乘积.

推论 1.2.1 (Slepian 不等式) 设 X 和 Y 为 $\mathcal{R}^N$ 中的零均值 Gauss 向量，满足 $EX_i^2=EY_i^2$ $(\forall i)$ 且

$$EX_iX_j\leq EY_iY_j\quad \forall i\neq j.$$

则对任何实数 λ_i 和 $i\leq N$ 有

$$P\Big\{\bigcup_{i=1}^N(Y_i>\lambda_i)\Big\}\leq P\Big\{\bigcup_{i=1}^N(X_i>\lambda_i)\Big\}.$$

特别地，由分部积分法我们有

$$E\max_{i\leq N}Y_i\leq E\max_{i\leq N}X_i.$$

下述 Gordon 的结果是 Slepian 不等式的一个有趣的推广.

推论 1.2.2 设 $X=(X_{ij})$ 和 $Y=(Y_{ij})$, $1\leq i\leq n, 1\leq j\leq m$ 为零均值 Gauss 向量，满足

$$\begin{aligned}EX_{ij}^2&=EY_{ij}^2\quad \forall i,j;\\ EX_{ij}X_{ik}&\leq EY_{ij}Y_{ik}\quad \forall i,j,k;\\ EX_{ij}X_{lk}&\geq EY_{ij}Y_{lk}\quad \forall i\neq l\text{ 和}j,k.\end{aligned}$$

则对任何实数 λ_{ij} 有

$$P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(Y_{ij}>\lambda_{ij})\Big\}\le P\Big\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(X_{ij}>\lambda_{ij})\Big\}.$$

这蕴涵了对任何 $\mathcal{R}$ 上的单调增加函数 g 有

$$E\Big\{\min_{i\le n}\max_{j\le m} g(Y_{ij})\Big\}\le E\Big\{\min_{i\le n}\max_{j\le m} g(X_{ij})\Big\},$$

和对任何实数 λ 有

$$P\Big\{\min_{i\le n}\max_{j\le m} Y_{ij}\ge\lambda\Big\}\le P\Big\{\min_{i\le n}\max_{j\le m} X_{ij}\ge\lambda\Big\}.$$

证明 令 $N=mn$. 对 $I\in\{1,\cdots,N\}$, 设 $i=i(I)$, $j=j(I)$ 为惟一的 $1\le i\le n$, $1\le j\le m$ 使得 $I=m(i-1)+j$. 考察 $\mathcal{R}^N$ 上如下形式的 Gauss 向量 X 和 Y : $X_I=X_{i(I),j(I)}$, $Y_I=Y_{i(I),j(I)}$. 令

$$A=\{(I,J):i(I)=i(J)\},\quad B=\{(I,J):i(I)\ne i(J)\}.$$

则定理 1.2.1 中第一组条件满足. 令 h 为下述集合的示性函数

$$\bigcup_{i=1}^{n}\bigcap_{I:i(I)=i}\{x\in\mathcal{R}^N;X_I>\lambda_{i,j(I)}\}.$$

通过取余集, 由定理 1.2.1 即得证推论.

Slepian 不等式不能用于 $\sup_{t\in T}|X_t|$. 为了说明这一点, 我们取 $T=\{1,2\}$, X_1 和 X_2 为标准的正态变量, 它们的相关系数为 ρ. 记 $P_\rho(\lambda)$ 为 $\max(X_1,X_2)>\lambda$ 的概率, Ψ 为标准正态分布的尾概率, 易知

$$\begin{aligned}P_{-1}(\lambda)=&P_{-1}\{X_1\vee X_2>\lambda\}=P\{|X|>\lambda\}=2\Psi(\lambda),\\ &P_0(\lambda)=2\Psi(\lambda)-\Psi^2(\lambda),\\ P_1(\lambda)=&P_1\{X_1\vee X_2>\lambda\}=\{X>\lambda\}=\Psi(\lambda).\end{aligned}$$

从而正如 Slepian 不等式所示，我们有 $P_{-1}(\lambda) \geq P_0(\lambda) \geq P_1(\lambda)$. 但是如果记 $\widehat{\mathrm{P}}_\rho(\lambda)$ 为 $\max(|X_1|, |X_2|) > \lambda$ 的概率，则 $\widehat{P}_{-1}(\lambda) = \widehat{P}_1(\lambda) = 2\Psi(\lambda)$, $\widehat{P}_0(\lambda) = 4\{\Psi(\lambda) - \Psi^2(\lambda)\}$, 从而对任何 $\lambda > 0$ 有

$$\widehat{P}_{-1}(\lambda) \leq \widehat{P}_0(\lambda), \quad \widehat{P}_0(\lambda) \geq \widehat{P}_1(\lambda),$$

因此 Slepian 引理中所要求的单调性不满足.

能够应用于 $\sup_{t \in T} |X(t)|$ 的一个不等式是 Anderson 不等式.

1.2.2 Anderson 不等式

定理 1.2.2 设 E 为 $\mathcal{R}^N$ 中的凸集，且关于原点对称. 设 $f(x) \geq 0$ 为一函数，满足

(i) $f(x) = f(-x)$,

(ii) 对每个 u $(0 < u < \infty)$, $K_u := \{x : f(x) \geq u\}$ 是凸集,

(iii) $\int_E f(x)\,dx < \infty$ (在 Lebesgue 积分意义下).

则对任何 $0 \leq h \leq 1$ 成立

$$\int_E f(x + hy)\,dx \geq \int_E f(x + y)\,dx.$$

证明这个定理之前，我们先给出一些推论.

推论 1.2.3 设 X 为 $\mathcal{R}^N$ 中的一零均值 Gauss 向量，E 为 $\mathcal{R}^N$ 中一个关于原点对称的凸集，$x \in \mathcal{R}^N$. 则对任何 $0 \leq |h| \leq 1$ 成立

$$P\{X + x \in E\} \leq P\{X + hx \in E\}.$$

证明 因为由对称性有 $P\{X - hx \in E\} = P\{-X + hx \in -E\} = P\{X + hx \in E\}$, 我们可设 $h \geq 0$. 令 $f(x) = (2\pi)^{-N/2} \exp\{-\frac{1}{2}x'\Sigma^{-1}x\}$ 为 X 的密度函数，其中 Σ 为正定矩阵. 由定理 1.2.2, 结论得证.

注 1.2.1 推论 1.2.3 也可由 (1.0.2) 得到. 为了说明这一点，我们只需考察 X 的分布为 γ_N 的情形. 这时在 (1.0.2) 中取 $A = E + x$, $B = E - x$ 和 $\lambda = (h + 1)/2$, 得

$$
\begin{aligned}
\operatorname{inv}\Phi\big(\gamma_N(E-hx)\big) &\geq \operatorname{inv}\Phi\big(\gamma_N(\lambda A+(1-\lambda)B)\big) \\
&\geq \lambda \operatorname{inv}\Phi\big(\gamma_N(E+x)\big)+(1-\lambda)\operatorname{inv}\Phi\big(\gamma_N(E-x)\big).
\end{aligned}
$$

由对称性，我们有

$$
\operatorname{inv}\Phi\big(\gamma_N(E-hx)\big) \geq \operatorname{inv}\Phi\big(\gamma_N(E-x)\big),
$$

由此得 $\gamma_N(E-hx) \geq \gamma_N(E-x)$, 这就是所要证的.

推论 1.2.4 设 X_1 和 X_2 为 $\mathcal{R}^N$ 中两个均值为零的 Gauss 向量，其协方差矩阵分别为 Σ_1 和 Σ_2. 若 $\Sigma_2-\Sigma_1$ 是半正定的，E 为一个关于原点对称的凸集，则

$$
P\{X_1 \in E\} \geq P\{X_2 \in E\}.
$$

证明 令 Y 为 $\mathcal{R}^N$ 中与 X_1 独立的零均值 Gauss 向量，其协方差矩阵为 $\Sigma_2-\Sigma_1$. 则 X_2 与 X_1+Y 同分布. 由推论 1.2.3, 我们有

$$
\begin{aligned}
P\{X_2 \in E\} = & P\{X_1+Y \in E\} = \int P\{X_1+y \in E\}\, dP_Y(y) \\
& \leq \int P\{X_1 \in E\}\, dP_Y(y) = P\{X_1 \in E\}.
\end{aligned}
$$

由推论 1.2.4 我们立即可得下述结果.

推论 1.2.5 设 $\{X_i(t); 0 \leq t \leq T\}$ (i=1,2) 为一零均值 Gauss 过程，其协方差矩阵为 $\Gamma_i(t,s)$. 假设 $\Gamma_2(t,s)-\Gamma_1(t,s)$ 为正定函数. 则

$$
P\left\{\int_0^T X_1^2(t)\, dt \leq x\right\} \geq P\left\{\int_0^T X_2^2(t)\, dt \leq x\right\}.
$$

若 $X_i(t)$ 为可分过程，则

$$
P\Big\{\sup_{0\leq t\leq T}|X_1(t)| \leq x\Big\} \geq P\Big\{\sup_{0\leq t\leq T}|X_2(t)| \leq x\Big\}.
$$

现在我们来证明定理 1.2.2.

定理 1.2.2 的证明 等价地我们只要证

$$\int_{E+hy} f(x)\,dx \geq \int_{E+y} f(x)dx.$$

我们先证明对任何 u 有

$$\mathrm{vol}\{(E+hy)\cap K_u\} \geq \mathrm{vol}\{(E+y)\cap K_u\},$$

其中 $\mathrm{vol}\{\cdot\}$ 表示集合的体积. 令 $\alpha=(h+1)/2$, 则 $\alpha y+(1-\alpha)(-y)=hy$. 从而由 K_u 的凸性和 $E+hy\supset\alpha(E+y)+(1-\alpha)(E-y)=\{\alpha E+(1-\alpha)E\}+hy$ 我们有 $(E+hy)\cap K_u\supset\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}$. 因此

$$\mathrm{vol}\{(E+hy)\cap K_u\}\geq\mathrm{vol}\big\{\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}\big\}.$$

由于 $(E+y)\cap K_u$ 和 $(E-y)\cap K_u$ 关于原点对称, 从而它们的体积相等. Brunn-Minkowski 定理 (参见 Bonnsen 和 Fenchel 1948) 告诉我们 $\mathrm{vol}^{1/N}\{(1-\theta)E_0+\theta E_1\}\geq(1-\theta)\mathrm{vol}^{1/N}(E_0)+\theta\,\mathrm{vol}^{1/N}(E_1)$ (其中 E_0 和 E_1 为非空集合, $0\leq\theta\leq1$). 因此

$$\mathrm{vol}\big\{\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}\big\}\geq\mathrm{vol}\{(E+y)\cap K_u\},$$

从而

$$H(u):=vol\{(E+hy)\cap K_u\}\geq vol\{(E+y)\cap K_u\}:=H^*(u).$$

由 Lebesgue 积分和 Lebesgue-Stielties 积分的定义得

$$\begin{aligned}\int_{E+hy} f(x)dx-\int_{E+y} f(x)dx&=-\int_0^\infty u\,dH(u)+\int_0^\infty udH^*(u)\\&=\int_0^\infty ud\{H^*(u)-H(u)\}.\end{aligned}$$

由分部积分法, 对任何 $b>a\geq0$ 有

$$\begin{aligned}\int_a^b ud\{H^*(u)-H(u)\}&=b\{H^*(b)-H(b)\}-a\{H^*(a)-H(a)\}\\&\quad+\int_a^b\{H(u)-H^*(u)\}\,du\geq b\{H^*(b)-H(b)\}.\end{aligned}$$

因为 $f(x)$ 在 E 上的积分有限，所以当 $b\to\infty$ 时，$bH(b)\to 0$ 且 $bH^*(b)\to 0$. 因而 $\int_0^\infty ud\{H^*(u)-H(u)\}\ge 0$, 结论得证.

利用 Anderson 不等式 (推论 1.2.3), 可得 Marcus (1968) 的下述关于 $\max_i |X_i|$ 的不等式.

定理 1.2.3 设 $X=(X_1,\cdots,X_N)$ 为 $\mathcal{R}^N$ 上的零均值 Gauss 向量，具有正定的协方差矩阵 Σ. 记 Σ_i 为 Σ 的第 i 个主子行列式. 定义 $\rho_i=\Sigma_{i-1}/\Sigma_i$, $i=1,\cdots,N$, $\Sigma_0=1$. 则

$$P\Big\{\max_{1\le i\le N}|X_i|\le a\Big\}\le\prod_{i=1}^N\sqrt{\frac{2}{\pi}}\int_0^{a(\rho_i)^{1/2}}e^{-t^2/2}\,dt.$$

证明 令 $A=\Sigma^{-1}$, $A=P'P$, 其中 $P=(P_{ij})$ 为对称正定矩阵. 令 $Y_i=\sum_j P_{ij}X_j$. 则 $Y_1,\cdots,Y_N$ 为独立的标准正态变量，且 Y_i 具有下述形式:

$$Y_i=(\rho_i)^{1/2}X_i+f_i(Y_{i-1},\cdots,Y_1),\quad i=1,\cdots,N.$$

则由推论 1.2.3, 有

$$\begin{aligned}
&P\Big\{\max_{1\le i\le N}|X_i|\le a\Big\}\\
=&P\Big\{\bigcap_{i=1}^N(|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\le a(\rho_i)^{1/2})\Big\}\\
=&E\Big(I\Big\{\bigcap_{i=1}^{N-1}\{|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\le a(\rho_i)^{1/2}\}\Big\}\\
&\quad\cdot P\{|Y_N-f_N(Y_{N-1},\cdots,Y_1)|\le a(\rho_N)^{1/2}|Y_{N-1},\cdots,Y_1\}\Big)\\
\le&E\Big(I\Big\{\bigcap_{i=1}^{N-1}\{|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\le a(\rho_i)^{1/2}\}\Big\}\\
&\quad\cdot P\{|Y_N|\le a(\rho_N)^{1/2}\}\Big)\\
\le&\cdots\le\prod_{i=1}^N P\{|Y_i|\le a(\rho_i)^{1/2}\}=\prod_{i=1}^N\sqrt{\frac{2}{\pi}}\int_0^{a(\rho_i)^{1/2}}e^{-t^2/2}\,dt.
\end{aligned}$$

推论 1.2.6 设 $X(t)$ 为一实值平稳 Gauss 过程，对某个 $\delta > 0$，其协方差函数 $\gamma(h) = EX(t)X(t+h)$ 为 $[0,\delta]$ 上的凸函数. 设 $t_0 < t_1 < \cdots < t_N, t_N - t_0 \le \delta$. 则

$$P\Big\{\max_{1\le i\le N}|\xi_i| \le a\Big\} \le \prod_{i=1}^{N}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}a/\alpha_{ii}} e^{t^2/2}\,dt,$$

其中 ξ_i 或者都为 $X(t_i) - X(t_{i-1})$，或者都为 $X(t_i) - X(t_0)$，$\alpha_{ii} = \sigma(t_i - t_{i-1})$，$i = 1,\cdots,N$，$\sigma^2(t) = E(X(t) - X(0))^2$.

证明 令 $\xi_i = X(t_i) - X(t_{i-1})$，$i = 1,\cdots,N$. 设 $A = (a_{ij}) = (E\xi_i\xi_j)$ 为 $(\xi_1,\cdots,\xi_N)$ 的协方差矩阵. 由 $\gamma(h)$ 的凸性，对 $i \ne j$ 有 $a_{ij} \le 0$. 对每个 i，记 Σ_i 为 A 的第 i 个主子行列式，记

$$S_u^i = \sum_{\substack{v=1\\ v\ne u}}^{i}|a_{uv}| = a_{uu} - \sum_{v=1}^{i} a_{uv} := \sigma_u^i a_{uu}, \quad 1 \le u \le i,$$

和

$$t_i^i = \max_{1\le k\le i-1}\sigma_k^i.$$

由 $\gamma(h)$ 的凸性易知

$$S_u^n = a_{uu} - E\big(X(t_u) - X(t_0)\big)\big(X(t_u) - X(t_{u-1})\big) < a_{uu}, \quad 1 \le u < n,$$

和

$$\begin{aligned} S_i^i &= \gamma(0) - \gamma(t_i - t_{i-1}) + \gamma(t_i - t_0) - \gamma(t_{i-1} - t_0) \\ &\le \gamma(0) - \gamma(t_i - t_{i-1}) = \frac{1}{2}a_{ii}. \end{aligned}$$

由此我们知矩阵 A 的对角线元素都为正，并且每一行上的非对角线元素的绝对值之和不超过这一行上的对角线元素值，从而 A 是正定的. 同样我们有

$$(a_{ii} + t_i^i S_i^i)\Sigma_{i-1} \ge \Sigma_i \ge (a_{ii} - t_i^i S_i^i)\Sigma_{i-1}.$$

因为 $S_k^i \le S_k^n \le a_{kk}$, 我们知 $0 \le t_i^i \le 1$. 从而

$$\frac{\Sigma_{i-1}}{\Sigma_i} \le \frac{2}{a_{ii}} = \frac{2}{\sigma^2(t_i - t_{i-1})}.$$

最后注意到由 $X(t_j) - X(t_{j-1}),\ j = 1, \cdots, i$ 的协方差矩阵得到的函数 Σ_{i-1}/Σ_i 与由 $X(t_j) - X(t_0),\ j = 1, \cdots, i$ 的协方差矩阵得到的函数 Σ_{i-1}/Σ_i 相同, 即得证结论.

1.2.3 Khatri-Šidák 不等式

设 $X = (X_1, \cdots, X_N)$ 为零均值 Gauss 向量满足 $EX_iX_j \le 0\ (i \ne j)$. Slepian 不等式 (推论 1.2.1) 告诉我们, 对任何实数 λ_i, $i \le N$ 成立

$$P\left\{\bigcap_{i=1}^{N}(X_i \le \lambda_i)\right\} \le \prod_{i=1}^{N} P\{X_i \le \lambda_i\}.$$

但是, Slepian 的证明不能推广到带绝对值的情形. 下述定理是 Slepian 不等式在带绝对值的情形时的一个类比.

定理 1.2.4 设 $(X_1, \cdots, X_N)$ 为 R^N 上的零均值 Gauss 向量, 则对任意的正数 λ_i, $i \le N$ 我们有

$$\begin{aligned} P\left\{\bigcap_{i=1}^{N}(|X_i| \le \lambda_i)\right\} &\ge P\left\{\bigcap_{i=1}^{N-1}(|X_i| \le \lambda_i)\right\} P\{|X_N| \le \lambda_N\} \\ &\ge \prod_{i=1}^{N} P\{|X_i| \le \lambda_i\}. \end{aligned}$$

我们可以把定理 1.2.4 改写成如下形式.

定理 1.2.4′ 设 $\{X(t); t \in T\}$ 为一零均值可分 Gauss 过程, $\{\lambda(t); t \in T\}$ 为一正实函数. 则对任何 $t_0 \in T$ 我们有

$$\begin{aligned} P\left\{\sup_{t \in T} \frac{|X(t)|}{\lambda(t)} \le 1\right\} \ge & P\left\{\sup_{t \in T \setminus \{t_0\}} \frac{|X(t)|}{\lambda(t)} \le 1\right\} \\ & \cdot P\left\{\frac{|X(t_0)|}{\lambda(t_0)} \le 1\right\}. \end{aligned}$$

定理 1.2.4 是由 Khatri (1967) 和 Šidák (1968) 得到的，因此叫作 Khatri-Šidák 不等式. 我们这里给出 Khatri 的证明，他的证明似乎简单一些. 事实上，定理 1.2.4 是下述命题的推论.

命题 1.2.1 设 $X=(X^{(1)},X^{(2)})$ 为 $\mathcal{R}^{m+n}$ 上的零均值 Gauss 向量，其中 $X^{(1)}=(X_1^{(1)},\cdots,X_m^{(1)})$, $X^{(2)}=(X_1^{(2)},\cdots,X_n^{(2)})$ 分别为 $\mathcal{R}^m$ 和 $\mathcal{R}^n$ 上的 Gauss 向量. 设 D_1 和 D_2 分别为 $\mathcal{R}^m$ 和 $\mathcal{R}^n$ 上的关于原点对称的凸集. 如果矩阵 $\mathrm{Cov}(X^{(1)},X^{(2)}):=E(X_1^{(1)},\cdots,X_m^{(1)})'(X_1^{(2)},\cdots,X_n^{(2)})$ 的秩至多为 1, 则有

$$P\{X^{(1)}\in D_1,X^{(2)}\in D_2\}\geq P\{X^{(1)}\in D_1\}P\{X^{(2)}\in D_2\}.$$

为证命题 1.2.1, 我们需要一个引理.

引理 1.2.1 设 $g(X)$ 和 $h(X)$ 为 $\mathcal{R}^N$ 上的随机向量 X 的两个函数. 如果对 $\mathcal{R}^N$ 中的任何两点 x_1 和 x_2 有 $(g(x_1)-g(x_2))(h(x_1)-h(x_2))\geq 0$, 则有

$$Eg(X)h(X)\geq Eg(X)Eh(X),$$

反过来，如果对 $\mathcal{R}^N$ 中的任何两点 x_1 和 x_2 有 $(g(x_1)-g(x_2))(h(x_1)-h(x_2))\leq 0$, 则有

$$Eg(X)h(X)\leq Eg(X)Eh(X).$$

证明 设 Y 为 X 的独立复制. 则 $(g(X)-g(Y))(h(X)-h(Y))\geq 0$, 从而 $E(g(X)-g(Y))(h(X)-h(Y))\geq 0$, 即 $Eg(X)h(X)\geq Eg(X)Eh(X)$.

命题 1.2.1 的证明 令 Σ_1 和 Σ_2 分别为 $X^{(1)}$ 和 $X^{(2)}$ 的协方差矩阵. 由 $\mathrm{Cov}(X^{(1)},X^{(2)})$ 的秩至多为 1 的事实，我们知存在两个向量 $a=(a_1,\cdots,a_m)\in\mathcal{R}^m$ 和 $b=(b_1,\cdots,b_n)\in\mathcal{R}^n$ 使得 $\mathrm{Cov}(X^{(1)},X^{(2)})=a'b$, 并且可将 X 写成如下形式

$$X^{(1)}=Y^{(1)}+ag,\quad X^{(2)}=Y^{(2)}+bg,$$

其中 g 为一标准正态变量, $Y^{(1)}$ ($Y^{(2)}$) 为零均值 Gauss 向量, 具有协方差矩阵 $\Sigma_1 - a'a$ ($\Sigma_2 - b'b$), 且 $Y^{(1)}, Y^{(2)}$ 和 g 相互独立. 推论 1.2.3 告诉我们, $P\{Y^{(1)} + ay \in D_1\}$ 和 $P\{Y^{(2)} + by \in D_2\}$ 都为 $|y|$ 的单调增加函数, 由引理 1.2.1 有

$$\begin{aligned}
P\{X^{(1)} \in D_1, X^{(2)} \in D_2\} &= P\{Y^{(1)} + ag \in D_1, Y^{(2)} + bg \in D_2\} \\
&= \int P\{Y^{(1)} + ay \in D_1, Y^{(2)} + by \in D_2\}\, dP_g(y) \\
&= \int P\{Y^{(1)} + ay \in D_1\} P\{Y^{(2)} + by \in D_2\}\, dP_g(y) \\
&\geq \int P\{Y^{(1)} + ay \in D_1\}\, dP_g(y) \int P\{Y^{(2)} + by \in D_2\}\, dP_g(y) \\
&= P\{Y^{(1)} + ag \in D_1\} P\{Y^{(2)} + bg \in D_2\} \\
&= P\{X^{(1)} \in D_1\} P\{X^{(2)} \in D_2\},
\end{aligned}$$

命题得证.

定理 1.2.4 的证明 在命题 1.2.1 中取 $X^{(1)} = (X_1, \cdots, X_{N-1})$, $X^{(2)} = X_N$, $D_1 = \bigcap_{i=1}^{N-1}\{|x_i| \leq \lambda_i\}$ 和 $D_2 = \{|x_N| \leq \lambda_N\}$, 得

$$P\left\{\bigcap_{i=1}^{N}(|X_i| \leq \lambda_i)\right\} \geq P\left\{\bigcap_{i=1}^{N-1}(|X_i| \leq \lambda_i)\right\} P\{|X_N| \leq \lambda_N\}.$$

由归纳法, 结论得证.

定理 1.2.5 设 $X = (X_1, \cdots, X_N)$ 为 $\mathcal{R}^N$ 中的零均值 Gauss 向量, 其协方差矩阵 Γ 满足 $a_{ij} = \alpha_i\alpha_j(a_{ii}a_{jj})^{1/2}$ ($|\alpha_i| \leq 1, i \neq j$), 且 $a_{ii} > 0\ \forall i$. 则对任何正数 $\lambda_i, i \leq N$ 有

$$P\{|X_i| \geq \lambda_i, i = 1, \cdots, N\} \geq \prod_{i=1}^{N} P\{|X_i| \geq \lambda_i\}.$$

证明 令 $\sigma_i^2 = a_{ii}$. 由假设, 可将协方差矩阵 Γ 写成 $\Gamma = T + \alpha'\alpha$, 其中 T 为一个 $N \times N$ 对角线矩阵, 其对角线元素为 $\sigma_i^2(1 - \alpha_i^2)$, $\alpha = (\sigma_1\alpha_1, \cdots, \sigma_N\alpha_N)$. 可将 X 写成

$$X = Y + \alpha g,$$

其中 $Y=(Y_1,\cdots,Y_N)$ 是 $\mathcal{R}^N$ 上的零均值 Gauss 向量, 其协方差矩阵为 T, g 为一标准正态变量且与 Y 独立. 从而 $Y_1,\cdots,Y_N,g$ 相互独立. 注意到由推论 1.2.3 可知对每个 i, $P\{|Y_i+\sigma_i\alpha_i y|\geq\lambda_i\}$ 是 $|y|$ 的单调增加函数, 由引理 1.2.1, 得

$$
\begin{aligned}
P\{|X_i|\geq\lambda_i, i=1,\cdots,N\} &= \int\prod_{i=1}^{N}P\{|Y_i+\sigma_i\alpha_i y|\geq\lambda_i\}\, dP_g(y)\\
&\geq\prod_{i=1}^{N}\int P\{|Y_i+\sigma_i\alpha_i y|\geq\lambda_i\}\, dP_g(y)=\prod_{i=1}^{N}P\{|X_i|\geq\lambda_i\},
\end{aligned}
$$

定理得证.

下面, 我们给出定理 1.2.4′ 的一个推广.

定理 1.2.4″ 设 $\{Y(t);t\in T\}=\{X_k(t);t\in T\}_{k=1}^{\infty}$ 为一列独立的零均值可分 Gauss 过程, $\{\lambda(t);t\in T\}$ 为一正实函数. 则对任何 $t_0\in T$ 有

$$
\begin{aligned}
&P\left\{\sup_{t\in T}\frac{\|Y(t)\|_{l^p}}{\lambda(t)}\leq 1\right\}\\
\geq&P\left\{\sup_{t\in T\setminus\{t_0\}}\frac{\|Y(t)\|_{l^p}}{\lambda(t)}\leq 1\right\}P\left\{\frac{\|Y(t_0)\|_{l^p}}{\lambda(t_0)}\leq 1\right\},
\end{aligned}
$$

其中 $p\geq 1$, $\|Y(t)\|_{l^p}^p=\sum_{k=1}^{\infty}|X_k(t)|^p$.

这一结果是下述命题的直接推论.

命题 1.2.2 设 $X=\{X_i(t);t\in T\cup\{t_0\}\}_{i=1}^N$ 为一列独立的零均值可分 Gauss 过程, D_1 为函数空间 $\mathcal{R}^{N\times T}=\{x(t)=(x_1(t),\cdots,x_N(t));t\in T\}$ 上的凸集, 在下述意义下对称, 即 $(x_1(\cdot),\cdots,x_N(\cdot))\in D_1$ 蕴涵了 $(\varepsilon_1x_1(\cdot),\cdots,\varepsilon_Nx_N(\cdot))\in D_1$ 对任何 $\varepsilon_i=\pm1, i=1\cdots,N$ 成立, D_2 为 $\mathcal{R}^N$ 中的凸集, 在下述意义下对称, 即 $(x_1,\cdots,x_N)\in D_2$ 蕴涵了 $(\varepsilon_1x_1,\cdots,\varepsilon_Nx_N)\in D_2$ 对任何

$\varepsilon_i = \pm 1, i = 1 \cdots, N$ 成立. 则

$$P\big\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\big\}$$
$$\geq P\big\{\{X(t); t \in T\} \in D_1\big\} P\{X(t_0) \in D_2\}.$$

证明 我们不妨设 T 是有限集. 令 $N_j(x_j)$, $N_{j,T}(x_j^{(T)})$ 和 $N_{j,0}(x_j^{(0)})$ 分别为 $\{X_j(t);\ t \in T \cup \{t_0\}\}$, $\{X_j(t);\ t \in T\}$ 和 $X_j(t_0)$ 的密度函数. 注意到对固定的 $\{X_2(t), \cdots, X_N(t); t \in T \cup \{t_0\}\}$, 集合 $D_1' = \big\{x_1(t); \{x(t); t \in T\} \in D_1\big\}$ 和 $D_2' = \{x_1(t_0); x(t_0) \in D_1\}$ 分别为 $\mathcal{R}^{1\times T}$ 和 $\mathcal{R}^1$ 中关于原点对称的凸集, 由命题 1.2.1 得

$$P\big\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\big\} = \int_{D_1 \times D_2} \prod_{j=1}^{N} N_j(x_j)\, dx_j$$
$$= \int_{D_1 \times D_2} N_1(x_1)\, dx_1 \prod_{j=2}^{N} N_j(x_j)\, dx_j$$
$$\geq \int_{D_1 \times D_2} N_{1,T}(x_1^{(T)})\, dx_1^{(T)} N_{1,0}(x_1^{(0)})\, dx_1^{(0)} \prod_{j=2}^{N} N_j(x_j)\, dx_j.$$

依次对 $x_2, x_3, \cdots, x_N$ 进行同样的处理我们得

$$P\big\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\big\}$$
$$\geq \int_{D_1 \times D_2} \prod_{j=1}^{N} \{N_{j,T}(x_j^{(T)})\, dx_j^{(T)} N_{j,0}(x_j^{(0)})\, dx_j^{(0)}\}$$
$$= P\big\{\{X(t); t \in T\} \in D_1\big\} P\big\{X(t_0) \in D_2\big\},$$

命题得证.

注 1.2.2 我们可把 Khatri-Šidáki 引理 (定理 1.2.4, 1.2.4$'$) 写成如下形式: 若 $(X_1, \cdots, X_n)$ 是零均值 Gauss 向量, 则

$$P(\max_{1 \leq i \leq n} |X_i| \leq 1) \geq P(|X_1| \leq 1) P(\max_{2 \leq i \leq n} |X_i| \leq 1). \qquad (1.2.1)$$

这是下述猜测的特殊情形：对任何 $1 \le k \le n$ 有

$$P(\max_{1\le i\le n}|X_i| \le 1) \ge P(\max_{1\le i\le k}|X_i| \le 1)P(\max_{k+1\le i\le n}|X_i| \le 1), \tag{1.2.2}$$

或等价地，设 A 和 B 为可分 Banach 空间 $\mathcal{E}$ 上的两个对称的凸集，X 和 Y 为 $\mathcal{E}$ 上的零均值 Gauss 向量，且 (X,Y) 是联合 Gauss 的，则

$$P(X \in A, Y \in B) \ge P(X \in A)P(Y \in B). \tag{1.2.3}$$

这就是 Gauss 相关性猜测 (Gaussian correlation conjecture, 参见：Gupta 等 1972, Tong 1980, Schechtman, Schumprecht 和 Zinn 1998 等等). (1.2.1) 说明猜测 (1.2.2) 对 $k=1$ 成立，命题 1.2.1 指出当 $\mathcal{E}$ 为 Euclidean 空间时，若 $\mathrm{Cov}(X,Y)$ 的秩至多为 1, 则 (1.2.3) 成立．在其它一些特殊情形，人们也证明了这一猜测正确．例如，Pitt (1977) 证明了当 $n=4, k=2$ 时，(1.2.2) 成立；Schechtman, Schumprecht 和 Zinn (1998) 证明了当集合 A, B 为对称椭球形的，或者集合不太大时，猜测正确；Harge (1998) 证明了若集合 A, B 中一个是对称椭球的，而另一个是简单对称凸的时，则猜测也正确等等．对于一般的情形，这一猜测正确与否还不得而知．Shao (1999) 给出了一个与 (1.2.2) 近似的不等式：

$$\begin{aligned}P(\max_{1\le i\le n}|X_i| \le 1)&\\ &\ge 2^{-k\vee(n-k)}P(\max_{1\le i\le k}|X_i| \le 1)P(\max_{k+1\le i\le n}|X_i| \le 1).\end{aligned} \tag{1.2.4}$$

Li (1999) 给出了相关性猜测的一个弱的形式：对任何 $0<\lambda<1$ 和 $\mathcal{E}$ 中的对称的凸集 A, B, 有

$$P(X \in A, Y \in B) \ge P(X \in \lambda A)P(Y \in (1-\lambda^2)^{1/2}B). \tag{1.2.5}$$

其中 X, Y 为 $\mathcal{E}$ 上的零均值 Gauss 向量，且 (X,Y) 是联合 Gauss 的.

(1.2.5) 可由 Anderson 不等式得到. 事实上, 令 $a=(1-\lambda^2)^{1/2}/\lambda$, 设 (X^*,Y^*) 为 (X,Y) 的独立复制. 易见 $X-aX^*$ 与 $Y+Y^*/a$ 独立. 由 Anderson 不等式 (推论 1.2.3) 得

$$\begin{aligned}P(X\in A,Y\in B)\geq&P\big((X,Y)+(-aX^*,Y^*/a)\in A\times B\big)\\=&P(X-aX^*\in A,Y+Y^*/a\in B)\\=&P(X-aX^*\in A)P(Y+Y^*/a\in B)\\\geq&P(X\in\lambda A)P\big(Y\in(1-\lambda^2)^{1/2}B\big).\end{aligned}$$

第二章　Gauss 过程的连续模和大增量的极限性质

从这一章起，我们开始研究 Gauss 过程及其相关过程的样本轨道性质. 这一章，我们着重研究实值 Gauss 过程，特别注重多参数过程的研究. 在 2.1 节，我们给出关于 Gauss 过程的连续性的一些基本结果. 在 2.2 节，我们研究一类简单而又非常重要的 Gauss 过程，即分数 Wiener 过程的连续模和大增量的极限性质，以此引入我们的正题. 2.3, 2.4 和 2.5 节研究几个特殊的两参数 Gauss 过程的连续模和大增量的极限性质. 在 2.3 节，我们研究"最简单"的两参数 Gauss 过程，即两参数 Wiener 过程，其增量既是平稳的又是独立的. 2.4 节研究的是两参数分数 Lévy-Wiener 过程，其增量是平稳的，但不再是独立的，这一过程是两参数 Lévy-Wiener 过程的推广. 在 2.5 节，我们考察的是两参数 Ornstein-Uhlenbeck 过程，其增量既不是平稳的也不是独立的，它是单参数 Ornstein-Uhlenbeck 过程的推广. 2.6 节考察较一般的两参数 Gauss 过程. 2.7 节考察 Gauss 过程的局部时过程 $L(x,t)$，这也是一类两参数随机过程.

§2.1　Gauss 过程的连续性

2.1.1　有界性和连续性

设 (T,d) 为一个拟距离空间. 对任意的 $\varepsilon>0$，记 $N(T,d;\varepsilon)$ 为距离熵 (entropy number)，即在距离 d 意义下，用来覆盖 T 的以 ε 为半径的开球的最小个数. 显然在距离 d 意义下，T 完全有界当且仅当对任何 $\varepsilon>0$ 有 $N(T,d;\varepsilon)<\infty$，这一性质在我们将要考察的条件下总是满足的. 记 $D=D(T)$ 为 (T,d) 的直径，即 $D=\sup\{d(s,t):s,t\in T\}$.

设 $X=\{X_t;t\in T\}$ 为一零均值 Gauss 过程，定义 T 上的正规距离为 $d_X(s,t)=\|X_t-X_s\|_2$, $s,t\in T$. 易知

$$E\sup_{s,t\in T}|X_s-X_t|=E\sup_{s,t}(X_s-X_t)=2E\sup_{t\in T}X_t.$$

从而对任何 $t_0\in T$ 有

$$\begin{aligned}E\sup_{t\in T}X_t\leq&E\sup_{t\in T}|X_t|\leq E|X_{t_0}|+E\sup_{s,t\in T}|X_s-X_t|\\ \leq&E|X_{t_0}|+2E\sup_{t\in T}X_t.\end{aligned}$$

这个不等式告诉我们当考察几乎处处有界性时，考察 $\sup_t X_t$ 和考察 $\sup_t|X_t|$ 是等价的.

下述结果告诉我们零均值的 Gauss 过程有界当且仅当它的最大值的矩有界.

定理 2.1.1 假设 X 为 T 上的零均值 Gauss 过程，(T,d_X) 有界. 则

$$P\{\sup_{t\in T}X_t<\infty\}=1\Longleftrightarrow E\sup_{t\in T}X_t<\infty.$$

证明 如通常一样，记 $\|X\|$ 为 $\sup_{t\in T}X_t$. 显然 $E\|X\|<\infty\Longrightarrow\|X\|<\infty$ a.s., 因此只要证 “$\Longrightarrow$”. 我们将证一个更强的结论，即对充分小的 $\alpha>0$ 有

$$Ee^{\alpha\|X\|^2}<M<\infty, \tag{2.1.1}$$

其中 M 为一绝对常数.

设 Y 和 Z 为 X 的两个独立的复制. 则因为 $(Y+Z)/\sqrt{2}$ 和 $(Y-Z)/\sqrt{2}$ 也是 X 的两个独立的复制，我们有 $(\|Y\|,\|Z\|)$ 和 $(\|Y+Z\|/\sqrt{2},\|Y-Z\|/\sqrt{2})$ 都是 $\|X\|$ 的两对独立的复制. 从而,

对任何实数对 (a,b) 有

$$\begin{aligned}
&P\{\|X\|\le a\}P\{\|X\|>b\}\\
&\quad = P\{\|Y+Z\|\le \sqrt{2}a, \|Y-Z\|>\sqrt{2}b\}\\
&\quad \le P\{\|Y\|>(b-a)\sqrt{2}, \|Z\|>(b-a)\sqrt{2}\}\\
&\quad \le \big(P\{\|Y\|>(b-a)\sqrt{2}\}\big)^2.
\end{aligned}$$

取 $a>0$ 使得 $q:=P\{\|X\|\le a\}\in(\frac{1}{2},1)$. 对

$$b=b_n=(\sqrt{2^{n+1}}-1)(\sqrt{2}+1)a$$

重复上述不等式易得, 对任何 $x\ge a$ 有

$$P\{\|X\|\ge x\}\le \exp\Big(-\frac{x^2}{24a^2}\log\frac{q}{1-q}\Big),$$

从而

$$Ee^{\alpha\|X\|^2}\le e^{\alpha a^2}+\int_a^\infty \exp\Big\{-\Big(\frac{1}{24a^2}\log\frac{q}{1-q}-\alpha\Big)x^2\Big\}\,dx.$$

取 $\alpha<(\log\frac{q}{1-q})/(24a^2)$ 得 (2.1.1).

注 2.1.1 在上面的证明中, 并没有用到 $\|\cdot\|$ 为最大范数的事实, 当 X 取值于一个可分 Banach 空间, $\|\cdot\|$ 为任意一个可测半范数时, 证明仍成立. 另外, 若 d 为 T 上的另一个距离, 且在这个距离意义下 T 有界, 如果要考察 Gauss 过程 X 相对于距离 d 的有界性, 只要简单地附加条件 “d_X 在 (T,d) 上有界” 即可.

下述定理将有界性和连续性紧密地联系在一起.

定理 2.1.2 设 $X=\{X_t;t\in T\}$ 几乎处处有界, 令 d 为 T 上的距离使得 d_X 是 d 一致连续的. 则 X 以概率 1, d 一致连续当且仅当

$$\lim_{\eta\to 0}E\sup_{d(s,t)<\eta}(X_s-X_t)=0.$$

为了证明定理 2.1.2, 我们需要熟知的 Borel-Cantelli 引理，这一引理在本书中将经常用到. 我们给出它的推广形式.

引理 2.1.1 设 $\{A_n; n \geq 1\}$ 为一事件序列, 如果 $\sum_{n=1}^{\infty} P(A_n) < \infty$, 则 $P(A_n, \text{ i.o.}) = 0$, 其中 $\{A_n, \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. 如果 $\sum_{n=1}^{\infty} P(A_n) = \infty$ 且

$$\liminf_{n\to\infty} \sum_{1\leq j<k\leq n} \left(P(A_jA_k) - P(A_j)P(A_k)\right) \Big/ \left(\sum_{j=1}^{n} P(A_j)\right)^2 = 0,$$

则 $P(A_n, \text{ i.o.}) = 1$.

证明 证明可在 Petrov(1995) 中找到. 由于证明较简单, 为了完整起见, 这里给出其详细证明. 首先, 引理的前一部分由下式即得:

$$\begin{aligned} P(A_n, \text{ i.o.}) =& P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) \leq P(\bigcup_{k=n}^{\infty} A_k) \\ &\leq \sum_{k=n}^{\infty} P(A_k) \to 0, \quad n \to \infty. \end{aligned}$$

这里用到了条件 $\sum_{n=1}^{\infty} P(A_k) < \infty$. 为证引理的后一部分, 令 $I_n = I_{A_n}$. 则 $EI_n = P(A_n)$. 从而

$$\begin{aligned} &\liminf_{n\to\infty} P\left\{\left|\sum_{k=1}^{n} I_k - \sum_{k=1}^{n} P(A_k)\right| \geq \frac{1}{2}\sum_{k=1}^{n} P(A_k)\right\} \\ &\leq \liminf_{n\to\infty} \frac{4\mathrm{Var}(\sum_{k=1}^{n} I_k)}{(\sum_{k=1}^{n} P(A_k))^2} \\ &= \liminf_{n\to\infty} \frac{8\sum_{1\leq k<l\leq n}\left(P(A_kA_l) - P(A_k)P(A_l)\right)}{(\sum_{k=1}^{n} P(A_k))^2} = 0. \end{aligned}$$

由此得

$$\liminf_{n\to\infty} P(B_n) = 0,$$

其中 $B_n = \{\sum_{k=1}^{n} I_k \leq \frac{1}{2}\sum_{k=1}^{n} P(A_k)\}$. 从而存在一个递增的整数序列 $\{n_m\}$ 使得 $\sum_{m=1}^{\infty} P(B_{n_m}) < \infty$. 由已证的引理前一部分

得：以概率 1，$\sum_{k=1}^{n_m} I_k \ge \frac{1}{2}\sum_{k=1}^{n_m} P(A_k)$ 至多对有限个 m 不成立，由此和条件 $\sum_{n=1}^{\infty} P(A_n)=\infty$ 得 $P(\sum_{k=1}^{\infty} I_k$ 发散$)=1$. 从而 $P(A_n,\ \text{i.o.})=1$ 得证.

定理 2.1.2 的证明 令 $\phi_d(\eta)=E\sup_{d(s,t)<\eta}(X_s-X_t)$. 先证必要性. 令 $U_\varepsilon=\{(s,t)\in T\times T; d(s,t)<\varepsilon\}$, $Y_{t,s}=X_t-X_s$. 则 $Y_{t,s}$ 为 $(T\times T, d_Y)$ 上的几乎处处有界的零均值 Gauss 过程，其中

$$d_Y\big((s,t),(s',t')\big)=\|(X_s-X_t)-(X_{s'}-X_{t'})\|_2\le d_X(s,t)+d_X(s',t').$$

由于 d_X 是 d 一致连续的，可取 $\varepsilon>0$ 使得 $(s,t),(s',t')\in U_\varepsilon \Longrightarrow d_Y\big((s,t),(s',t')\big)\le 1$. 从而 (U_ε,d_Y) 有界. 因此，由定理 2.1.1 得

$$E\sup_{d(s,t)<\varepsilon}|X_t-X_s|<\infty.$$

另外，对几乎每个 ω 我们有

$$\lim_{\eta\to 0}\sup_{d(s,t)<\eta}|X_t(\omega)-X_s(\omega)|=0,$$

从而由控制收敛定理知 $\lim_{\eta\to\infty}\phi_d(\eta)=0$.

反过来，可找到序列 η_n 使得 $\phi_d(\eta_n)\le 4^{-n}$. 考察事件 $A_n=\{\sup_{d(s,t)<\eta_n}|X_s-X_t|>2^{-n}\}$. 因为 $\sum_{n=1}^{\infty}P(A_n)\le\sum_{n=1}^{\infty}2^{-n}<\infty$, 所以由 Borel-Cantelli 引理知 X 几乎处处 d 一致连续. 定理得证.

2.1.2 Fernique 条件

现在考察 $T=[0,1]^k$ 的情形，并假设 X 为零均值 Gauss 过程, 其方差函数为 Γ. 对任意的 $(s,t)\in T\times T$ 定义 $d(s,t)=|s-t|=\sup_{1\le i\le k}|s_i-t_i|$. 令 $\varphi:[0,1]\to\mathcal{R}_+$ 为由 (1.1.8) 定义的函数，即

$$\varphi(h)=\sup_{\substack{(s,t)\in T\times T\\ d(s,t)\le h}}\|X_s-X_t\|_2. \tag{2.1.2}$$

定理 2.1.3 存在绝对常数 K 使得

$$E\sup_{t\in T}|X_t|\le K\left\{\sup_{t\in T}\|X_t\|_2+\int_1^\infty\varphi(e^{-x^2})\,dx\right\}. \tag{2.1.3}$$

进一步，如果 $\int_1^\infty\varphi(e^{-x^2})dx<\infty$，则 X 几乎处处一致连续（这里，连续是指 d 连续）.

证明 由定理 1.1.3 和注 1.1.1，得

$$\begin{aligned}E\sup_{t\in T}|X_t|\le&\left\{\sup_{t\in T}\|X_t\|_2+(2+\sqrt{2})\int_1^\infty\varphi\Big(\frac{1}{2}p^{-u^2}\Big)\,du\right\}\\&\cdot\left\{\sqrt{1+4k\log p}+\frac{5}{2\sqrt{e}}\right\}\\\le&K\left\{\sup_{t\in T}\|X_t\|_2+\int_1^\infty\varphi(e^{-u^2})\,du\right\},\end{aligned}$$

并且对任何 $t_0\in T$ 有

$$\begin{aligned}E\sup_{d(t,t_0)\le h}|X_t|\le&\left\{\varphi(h)+(2+\sqrt{2})\int_1^\infty\varphi(hp^{-u^2})\,du\right\}\\&\cdot\left\{\sqrt{1+4k\log p}+\frac{5}{2\sqrt{e}}\right\}\\\le&K\left\{\varphi(h)+\int_1^\infty\varphi(he^{-u^2})\,du\right\}.\end{aligned} \tag{2.1.4}$$

从而定理的前一部分得证，并且我们得到：对任何 $t_0\in T$, X 在 t_0 处几乎处处连续. 但是，我们要证明的是 X 一致连续. 为此，我们需要一些引理.

引理 2.1.2 设 $\{X_i(t);t\in T_i\}$, $i=1,2,\cdots,N$ 为零均值 Gauss 过程. 则

$$E\max_{i\le N}\sup_{t\in T_i}X_i(t)\le 2\max_{i\le N}E\sup_{t\in T_i}X_i(t)+3(\log N)^{1/2}\max_{i\le N}\sup_{t\in T_i}\|X_i(t)\|_2.$$

特别地，对任何实值零均值 Gauss 变量 $X_i, i=1,2,\cdots,N$ 我们有

$$E\max_{i\le N}X_i\le 5(\log N)^{1/2}\max_{i\le N}(EX_i^2)^{1/2}.$$

证明 记 $\sup_{t\in T_i} X_i(t)$ 为 $\|X_i\|$, $\sup_{t\in T_i}(EX_i^2(t))^{1/2}$ 为 $\sigma(X_i)$. 我们不妨设要证的不等式右边是有限的，否则不等式显然成立. 对 $\delta \geq 0$, 由分部积分法和 Borell 不等式 (见定理 1.1.1) 得

$$\begin{aligned}E\max_{i\leq N}\big|\|X_i\|-E\|X_i\|\big| &\leq \delta+\sum_{i=1}^{N}\int_{\delta}^{\infty}P\{\big|\|X_i\|-E\|X_i\|\big|>u\}\,du\\ &\leq \delta+N\int_{\delta}^{\infty}\exp\Big(-\frac{u^2}{2\max_{i\leq N}\sigma(X_i)}\Big)\,du.\end{aligned}$$

从而，令 $\delta=(2\log N)^{1/2}\max_{i\leq N}\sigma(X_i)$ 得

$$E\max_{i\leq N}\|X_i\|\leq 2\max_{i\leq N}E\|X_i\|+3(\log N)^{1/2}\max_{i\leq N}\sigma(X_i),$$

引理得证.

引理 2.1.3 设 $Y=\{Y_t; t\in T\}$ 为距离空间 (T,d) 上的零均值 Gauss 过程. 令 $d_Y(s,t)=\|Y_s-Y_t\|_2$. 则对任何 $\eta>0$ 有

$$\begin{aligned}E\sup_{d(s,t)<\eta}|Y_s-Y_t|\leq K\Big\{&\sup_{t\in T}E\sup_{d(s,t)<\eta}|Y_s-Y_t|\\ &+\sup_{d(s,t)<3\eta}d_Y(s,t)\big(\log N(T,d;\eta)\big)^{1/2}\Big\},\end{aligned}$$

其中 $K>0$ 为常数.

证明 给定 $\eta>0$, 令 $N=N(T,d;\eta)$ (假设它有限且大于 2). 令 $U=\{u_1,\cdots,u_N\}$ 为 T 的子集使得以 u_i 为中心 $\eta>0$ 为半径的 d 球覆盖 T. 显然

$$\sup_{d(s,t)<\eta}|Y_s-Y_t|\leq 2\max_{u\in U}\Big(\sup_{d(t,u)<\eta}|Y_t-Y_u|\Big)+\max_{\substack{u,v\in U\\ d(u,v)<3\eta}}|Y_u-Y_v|.$$

由引理 2.1.2, 得

$$\begin{aligned}E\max_{u\in U}\Big(\sup_{d(t,u)<\eta}|Y_t-Y_u|\Big)&=E\max_{u\in U}\Big(\sup_{d(t,u)<\eta}(Y_t-Y_u)\Big)\\ &\leq 2\max_{u\in U}E\Big(\sup_{d(t,u)<\eta}(Y_t-Y_u)\Big)+3\max_{d(s,t)<\eta}d_Y(s,t)(\log N)^{1/2};\end{aligned}$$

同理,

$$E \max_{\substack{u,v\in U \\ d(u,v)<3\eta}} |Y_u - Y_v| \le 5 \max_{d(s,t)<3\eta} d_Y(s,t)(\log N^2)^{1/2}.$$

从而引理得证.

定理 2.1.3 的证明的续 由 (2.1.4) 和引理 2.1.3, 得

$$\begin{aligned} &E \sup_{d(s,t)\le h} |X_t - X_s| \\ &\quad \le K\left\{\varphi(h) + \int_1^\infty \varphi(he^{-u^2})\, du + \varphi(3h)(\log h^{-1})^{1/2}\right\}. \end{aligned} \quad (2.1.5)$$

因为 $\int_1^\infty \varphi(e^{-u^2})\, du < \infty$, 我们有 $u\varphi(e^{-u^2}) \to 0$ $(u \to \infty)$, 从而 $\varphi(h)(\log h^{-1})^{1/2} \to 0$ ($h \to 0$). 因此, 我们得

$$\lim_{h\to 0} E \sup_{d(s,t)\le h} |X_t - X_s| = 0.$$

由定理 2.1.2 知 X 几乎处处连续. 定理得证.

若 $X = \{X(t); t \in [0,1]\}$ 为一个零均值、具有平稳增量的 Gauss 过程, 且 $\varphi(h) = \|X(t+h) - X(t)\|_2$ 为 h 的单调非降函数. 我们以后还将证明 Fernique 条件 $\int_1^\infty \varphi(e^{-u^2})\ du < \infty$ 也是 X 几乎处处有界或连续的必要条件.

2.1.3 主测度条件

如通常一样, 设 T 为一个零均值 Gauss 过程的参数空间, d_X 为 T 上的正规距离. 设 m 为 T 上的概率测度, $g: [0,1] \to \mathcal{R}_+$ 为一个函数, 定义为

$$g(t) = (\log t^{-1})^{1/2}, \quad 0 \le t \le 1.$$

令 $B(t,\varepsilon)$ 表示在距离 d_X 意义下以 t 为中心的 ε 球.

定义 2.1.1 一个概率测度 m 被称为 (相对于 (T, d_X) 的) 主测度 (majorizing measure), 如果

$$\sup_{t\in T}\int_0^\infty g\big(m(B(t,\varepsilon))\big)\,d\varepsilon < \infty. \tag{2.1.6}$$

下述关于 Gauss 过程的有界性和连续性的一般性结果是由 Talagrand (1987) 得到的, 它渊源于 Fernique(1978). 由于其证明太长, 这里不再叙述, 读者可参看 Ledoux 和 Talagrand (1991).

定理 2.1.4 设 $X = \{X_t; t \in T\}$ 为一个零均值 Gauss 过程. 则对某个绝对常数 $K > 0$ 和 (T, d_X) 上的任意概率测度 m 有

$$E\sup_{t\in T}X_t \le K\sup_{t\in T}\int_0^\infty g\big(m(B(t,\varepsilon))\big)\,d\varepsilon, \tag{2.1.7}$$

并且存在 (T, d_X) 上的一个概率测度 m 使得

$$\sup_{t\in T}\int_0^\infty g\big(m(B(t,\varepsilon))\big)\,d\varepsilon \le KE\sup_{t\in T}X_t. \tag{2.1.8}$$

也就是说, X 在 (T, d_X) 上几乎处处有界当且仅当 (T, d_X) 有一个主测度.

进一步, X 在 (T, d_X) 上几乎处处有界且 (一致) 连续当且仅当 (T, d_X) 完全有界且存在一个 (T, d_X) 上的概率测度 m 使得

$$\lim_{\eta\to 0}\sup_{t\in T}\int_0^\eta g\big(m(B(t,\varepsilon))\big)\,d\varepsilon = 0. \tag{2.1.9}$$

注 2.1.2 注意到, 若 $\varepsilon > 2D$, 其中 $D = D(T)$ 为 (T, d_X) 的直径, 则 $m(B(t,\varepsilon)) = 1$, 因而 (2.1.7) 和 (2.1.8) 中的积分上限实际上是 $D/2$. 另外, 设在 T 上有另一个距离 d 且 T 在 d 意义下是紧致的, 如果考察的是零均值 Gauss 过程 X 相对于距离 d 的连续性, 我们只要简单地附加条件: d_X 在 (T, d) 上连续, 也即 X 在

L_2 意义下 (或依概率) 连续. 事实上, 若 (T,d) 为任意一个紧致空间, 则零均值 Gauss 过程 $X=\{X_t;t\in T\}$ 在 (T,d) 上几乎处处连续当且仅当它在 (T,d_X) 上几乎处处连续且 d_X 在 (T,d) 上连续. 上述论断的充分性显然. 下设 X 是 d 连续的, 则 d_X 也是 d 连续的. 由紧致性, X 和 d_X 都是 d 一致连续的. 由定理 2.1.2 得

$$\lim_{\varepsilon\to 0} E\sup_{d(s,t)\le\varepsilon}|X_s-X_t|=0.$$

对任意的 $\eta>0$, 令 $A_\eta=\{(s,t)\in T\times T: d_X(s,t)\le\eta\}$. 这是 $T\times T$ 上的闭集且 $\bigcap_\eta A_\eta=A_0$. 易知 $\{(s,t)\in T\times T, d(s,s')\le\varepsilon, d(t,t')\le\varepsilon, (s',t')\in A_0\}$ 为闭集并且包含 A_0. 固定 $\varepsilon>0$. 由紧致性, 存在 $\eta>0$ 使得: 对任何 $(s,t)\in A_\eta$, 存在 $(s',t')\in A_0$ 满足 $d(s,s')\le\varepsilon$ 和 $d(t,t')\le\varepsilon$ 使得

$$|X_s-X_t|\le|X_s-X_{s'}|+|X_{s'}-X_{t'}|+|X_{t'}-X_t|$$

成立. 因为 $(s',t')\in A_0$, 我们知 $X_{s'}=X_{t'}$ 以概率 1 成立. 从而

$$E\sup_{d_X(s,t)\le\eta}|X_s-X_t|\le 2E\sup_{d(s,t)\le\varepsilon}|X_s-X_t|.$$

因此

$$\lim_{\eta\to 0} E\sup_{d_X(s,t)\le\eta}|X_s-X_t|=0.$$

再次由定理 2.1.2, 我们得 X 几乎处处 d_X 一致连续.

下述推论是 Dudley (1973) 的结果.

推论 2.1.1 设 $X=\{X_t;t\in T\}$ 为零均值 Gauss 过程. 则

$$E\sup_{t\in T}X_t\le K\int_0^\infty\left(\log N(T,d_X;\varepsilon)\right)^{1/2}d\varepsilon, \tag{2.1.10}$$

其中 $K>0$ 为绝对常数. 进一步, 若上式右边的熵积分收敛, 则 X 在 (T,d_X) 上几乎处处有界且 (一致) 连续.

易知 (2.1.10) 中熵积分的积分上限实际上是 D.

证明 可设直径 $D = D(T)$ 有限, 否则对某个 $\varepsilon > 0$ 有 $N(t, d_X; \varepsilon) = \infty$, 从而没有什么要证明的. 我们将证明存在 T 上的一个概率测度 m 使得

$$\sup_{t\in T}\int_0^D g\big(m(B(t,\varepsilon))\big)\,d\varepsilon \le K\int_0^D \big(\log N(T,d_X;\varepsilon)\big)^{1/2}\,d\epsilon, \tag{2.1.11}$$

其中 K 为绝对常数. 令 l_0 为使得 $2^{-l_0} \ge D$ 成立的最大整数. 对任意的 $l \ge l_0$, 令 $T_l \subset T$ 为能够覆盖 T 的个数最小的一族球 $\{B(t,2^{-l}), t \in T_l\}$ 的球心组成的集合. 由定义知 $Card\ T_l = N(T, d_X; 2^{-l})$. 考察 T 上的概率测度 m:

$$m = \sum_{l>l_0} 2^{-l+l_0} N(T,d_X;2^{-l})^{-1}\sum_{t\in T_l}\delta_t,$$

其中 δ_t 为在 t 处的单点测度 (Dirac 测度). 显然, 对任何 t 和 $l > l_0$ 有 $m(B(t,2^{-l})) \ge 2^{-l+l_0}N(T,d_X;2^{-l})^{-1}$. 从而

$$\begin{aligned}
\int_0^D g\big(m(B(t,\varepsilon))\big)\,d\varepsilon &\le \sum_{l>l_0} 2^{-l} g\big(m(B(t,2^{-l}))\big)\\
&\le \sum_{l>l_0} 2^{-l}\big(\log(2^{l-l_0}N(T,d_X;2^{-l}))\big)^{1/2}\\
&\le \sum_{l>l_0} 2^{-l}\big(\log 2^{l-l_0}\big)^{1/2} + \sum_{l>l_0} 2^{-l}\big(\log N(T,d_X;2^{-l})\big)^{1/2}\\
&\le 2^{-l_0+1}\int_0^{1/2}(\log x^{-1})^{1/2}\,dx + 2\int_0^D \big(\log N(T,d_X;\epsilon)\big)^{1/2}\,d\epsilon\\
&\le 4D\int_0^{1/2}\big(\log x^{-1}\big)^{1/2}\,dx + 2\int_0^D\big(\log N(T,d_X;\varepsilon)\big)^{1/2}\,d\varepsilon\\
&\le 2\Big(1+2\int_0^{1/2}\big(\log x^{-1}\big)^{1/2}\,dx\Big)\int_0^D\big(\log N(T,d_X;\varepsilon)\big)^{1/2}\,d\varepsilon.
\end{aligned}$$

(2.1.11) 得证. 同理, 对任何 $\eta \le D$, 若令 l_0 为满足 $2^{-l_0} > \eta$ 的最

大整数，则得

$$\int_0^\eta g\big(m(B(t,\varepsilon))\big)\,d\varepsilon \le 2\Big(1+2\int_0^{1/2}\Big(\log\frac{1}{x}\Big)^{1/2}dx\Big)\cdot\int_0^\eta\big(\log N(T,d_X;\varepsilon)\big)^{1/2}\,d\varepsilon. \quad (2.1.12)$$

由定理 2.1.4, (2.1.10) 得证. 当熵积分收敛时， m 为满足 (2.1.9) 的主测度，从而 X 在 (T,d_X) 上几乎处处有界且 (一致) 连续. 推论得证.

当 X 平稳时，上述推论的逆也可以成立. 为了叙述这一结果，我们只考察 T 为 $\mathcal{R}_+^k$ 的子集的情形，并记 $T_1+T_2=\{t+s:t\in T_1, s\in T_2\}(\forall T_1,T_2\subset\mathcal{R}_+^k)$, $T'=T+T$, $T''=T+T+T$, $t+T=\{t+s:s\in T\}$ ($\forall t\in\mathcal{R}_+^k, T\subset\mathcal{R}_+^k$).

定理 2.1.5 设 $X=\{X_t;t\in\mathcal{R}_+^k\}$ 为零均值 Gauss 过程. 假设 X 具有平稳增量，即 d_X 在下述意义下是平移不变的: $d_X(u+s,u+t)=d_X(s,t)$ ($u,s,t\in\mathcal{R}_+^k$). 设 T 为 $\mathcal{R}_+^k$ 中具有非空内部的紧子集. 则 X 在 T 上几乎处处有界且 (一致) 连续当且仅当 d_X 在 $T\times T$ 上连续，并且

$$H(T,d_X):=\int_0^\infty\big(\log N(T,d_X;\varepsilon)\big)^{1/2}d\varepsilon<\infty. \quad (2.1.13)$$

进一步，存在绝对常数 $K>0$ 使得

$$K^{-1}\{H(T,d_X)-\big(\log(|T''|/|T|)\big)^{1/2}D(T)\} \le E\sup_{t\in T}X_t\le KH(T,d_X), \quad (2.1.14)$$

其中 $|\cdot|$ 为 $\mathcal{R}_+^k$ 上的 Lebesgue 测度.

证明 由推论 2.1.1 即得不等式的右边. 由定理 2.1.4, 只要证明对 $T\subset\mathcal{R}_+^k$ 有

$$\frac{1}{2}\int_0^D\Big(\log\Big(\frac{|T|}{|T''|}N(T,d_X;\varepsilon)\Big)\,d\varepsilon\Big)^{1/2}\le\sup_{t\in T}\int_0^D g\big(m(B(t,\varepsilon))\big)\,d\varepsilon, \quad (2.1.15)$$

其中 m 为 T 上的概率测度, $D = D(T)$ 为 T 的 d_X 直径. 令 M 为不等式 (2.1.15) 右边的值, 记 $\lambda(\cdot) = |\cdot|$ 为 $\mathcal{R}_+^k$ 上的 Lebesgue 测度. 由于 $g(x)$ 为凸函数, 由 Jensen 不等式得

$$\begin{aligned} M &\geq \int_0^D \frac{1}{|T|} \int_T g\big(m(B(t,\varepsilon))\big)\, d\lambda(t)\, d\varepsilon \\ &\geq \int_0^D \left(\log \frac{|T|}{\int_T m(B(t,\varepsilon))\, d\lambda(t)}\right)^{1/2} d\varepsilon. \end{aligned}$$

由 Fubini 定理和平移不变性, 得

$$\int_T m(B(t,\varepsilon))\, d\lambda(t) = \int_T |T \cap B(t,\varepsilon)|\, dm(s) \leq |T' \cap B(0,\varepsilon)|.$$

从而,

$$M \geq \int_0^D \left(\log \frac{|T|}{|T' \cap B(0,\varepsilon)|}\right)^{1/2} d\varepsilon.$$

设 $\{t_1, \cdots, t_p\}$ 为 T 中满足条件 $\big(t_i + B(0,\varepsilon)\big) \cap \big(t_j + B(0,\varepsilon)\big) = \emptyset$ $\forall i \neq j$ 的最大的点集. 若 $t \in T$, 由最大性, 存在 $i = 1, \cdots, p$ 使得 $\big(t + B(0,\varepsilon)\big) \cap \big(t_i + B(0,\varepsilon)\big) \neq \emptyset$. 从而 $t \in \{t_i + B(0,2\varepsilon)\}$. 由此得 $T \subset \cup_{i=1}^p \{t_i + B(0,2\varepsilon)\}$, 从而 $N(T, d_X; 2\varepsilon) \leq p$. 注意到 $t_i + T' \cap B(0,\varepsilon) \subset T''$, 我们有

$$p|T' \cap B(0,\varepsilon)| = \sum_{i=1}^p |t_i + T' \cap B(0,\varepsilon)| = \left|\bigcup_{i=1}^p \{t_i + T' \cap B(0,\varepsilon)\}\right| \leq |T''|.$$

因此

$$N(t, d_X; 2\varepsilon) \leq \frac{|T''|}{|T' \cap B(0,\varepsilon)|}.$$

(2.1.15) 得证. 定理证毕.

通常情况下推论 2.1.2 的逆不成立. 下面是一个反例. 令 $T = \mathcal{Z}_+$, 这时的过程实际上是一个序列 $\{X_n; n \geq 1\}$. 设 X_n 独立且

$$\sigma_n = \sigma(X_n) = (EX_n^2)^{1/2} \leq (1 + \log n)^{-1/2}.$$

则序列 $\{X_n\}$ 几乎处处有界，且有绝对常数 K, 成立

$$E\sup_n |X_n| \le K. \tag{2.1.16}$$

事实上，注意到对任意的 $n \ge 1$ 和 $x \ge 2$ 有

$$P\{|X_n| \ge x\} \le e^{-x^2/(2\sigma_n^2)} \le e^{-\frac{1}{2}x^2(1+\log n)} \le Cn^{-x^2/2},$$

从而

$$P\{\sup_n |X_n| \ge x\} \le \sum_{n=1}^{\infty} P\{|X_n| \ge x\} \le C\sum_{n=1}^{\infty} n^{-\frac{1}{2}x^2} \le Ce^{-\frac{1}{2}x^2},$$

对上式进行积分即得 (2.1.16).

但是另一方面，序列 $\{X_n; n \ge 1\}$ 不具有有限的熵积分. 事实上，给定 $\varepsilon > 0$, 对 $n < n_\varepsilon = \exp\big(-1 + 1/(2\varepsilon^2)\big)$ 我们有 $\sigma(X_n) > \varepsilon\sqrt{2}$, 从而

$$d_X(n,m) > 2\varepsilon, \quad \forall n, m < n_\varepsilon,$$

这意味着如果 $n, m < n_\varepsilon$, 则 n 和 m 不在同一个 ε 球中，由此知 $N(T, d_X; \varepsilon) \ge n_\varepsilon - 1$, 因此

$$\inf_{\varepsilon > 0} \varepsilon\big(\log N(T, d_X; \varepsilon)\big)^{1/2} > 0,$$

从而熵积分 $\int \big(\log N(T, d_X; \varepsilon)\big)^{1/2} d\varepsilon$ 不可能有限.

在 $k = 1$ 的特殊情形，下述定理告诉我们 Fernique 条件也是具有平稳增量的零均值 Gauss 过程有界且连续的必要条件.

定理 2.1.6 设 $X = \{X_t; t \in [0,1]\}$ 为具有平稳增量的零均值 Gauss 过程. 假设 $\sigma^2(t) = E(X_t - X_0)^2$ 是单调增加的. 则 X 几乎处处有界且 (一致) 连续当且仅当

$$\int_0^{1/2} \frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\, du < \infty.$$

而且, 存在常数 K 使得

$$K^{-1}\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\,du \le \sup_{t\in[0,1]} X_t \le K\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\,du. \tag{2.1.17}$$

证明 注意到

$$\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\,du = \int_{(\log 2)^{1/2}}^{\infty}\sigma(e^{-u^2})\,du,$$

由定理 2.1.3, 只要证明左边的不等式. 注意到 $D=\sup_{s,t\in[0,1]}\sigma(|t-s|)=\sigma(1)$ 和 $N([0,1],d_X;\varepsilon)=1\wedge\dfrac{1}{2\mathrm{inv}\sigma(\varepsilon)}$, 由定理 2.1.5 有

$$\begin{aligned}\int_0^{\sigma(1)}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}d\varepsilon &\le CE\sup_{t\in[0,1]}X_t + C\sigma(1)\\ &\le CE\sup_{t\in[0,1]}X_t + C(E(X_1-X_0)^2)^{1/2}\\ &\le CE\sup_{t\in[0,1]}X_t + CE|X_1-X_0|\\ &\le CE\sup_{t\in[0,1]}X_t + CE\sup_{s,t\in[0,1]}|X_s-X_t|\\ &\le CE\sup_{t\in[0,1]}X_t.\end{aligned}$$

可设 $E\sup_{t\in[0,1]}X_t<\infty$, 否则没有什么要证明的. 因此上述积分收敛. 由分部积分法得

$$\begin{aligned}&\int_0^{\sigma(1)}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}d\varepsilon\\ &=\varepsilon\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\Big|_0^{\sigma(1)}-\int_0^{\sigma(1)}\varepsilon\, d\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\\ &\ge\varepsilon\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\Big|_0^{\sigma(1)}+\int_0^{\infty}\sigma(e^{-u^2})\,du.\end{aligned}$$

由于 $\mathrm{inv}\sigma(\varepsilon)$ 非降, 由积分的收敛性易知当 $\varepsilon\to 0$ 时, $\varepsilon\left(\log\dfrac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}$

$\to 0$. 从而

$$\int_0^{\sigma(1)} \left(\log \frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2} d\varepsilon \geq \int_0^{\infty} \sigma(e^{-u^2})\, du$$
$$\geq \int_{(\log 2)^{1/2}}^{\infty} \sigma(e^{-u^2})\, du - (\log 2)^{1/2}\sigma(1).$$

因此

$$\int_0^{1/2} \frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\, du \leq KE \sup_{t\in[0,1]} X_t + \sigma(1) \leq KE \sup_{t\in[0,1]} X_t.$$

定理得证.

2.1.4 向量值 Gauss 过程

设 B 为一个可分的 Banach 空间, 其对偶空间记为 B'. 一个取值于 B 的过程 $X=\{X_t;t\in T\}$ 是以 T 为指标的取值于 B 的一族 Borel 随机变量 $\{X_t;t\in T\}$. X 称为 (零均值)Gauss 的, 若每个有限样本 $(X_{t_1},\cdots,X_{t_N})$, $t_i\in T$ 为 B^N 中的 (零均值)Gauss 元. 下述定理是由 Fernique (1990) 得到的.

定理 2.1.7 设 (T,d) 为一个距离空间, $X=\{X_t;t\in T\}$ 为取值于 B 的零均值 Gauss 过程. 假设存在一个取值于 B 的零均值 Gauss 变量 ξ 使得

$$Ef^2(X_t)\leq Ef^2(\xi),\quad \forall (f,t)\in B'\times T,$$

则存在绝对常数 $C>0$ 使得

$$E\sup_{t\in T}\|X_t\|\leq C\left\{E\|\xi\| + \sup_{\|f\|\leq 1}\int_0^{\infty}\left(\log N(T,d_{f(X)};\epsilon)\right)^{1/2} d\epsilon\right\}. \tag{2.1.18}$$

此外, 假设 (T,d) 是紧距离空间, 若对任何 $f\in B'$, $\|f(X_t)-f(X_s)\|_2$ 在 (T,d) 上连续, 并且

$$\lim_{\eta\to 0}\sup_{\|f\|\leq 1}\int_0^{\eta}\left(\log N(T,d_{f(X)};\epsilon)\right)^{1/2} d\epsilon = 0, \tag{2.1.19}$$

则 X 在 (T,d) 上几乎处处连续.

证明 首先证明 (2.1.18). 令 B_1' 为 B' 中的单位球. 我们把 X 看作是以 $T^*=B_1'\times T$ 为指标集的过程. 由定理 2.1.5, 以 B_1' 为指标集的实值 Gauss 过程 $\xi=\{f(\xi);f\in B_1'\}$ 有主测度, 即存在 (B_1',d_ξ) 上的概率测度 μ 使得

$$\sup_{f\in B_1'}\int_0^\infty g\big(\mu(B(f,\varepsilon)\big)\,d\varepsilon\le CE\sup_{f\in B_1'}f(\xi)=CE\|\xi\|,$$

其中 $B(f,\varepsilon)$ 为 ε 球, 其对应的距离为其中心 f 所在空间上对应的正规距离, 即 $d_\xi(f,g)=\|f(\xi)-g(\xi)\|_2$, $f,g\in B_1'$. (我们还将用到这种球的概念). 令 $D=D(T^*)$ 为 $B_1'\times T$ 的 d_X 直径, 其中

$$d_X\big((f,t),(g,s)\big)=\|f(X_t)-g(X_s)\|_2,\quad f,g\in B',\quad s,t\in T.$$

对每个 $f\in B_1'$, 令 $S(f,u)$ 为能够覆盖 T 的个数最小的一族球 $B_{f(X)}(t,u)$, $t\in S(f,u)$ 的球心. 由定义, $Card\,S(f,u)=N(T,d_{f(X)};u)$. 考察 T 上的概率测度 π_f:

$$\pi_f=\sum_{p=1}^\infty 2^{-p}\sum_{s\in S(f,D/2^p)}\frac{\delta_s}{N(T,d_{f(X)};D/2^p)},$$

其中 δ_t 为在 t 处的 Dirac 测度. 由 μ 和 $\{\pi_f;f\in B_1'\}$, 我们构造 $B_1'\times T$ 上的一个概率测度 λ 为

$$\lambda=\int(\delta_f\otimes\pi_f)\,d\mu(f).$$

由定理 2.1.5 得

$$\begin{aligned}E\sup_{t\in T}\|X_t\|&=E\sup_{(f,t)\in B_1'\times T}f(X_t)\\&\le K\sup_{(f,t)\in B_1'\times T}\int_0^\infty g\left(\lambda\Big(B\big((f,t),\varepsilon\big)\Big)\right)\,d\varepsilon.\end{aligned}$$

给定 $(f,t)\in B_1'\times T$ 和 $p\geq 1$, 对任意的 $(g,s)\in B_1'\times T$, 由三角不等式有

$$
\begin{aligned}
d_X\big((f,t),(g,s)\big) &\leq \|f(X_t)-g(X_t)\|_2+\|g(X_t)-g(X_s)\|_2\\
&\leq d_\xi(f,g)+d_{g(X)}(t,s),\\
d_{g(X)}(s,t) &\leq \|f(X_s)-f(X_t)\|_2+\|f(X_s)-g(X_s)\|_2\\
&\quad+\|f(X_t)-g(X_t)\|_2\\
&\leq d_{f(X)}(s,t)+2d_\xi(f,g).
\end{aligned}
$$

从而

$$
\begin{aligned}
&d_{g(X)}(t,s)\leq D/2^{p+1},\\
&d_\xi(g,f)\leq D/2^{p+3}\Longrightarrow d_X\big((f,t),(g,s)\big)\leq D/2^p.
\end{aligned}
$$

因此

$$
\begin{aligned}
&\lambda\big(B\big((f,t),D/2^p\big)\big)\\
&\quad\geq\int\pi_g\left(B_{g(X)}(t,D/2^{p+1})\right)I\{g;g\in B(f,D/2^{p+3})\}\,d\mu(g)\\
&\quad\geq\int\frac{1}{2^{p+1}N(T,d_{g(X)};D/2^{p+1})}I\{g;g\in B(f,D/2^{p+3})\}\,d\mu(g).
\end{aligned}
$$

注意到

$$
d_{f(X)}(s,t)\leq D/2^{p+2},\ \ d_\xi(f,g)\leq D/2^{p+3}\Longrightarrow d_{g(X)}(s,t)\leq D/2^{p+1},
$$

并且 $\{B_{f(X)}(s,D/2^{p+2});s\in S(f,D/2^{p+2})\}$ 覆盖 T, 我们得: 若 $d_\xi(f,g)\leq D/2^{p+3}$, 则 $\{B_{g(X)}(s,D/2^{p+1});s\in S(f,D/2^{p+2})\}$ 覆盖 T. 由此得

$$
\begin{aligned}
&d_\xi(f,g)\leq D/2^{p+3}\\
&\Longrightarrow N(T,d_{g(X)};D/2^{p+1})\leq N(T,d_{f(X)};D/2^{p+2}).
\end{aligned}
$$

从而

$$
\lambda\big(B\big((f,t),D/2^p\big)\big)\geq\frac{1}{2^{p+1}N(T,d_{f(X)};D/2^{p+2})}\mu\big(B(f,D/2^{p+3})\big).
$$

因此，对每个 $(f,t)\in B_1'\times T$，有

$$\int g\left(\lambda\Big(B\big((f,t),\varepsilon\big)\Big)\right)\,d\varepsilon \le \sum_{p=0}^{\infty}\frac{D}{2^p}g\left(\lambda\Big(B\big((f,t),D/2^p\big)\Big)\right)$$

$$\le \sum_{p=0}^{\infty}\frac{D}{2^p}g\left(\frac{\mu\big(B(f,D/2^{p+3})\big)}{2^{p+1}N(T,d_{f(X)};D/2^{p+2})}\right)$$

$$\le \sum_{p=0}^{\infty}\frac{D}{2^p}\Big\{(\log 2^{p+1})^{1/2}+g\left(\mu\big(B(f,D/2^{p+3})\big)\right)$$

$$+\big(\log N(T,d_{f(X)};D/2^{p+2})\big)^{1/2}\Big\}$$

$$\le K\left\{D+\int_0^{\left(\mu\left(B(f,u)\right)\right)\,du}\right.$$

$$\left.+\int_0^D\big(\log N(T,d_{f(X)};u)\big)^{1/2}\,du\right\}.$$

从而

$$E\sup_{t\in T}\|X_t\|$$

$$\le K\left\{D+E\|\xi\|+\sup_{f\in B_1'}\int_0^D\big(\log N(T,d_{f(X)};u)\big)^{1/2}\,du\right\}.$$

注意到

$$D\le \sup_{\substack{f,g\in B_1'\\ s,t\in T}}\|f(X_s)-g(X_t)\|_2\le 2\sup_{(f,t)\in B_1'\times T}\|f(X_t)\|_2$$

$$\le 2\sup_{f\in B_1'}\|f(\xi)\|_2=(2\pi)^{1/2}\sup_{f\in B_1'}E|f(\xi)|\le(2\pi)^{1/2}E\|\xi\|,$$

我们知对某个常数 C 有

$$E\sup_{t\in T}\|X_t\|$$

$$\le C\left\{E\|\xi\|+\sup_{f\in B_1'}\int_0^{(2\pi)^{1/2}E\|\xi\|}\big(\log N(T,d_{f(X)};u)\big)^{1/2}\,du\right\}.$$

这一个不等式蕴涵了 (2.1.18).

现在, 假设 (2.1.19) 成立. 若 F 为 B 的一个有限维子空间, 令 T_F 为商映射 $B \to B/F$, 由上述已经证明的不等式得

$$
\begin{aligned}
&E\sup_{t\in T}\|T_F(X_t)\| \\
\le & C\left\{E\|T_F(\xi)\| + \sup_{f\in B_1'}\int_0^{\sqrt{2\pi}E\|T_F(\xi)\|}\left(\log N(T, d_{f(X)}; u)\right)^{\frac{1}{2}}du\right\}.
\end{aligned}
$$

若 F_n 为 B 的一列有限维子空间, 满足 $F_n \uparrow B$, 则

$$
\lim_{n\to\infty} E\|T_{F_n}(\xi)\| = 0.
$$

因此我们有

$$
\lim_{n\to\infty}\sup_{t\in T}\|T_{F_n}(X_t)\| = 0 \quad \text{a.s.}, \qquad n\to\infty.
$$

另外, 对每个 $f \in B_1'$, 由推论 2.1.1 和 $\|f(X_t)-f(X_t)\|_2$ 的连续性假设知 $f(X_t)$ 在 (T,d) 上几乎处处连续. 由此得取值于有限维空间 F_n 的过程 $X - T_{F_n}(X)$ 在 (T,d) 上几乎处处连续. 从而 X 在 (T,d) 上几乎处处连续. 定理得证.

下述结论是定理 2.1.7 和定理 2.1.5 的简单推论.

推论 2.1.2 设 $X=\{X_t; t\in T\}$ 为取值于 B 的零均值平稳 Gauss 过程, 其中 T 为 $\mathcal{R}_+^k$ 中的具有非空内部的紧子集. 那么对某个常数 $C>0$ 有

$$
\begin{aligned}
&\frac{1}{2}\Big(E\|X_0\| + \sup_{\|f\|\le 1} E\sup_{t\in T} f(X_t)\Big) \le E\sup_{t\in T}\|X_t\| \\
\le & C\Big(1+\log\frac{|T''|}{|T|}\Big)\Big(E\|X_0\| + \sup_{\|f\|\le 1} E\sup_{t\in T} f(X_t)\Big). \quad (2.1.20)
\end{aligned}
$$

进一步, X 在 T 上几乎处处连续当且仅当对每个 $f\in B'$, $\|f(X_t)-f(X_s)\|_2$ 在 $T\times T$ 上连续, 且

$$
\lim_{\eta\to\infty}\sup_{\|f\|\le 1} E\sup_{t\in\eta T} f(X_t) = 0. \qquad (2.1.21)
$$

2.1.5 独立 Ornstein-Uhlenbeck 过程的无穷级数

设 $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ 为一列独立的 Ornstein-Uhlenbeck (O-U) 过程, 其系数为 $\gamma_k \geq 0, \lambda_k > 0$, 也即 $X_k(\cdot)$ 为平稳的零均值 Gauss 过程满足 $EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|)$, $k = 1, 2, \cdots$.

过程 $Y(\cdot)$ 是由 Dawson (1972) 作为下述随机微分方程无穷组列的平稳解引入的:

$$d\,X_i(t) = -\lambda_i X_i(t)\,dt + (2\gamma_i)^{1/2}\,dW_i(t) \quad (i = 1, 2, \cdots), \quad (2.1.22)$$

其中 $\{W_i(t); -\infty < t < \infty\}$ 为独立的标准 Wiener 过程. 自从 Dawson (1972) 引入后, 这类过程已经在文献中得到了广泛的研究. 由于它们在纯数学和应用数学的许多领域里频频出现, 它们显得非常重要. 自然, 它们在随机微分方程的研究中继续起着重要的作用 (参见 Dawson (1975), Ricciardi 和 Sacerdote (1979), Walsh (1981), Antoniadis 和 Carmona (1987) 及 Itô (1984)). 它们也出现在构造量子场理论 (参见 Carmona (1977) 和 Gross (1977))、无穷粒子系统理论 (参见 Holley 和 Strook (1978)) 以及无穷维扩散过程等 (参见 Itô (1984), Kuo (1975), Piech (1975), Strook (1981) 和 Schmuland (1988)) 的研究中.

Walsh (1981) 利用随机过程给出了神经反射的数学模型并研究了他引入的过程的许多解析性质. 其中一个有趣的过程是 $Y(\cdot)$ 的坐标分量过程的无穷级数 (独立 O-U 过程的无穷级数), 即过程 $X(\cdot)$:

$$\{X(t); -\infty < t < \infty\} = \Big\{\sum_{k=1}^{\infty} X_k(t); -\infty < t < \infty\Big\}, \quad (2.1.23)$$

其中 $X_k(\cdot)$ 为 $Y(\cdot)$ 的 O-U 分量.

Csáki, Csörgő, Lin 和 Révész (1991) 研究了过程 $X(\cdot)$ 的样本轨道性质. 如果级数 $\sum_{j=1}^{\infty}\gamma_j/\lambda_j$ 收敛, 那么对每一固定的 t, $X(t)$

是一均值为零、方差为 $\sum_{j=1}^{\infty}\gamma_j/\lambda_j$ 的随机变量. 然而, 如果仅仅有这一假设, $X(\cdot)$ 还不一定是 $\mathcal{R}$ 上 a.s. 连续的 Gauss 过程. 下面的定理给出关于 $X(\cdot)$ 的连续性的结果. $Y(\cdot)$ 也可以直接当做 l^p 值 Gauss 过程来研究, 我们将在下一章讨论.

定理 2.1.8 设 $X(\cdot)$ 如 (2.1.23) 所示, 定义

$$\begin{aligned}&\{X(t,n);-\infty<t<\infty,n=1,2,\cdots\}\\&\qquad=\Big\{\sum_{k=1}^{n}X_k(t);-\infty<t<\infty,n=1,2,\cdots\Big\}.\end{aligned}\tag{2.1.24}$$

假设对某个 $\delta>0$ 有

$$\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\big(\log(\lambda_k\vee e)\big)^{1+\delta}<\infty,\tag{2.1.25}$$

则以概率 1, $X(t,n)\to X(t)$ 在任何有限区间上对于 t 一致收敛, 即, 对任意的 $\varepsilon>0$, $T>0$ 和几乎所有的 $\omega\in\Omega$, 存在整数 $n_0=n_0(\varepsilon,T,\omega)$ 使得当 $n\ge n_0$ 时有

$$\sup_{|t|\le T}|X(t,n,\omega)-X(t,\omega)|\le\varepsilon.\tag{2.1.26}$$

从而, Gauss 过程 $\{X(t);-\infty<t<\infty\}$ 以概率 1 连续.

证明 根据 Itô-Nisio 定理 (参见 Ledoux 和 Talagrand 1991 的定理 6.1), 为证 (2.1.26), 只要证明当 $n\to\infty$ 时

$$\sup_{|t|\le T}|X(t,n)-X(t)|=\sup_{|t|\le T}\Big|\sum_{k=n+1}^{\infty}X_k(t)\Big|$$

依概率趋于 0. 从而只要证明对任意的 $\varepsilon>0$ 有

$$\lim_{n\to\infty}P\Big\{\sup_{|t|\le T}\Big|\sum_{k=n+1}^{\infty}X_k(t)\Big|>\varepsilon\Big\}=0.$$

因此，只要证明在条件 (2.1.25) 下，对任意的 $\varepsilon > 0$ 和 $0 < \eta < 1$ 存在 $n_0 = n_0(\varepsilon, \eta)$ 使得当 $m > n \geq n_0$ 时，

$$P\left\{\sup_{|t|\leq T}\Big|\sum_{k=n+1}^{m} X_k(t)\Big| > \varepsilon\right\} \leq \eta,$$

令

$$X_{m,n}(t) = X(t,m) - X(t,n) = \sum_{k=n+1}^{m} X_k(t),$$

那么，对每个 $m > n$，过程 $\{X_{m,n}(t); -\infty < t < \infty\}$ 为平稳的零均值 Gauss 过程满足

$$\begin{aligned}
EX_{m,n}^2(t) &= \sum_{k=n+1}^{m} \gamma_k/\lambda_k,\\
EX_{m,n}(t)X_{m,n}(s) &= \sum_{k=n+1}^{m} (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|),\\
\sigma_{m,n}^2(h) :&= E\big(X_{m,n}(t+h) - X_{m,n}(t)\big)^2\\
&= 2\sum_{k=n+1}^{m} (\gamma_k/\lambda_k)\big(1 - \exp(-\lambda_k h)\big)
\end{aligned}$$

对任何 $t, s \in \mathcal{R}$ 和 $h > 0$ 成立. 记

$$K_1 = \{k : \lambda_k < e^{u^2/2}\}, \quad K_2 = \{k : \lambda_k \geq e^{u^2/2}\}.$$

则

$$\begin{aligned}
&\int_1^{\infty}\Big(\sum_{\substack{k=n+1\\ k\in K_1}}^{m} \frac{\gamma_k}{\lambda_k}\big(1 - \exp(-\lambda_k e^{-u^2})\big)\Big)^{1/2} du\\
&\leq \int_1^{\infty}\Big(\sum_{k=n+1}^{m} \frac{\gamma_k}{\lambda_k}\cdot\lambda_k e^{-u^2} I_{\{\lambda_k < e^{u^2/2}\}}\Big)^{1/2} du\\
&\leq \int_1^{\infty}\Big(\sum_{k=n+1}^{m} \frac{\gamma_k}{\lambda_k}\Big)^{1/2} e^{-u^2/4}\, du \leq 4\Big(\sum_{k=n+1}^{m} \frac{\gamma_k}{\lambda_k}\Big)^{1/2},
\end{aligned}$$

$$\int_1^\infty \Big(\sum_{\substack{k=n+1\\k\in K_2}}^m \frac{\gamma_k}{\lambda_k}\big(1-\exp(-\lambda_k e^{-u^2})\big)\Big)^{1/2}du$$
$$\le \int_1^\infty \Big(\sum_{k=n+1}^m \frac{\gamma_k}{\lambda_k} I_{\{\lambda_k\ge e^{u^2/2}\}}\Big)^{1/2}du$$
$$\le \int_1^\infty \Big(\sum_{k=n+1}^m \frac{\gamma_k}{\lambda_k}(\log^+\lambda_k)^{1+\delta}\cdot\Big(\frac{2}{u^2}\Big)^{1+\delta}\Big)^{1/2}du$$
$$\le 2^{(1+\delta)/2}\delta^{-1}\Big(\sum_{k=n+1}^m \frac{\gamma_k}{\lambda_k}(\log^+\lambda_k)^{1+\delta}\Big)^{1/2}.$$

从而

$$\int_1^\infty \sigma_{m,n}(e^{-u^2})\,du \le C\Big\{\sum_{k=n+1}^m \frac{\gamma_k}{\lambda_k}\Big(\log(\lambda_k\vee e)\Big)^{1+\delta}\Big\}^{1/2}.$$

由定理 2.1.3 得

$$E\sup_{0\le t\le 1}|X_{m,n}(t)| \le K\Big\{\sum_{k=n+1}^m \frac{\gamma_k}{\lambda_k}\Big(\log(\lambda_k\vee e)\Big)^{1+\delta}\Big\}^{1/2} =: K\eta_{m,n},$$

其中 K 为常数. 由于过程 $\{X_{m,n}(t); -\infty<t<\infty\}$ 是平稳的, 我们有

$$E\sup_{|t|\le T}|X_{m,n}(t)| \le 2(T+1)E\sup_{0\le t\le 1}|X_{m,n}(t)| \le 2(T+1)K\eta_{m,n}.$$

由条件 (2.1.25) 知 $\eta_{m,n}\to 0$ $(n\to\infty)$. 从而, 对充分大的 n_0, 当 $m>n\ge n_0$ 时有

$$P(\sup_{|t|\le T}|X_{m,n}(t)|>\varepsilon) \le \frac{2K(T+1)}{\varepsilon}\eta_{m,n} < \eta.$$

定理 2.1.8 得证.

§2.2 分数 Wiener 过程

从这一节开始，我们研究 Gauss 过程的连续模与大增量的极限性质. 我们首先考察一个简单而又重要的 Gauss 过程，即分数 Wiener 过程，或者叫做分数 Brown 运动.

阶为 α $(0<\alpha<1)$ 的分数 Wiener 过程 $\{Z(t);t\in\mathcal{R}\}$ 是一个零均值、具有平稳增量的实值 Gauss 过程，满足 $Z(0)=0$ 且 $\sigma^2(|t|)=EZ^2(t)=|t|^{2\alpha}$ $(t\in\mathcal{R})$. 显然，当 $\alpha=1/2$ 时，$\{Z(t);t\in R\}$ 为标准 Wiener 过程，记之为 $\{W(t);t\in\mathcal{R}\}$. 易知

$$\{Z(t);t\in\mathcal{R}\}\text{与}$$
$$\left\{\int_{\mathcal{R}}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\,dW(x);t\in\mathcal{R}\right\}\text{同分布,}$$

其中 $K_\alpha^2=\int_{\mathcal{R}}\left(|x-1|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\right)^2dx$, 且 $\alpha=1/2$ 时，${K_\alpha}^{-1}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}$ 被视作 $I_{(0,t]}$. 从而，这样一个分数 Wiener 过程 $\{Z(t);t\in\mathcal{R}\}$ 存在且可以写成

$$Z(t)=\int_{\mathcal{R}}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\,dW(x),\quad t\in\mathcal{R}.$$

分数 Wiener 过程是 Wiener 过程直接而简单的推广，它是由 Mandelbrot 和 Van Ness (1968) 引入的，至今它在计量经济学、金融统计和管理科学中有着重要而广泛的应用. 例如，它在粮食产量趋势预测、新增固定资产分析、国民收入分析、股票收益、长程相关统计及 α 指数计算、分数阶 ARIMA 模型及其在价格指数预测的应用等等领域中都起着重要的作用. 分数 Wiener 过程保持了 Wiener 过程的许多性质，例如，连续性 (由定理 2.1.3), 增量的平稳性等等. 但是，除非它本身是一个 Wiener 过程，即 $\alpha=1/2$ 的情形，它不再具有独立增量性. 在这一节，我们感兴趣的是它的

连续模和大增量的极限性质. 由于 $\{Z(t); t \le 0\}$ 与 $\{Z(t); t \ge 0\}$ 同分布, 我们只要考察 $\{Z(t); t \ge 0\}$ 即可.

2.2.1 $Z(\cdot)$ 的连续模

以下叙述 Lévy 形式的连续模 (洪圣岩 1990).

定理 2.2.1 我们有

$$\begin{aligned}&\lim_{h\to 0}\sup_{0\le t\le 1-h}\frac{|Z(t+h)-Z(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\\&\quad=\lim_{h\to 0}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\frac{|Z(t+s)-Z(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}=1\quad \text{a.s.}\end{aligned}\tag{2.2.1}$$

这里和书中的余下部分, 除非特别说明, $\log x$ 表示 $\log(x\vee e)$.

论证定理 2.2.1 之前, 我们要证明一个引理. 首先, 我们引入拟增, 拟降和正则变化 (正变) 函数的概念.

定义 2.2.1 一个 (a,b) 上的函数 $f(x)$ 称为是拟增 (拟降) 的, 如果存在正常数 C_0 使得

$$f(x)\le C_0 f(y)\quad \forall\, a<x<y<b,$$
$$(f(x)\ge C_0 f(y)\quad \forall\, a<x<y<b).$$

定义 2.2.2 一个函数 $f(x)$ 称为在 0 点 (无穷处) 以正指数 α 正则变化, 如果

$$\lim_{s\to 0}f(\theta s)/f(s)=\theta^\alpha\quad \forall\,\theta>0\qquad (\lim_{s\to\infty}f(\theta s)/f(s)=\theta^\alpha\quad \forall\,\theta>0).$$

易知一个 $[0,1]$ ($[0,\infty)$) 上的函数 $f(x)$ 在 0 点 (无穷处) 以指数 α 正则变化, 则 $f(s)/s^{\alpha/2}$ 为 $[0,1]$ ($[1,\infty)$) 上的拟增函数.

引理 2.2.1 假设 $\{X(t); t\ge 0\}$ 为一个可分的 Gauss 过程满足

$$E(X(t)-X(s))^2\le \Lambda^2(|t-s|),$$

其中 $\Lambda(x)$ 为非降连续函数，使得对某 $\alpha>0$ 和 $h_0>0$, $\Lambda(x)/x^\alpha$ 在 $(0,h_0)$ 上拟增. 则对任意的 $\varepsilon>0$, 存在正常数 C_1, C_2 使得对任何 $x\geq C_1$, $0<h\leq h_0$ 和 $T>h$ 成立

$$P\Big\{\sup_{0\leq t\leq t'\leq T}\sup_{t'-t\leq h}|X(t')-X(t)|\geq x\Lambda(h)\Big\}\leq C_2\frac{T}{h}e^{-x^2/(2+\varepsilon)}. \tag{2.2.2}$$

证明 由推论 1.1.1, 对任意的 $y\geq\sqrt{1+4\log p}$ 和任何 $0<\Delta<h_0$, $t\geq 0$ 成立

$$P\Big\{\sup_{0\leq s\leq\Delta}|X(t+s)-X(t)|\geq y\Big(\Lambda(\Delta)+(2+\sqrt{2})\cdot\int_1^\infty\Lambda\Big(\frac{\Delta}{2}p^{-u^2}\Big)\,du\Big)\Big\}\leq\frac{5}{2}p^2\int_y^\infty e^{-u^2/2}\,du,$$

其中 $p\geq 2$ 为一整数.

因 $\Lambda(x)/x^\alpha$ 在 $(0,h_0)$ 上拟增，所以存在 $c_0\geq 1$ 使得

$$\Lambda(ht)\leq c_0t^\alpha\Lambda(h),\quad \forall\, 0\leq t\leq 1, 0<h<h_0.$$

从而

$$\int_1^\infty\Lambda\Big(\frac{\Delta}{2}p^{-u^2}\Big)\,du\leq c_0\Lambda(\Delta)\int_1^\infty p^{-\alpha u^2}\,du\leq c_0\Lambda(\Delta)\frac{2}{\alpha p^\alpha\log p}.$$

因此对 $y\geq\sqrt{1+4\log p}$ 有

$$P\left\{\sup_{0\leq s\leq\Delta}|X(t+s)-X(t)|\geq y\Lambda(\Delta)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\right\}\leq\frac{5}{2}p^2\int_y^\infty e^{-u^2/2}\,du.$$

给定 $0<\delta<h_0$, 令 $N=(2c_0/\delta)^{1/\alpha}/h$. 注意到 $\Lambda(\frac{1}{N})\leq c_0(\frac{\delta}{2C_0})\Lambda(h)=\frac{\delta}{2}\Lambda(h)$, 对 $y\geq\sqrt{1+4\log p}$ 我们有

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le t'\le T}\ \sup_{t'-t\le h}|X(t')-X(t)|\ge(1+\delta)y\Lambda(h)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}\\
&\quad\le P\Big\{\sup_{0\le i\le[NT]}\ \sup_{0\le s\le h}\Big|X\Big(\frac{i}{N}+s\Big)-X\Big(\frac{i}{N}\Big)\Big|\\
&\quad\ge y\Lambda(h)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}+P\Big\{\sup_{0\le i\le[NT]}\ \sup_{0\le s\le 1/N}\Big|X\Big(\frac{i}{N}+s\Big)\\
&\qquad-X\Big(\frac{i}{N}\Big)\Big|\ge\frac{\delta}{2}y\Lambda(h)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}\\
&\quad\le 2(NT+1)\frac{5}{2}p^2\int_y^\infty e^{-u^2/2}\,du\\
&\quad\le 5\Big(\Big(\frac{2c_0}{\delta}\Big)^{1/\alpha}+1\Big)\frac{T}{h}p^2\int_y^\infty e^{-u^2/2}\,du.
\end{aligned}
$$

对任意给定的 $\varepsilon>0$, 现在取 $\delta>0$ 充分小和 p 充分大使得 $(1+\delta)(1+\frac{8c_0}{\alpha p^\alpha\log p})\le\sqrt{1+\varepsilon/2}$. 令 $x=y(1+\delta)(1+\frac{8c_0}{\alpha p^\alpha\log p})$, 则

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le t'\le T}\ \sup_{t'-t\le h}|X(t')-X(t)|\ge x\Lambda(h)\Big\}\\
&\qquad\le C_2\frac{T}{h}\int_{\frac{x}{\sqrt{1+\varepsilon/2}}}^\infty e^{-u^2/2}\,du\le C_2\frac{T}{h}e^{-x^2/(2+\varepsilon)}
\end{aligned}
$$

对 $x\ge c_1=:2\sqrt{1+4\log p}$ 成立. 引理得证.

现在我们证明定理 2.2.1. 令

$$
A_h=\sup_{0\le t\le 1-h}\ \sup_{0\le s\le h}|Z(t+s)-Z(t)|.
$$

由引理 2.2.1, 对充分小的 h 有

$$
\begin{aligned}
&P\Big\{\frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\ge 1+\varepsilon\Big\}\\
&\qquad\le\frac{c}{h}\exp\Big(-\frac{2(\log h^{-1})(1+\varepsilon)^2}{2+\varepsilon}\Big)\le ch^\varepsilon.
\end{aligned}
$$

取 $A > 1/\varepsilon$, 令 $h = h_n = n^{-A}$. 则

$$\sum_{n=1}^{\infty} P\left\{ \frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}} \geq 1+\varepsilon \right\} \leq \sum_{n=1}^{\infty} cn^{-A\varepsilon} < \infty.$$

由 Borel-Cantelli 引理得

$$\limsup_{n\to\infty} \frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}} \leq 1+\varepsilon \quad \text{a.s.}$$

当 $h_{n+1} < h \leq h_n$ 时, 有

$$\begin{aligned}\limsup_{h\to 0} \frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}} &\leq \limsup_{n\to\infty} \frac{A_{h_n}}{\{2\sigma^2(h_{n+1})\log h_n^{-1}\}^{1/2}} \\ &= \limsup_{n\to\infty} \frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}} \cdot \frac{\sigma(h_n)}{\sigma(h_{n+1})} \leq 1+\varepsilon \quad \text{a.s.}\end{aligned}$$

从而

$$\limsup_{h\to 0} \frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}} \leq 1 \quad \text{a.s.} \tag{2.2.3}$$

下面我们证明

$$\liminf_{h\to 0} \sup_{0\leq t\leq 1-h} \frac{Z(t+h)-Z(t)}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}} \geq 1 \quad \text{a.s.} \tag{2.2.4}$$

给定 $\varepsilon > 0$ 和 $\delta > 0$, 令 m 为整数使得对 $l \geq m$ 成立 $\frac{1}{2}\big((l+1)^{2\alpha} + (l-1)^{2\alpha} - 2l^{2\alpha}\big) < \delta^2$. 易知对任何 $l \neq k$ 有

$$\begin{aligned}&E\Big(Z\Big(\frac{km+1}{n}\Big) - Z\Big(\frac{km}{n}\Big)\Big)\Big(Z\Big(\frac{lm+1}{n}\Big) - Z\Big(\frac{lm}{n}\Big)\Big) \\ =&\frac{1}{2}\Big(\frac{1}{n}\Big)^{2\alpha}\big((|l-k|m+1)^{2\alpha} + (|l-k|m-1)^{2\alpha} - 2(|l-k|m)^{2\alpha}\big) \\ \leq&\sigma^2\Big(\frac{1}{n}\Big)\delta^2.\end{aligned}$$

定义 $\xi_i = \big(Z(\frac{im+1}{n}) - Z(\frac{im}{n})\big)/\sigma(\frac{1}{n})$, $i = 0,\cdots,[\frac{n}{m}]-1$, 令 τ, η_i $(i = 0,\cdots,[\frac{n}{m}]-1)$ 为独立的零均值正态变量且 $E\eta_i^2 = 1-\delta^2$,

$E\tau^2=\delta^2$. 则 $E\xi_i^2=E(\eta_i+\tau)^2=1$, $E\xi_i\xi_j\le\delta^2=E(\eta_i+\tau)(\eta_j+\tau)$, $i\neq j$. 由 Slepian 引理 (推论 1.2.1) 我们有

$$\begin{aligned}
&P\left\{\max_{0\le i\le[\frac{n}{m}]-1}\frac{Z(\frac{im+1}{n})-Z(\frac{im}{n})}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\le 1-3\varepsilon\right\}\\
\le&P\left\{\max_{0\le i\le[\frac{n}{m}]-1}\frac{\eta_i+\tau}{(2\log n)^{1/2}}\le 1-3\varepsilon\right\}\\
\le&P\left\{\max_{0\le i\le[\frac{n}{m}]-1}\eta_i\le(1-2\varepsilon)(2\log n)^{1/2}\right\}+P\{\tau>\varepsilon(2\log n)^{1/2}\}\\
=&\prod_{i=0}^{[\frac{n}{m}]-1}P\left\{N(0,1)\le(1-2\varepsilon)(\frac{2\log n}{1-\delta^2})^{1/2}\right\}\\
&+P\Big\{N(0,1)\ge\frac{\varepsilon}{\delta}(2\log n)^{1/2}\Big\}.
\end{aligned}$$

取 $\delta>0$ 充分小使得 $\frac{1-2\varepsilon}{\sqrt{1-\delta^2}}<1-\frac{3}{2}\varepsilon$ 且 $\frac{\varepsilon}{\delta}>2$. 由 (1.1.1) 得, 对充分大的 n 有

$$\begin{aligned}
&P\left\{N(0,1)\le(1-2\varepsilon)\Big(\frac{2\log n}{1-\delta^2}\Big)^{1/2}\right\}\\
&\quad\le P\left\{N(0,1)\le\Big(1-\frac{3}{2}\varepsilon\Big)(2\log n)^{1/2}\right\}\\
&\quad\le 1-\frac{1}{n^{1-\varepsilon}}\le\exp\Big(-\frac{1}{n^{1-\varepsilon}}\Big)
\end{aligned}$$

和 $P\left\{N(0,1)\ge\frac{\varepsilon}{\delta}(2\log n)^{1/2}\right\}\le n^{-2}$. 我们得

$$\begin{aligned}
&P\left\{\max_{0\le i\le[\frac{n}{m}]-1}\frac{Z(\frac{im+1}{n})-Z(\frac{im}{n})}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\le 1-3\varepsilon\right\}\\
&\quad\le\exp\Big(-[\frac{n}{m}]\frac{1}{n^{1-\varepsilon}}\Big)+\frac{1}{n^2}\le\exp\Big(-\frac{1}{2m}n^{\varepsilon}\Big)+\frac{1}{n^2}.
\end{aligned}$$

因此

$$\sum_{n=1}^{\infty}P\left\{\max_{0\le i\le[\frac{n}{m}]-1}\frac{Z(\frac{im+1}{n})-Z(\frac{im}{n})}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\le 1-3\varepsilon\right\}<\infty.$$

由 Borel-Cantelli 引理得

$$\begin{aligned}&\liminf_{n\to\infty}\sup_{0\le t\le 1-\frac{1}{n}}\frac{Z(t+\frac{1}{n})-Z(t)}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\\&\geq\liminf_{n\to\infty}\max_{0\le i\le[\frac{n}{m}]-1}\frac{Z(\frac{im+1}{n})-Z(\frac{im}{n})}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\geq 1-3\varepsilon\quad\text{a.s.}\end{aligned}\tag{2.2.5}$$

现考察 $h_{n+1}<h\le h_n$, 其中 $h_n=1/n$, 我们得

$$\begin{aligned}&\liminf_{h\to 0}\sup_{0\le t\le 1-h}\frac{Z(t+h)-Z(t)}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\\&\geq\liminf_{n\to\infty}\sup_{0\le t\le 1-\frac{1}{n+1}}\frac{Z(t+\frac{1}{n+1})-Z(t)}{\{2\sigma^2(\frac{1}{n+1})\log(n+1)\}^{1/2}}\\&\quad\cdot\frac{\{2\sigma^2(\frac{1}{n+1})\log(n+1)\}^{1/2}}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}}\\&\quad-\limsup_{n\to\infty}\sup_{0\le t\le 1-\frac{1}{n+1}}\sup_{0\le s\le\frac{1}{n(n+1)}}\frac{|Z(t+s)-Z(t)|}{\{2\sigma^2(\frac{1}{n})\log n\}^{1/2}},\end{aligned}$$

其中, 由 (2.2.3), 后一个随机变量 $\overset{\text{a.s.}}{=}o(1)$, 由 (2.2.5), 前一个随机变量 $\geq 1-3\varepsilon$ a.s. 从而 (2.2.4) 得证.

2.2.2 $Z(\cdot)$ 的增量有多大?

在定理 2.2.1 中我们看到, 当 h 充分小时, 分数 Wiener 过程 $Z(\cdot)$ 在 $[0,1]$ 中以长度为 h 的子区间上的增量有多大. 下述定理 2.2.2 研究当 $T\to\infty$ 时, 分数 Wiener 过程在 $[0,T]$ 中以 a_T 为长度的子区间上的增量有多大, 其中 a_T 为 T 的不减函数.

定理 2.2.2 (Ortega 1984) 设 a_T $(0<a_T\le T)$ 为 T 的函数满足

(i) a_T 非降,

(ii) T/a_T 非降.

则

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|Z(t+s)-Z(t)|\\&=\limsup_{T\to\infty}\beta_T|Z(T+a_T)-Z(T)|=1\quad\text{a.s.}\end{aligned}\tag{2.2.6}$$

其中

$$\beta_T = \{2\sigma^2(a_T)(\log T/a_T + \log\log T)\}^{-1/2}.$$

若还有

(iii) $\quad \lim_{T\to\infty}(\log T/a_T)(\log\log T)^{-1} = \infty,$

则

$$\begin{aligned}&\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|Z(t+s)-Z(t)|\\&\quad=\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\beta_T|Z(t+a_T)-Z(T)|=1\quad \text{a.s.}\end{aligned}\tag{2.2.7}$$

证明见 Ortega(1984) 或林正炎、陆传荣 (1992).

还有另一类增量，即滞后增量. Wiener 过程 $W(\cdot)$ 的滞后增量是由 Hanson 和 Russo (1983a, b) 引入并研究的，其推广的结果可以参看 Chen, Kong 和 Lin (1986). 下述定理是关于分数 Wiener 过程 $Z(\cdot)$ 滞后增量的结果.

定理 2.2.3 我们有

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{0<t\le T}|Z(T)-Z(T-t)|/d(T,t)\\&\quad=\limsup_{T\to\infty}\sup_{0<t\le T}\sup_{t\le s\le T}|Z(s)-Z(s-t)|/d(T,t)\\&\quad=\limsup_{T\to\infty}\sup_{0<t\le T}\sup_{0\le s\le t}|Z(s)-Z(T-s)|/d(T,t)=1\ \text{a.s.}\end{aligned}\tag{2.2.8}$$

其中 $d(T,t) = \{2\sigma^2(t)(\log T/t + \log\log t)\}^{1/2}$.

定理 2.2.3 由陆传荣 (1986) 和洪圣岩 (1990) 得到，其证明与 Wiener 过程的类似，这里从略，读者可参看林正炎和陆传荣 (1992, p. 10).

注 2.2.1 对分数 Wiener 过程 $\{Z(t); t \ge 0\}$，我们也有与 Wiener 过程一样的增量一般形式，王文胜 (1997) 证明了：设 $\{Z(t); t \ge 0\}$ 是阶为 α $(0 < \alpha < 1)$ 的分数 Wiener 过程， a_T, b_T, c_T 是 T 的非负函数，满足

(i) $a_T + b_T \ge T$ 且对充分大的 T, $c_T \ge T$,

(ii) 存在常数 A 使得对任一 $T \ge 1$ 有

$$b_T - b_{T-1} \le A a_T, \qquad a_T + b_T \le A(a_{T-1} + b_{T-1}).$$

那么

$$\limsup_{T\to\infty} \sup_{0\le t} \sup_{0<s} \sup_{0\le r\le s} \frac{|Z(t+r) - Z(t)|}{d(t+s\vee c_T, s)} = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty} \beta(a_T + b_T, a_T)|Z(a_T+b_T) - Z(b_T)| = 1 \quad \text{a.s.},$$

其中 $\beta(M,m) = \{2\sigma^2(m)(\log(M/m) + \log\log M)\}^{-1/2}$. 进一步, 若对任意的 $0 \le \epsilon < 1$ 有

$$\sum_{N=1}^{\infty} \exp\left\{ - b_N \Big/ \left(a_N^{\epsilon} \left((a_N + b_N)\log(a_N + b_N) \right)^{1-\epsilon} \right) \right\} < \infty,$$

$$\lim_{T\to\infty} b_T/b_{[T]} = \lim_{T\to\infty} a_T/a_{[T]} = 1,$$

那么

$$\lim_{T\to\infty} \sup_{0\le t} \sup_{0<s} \sup_{0\le r\le s} \frac{|Z(t+r) - Z(t)|}{d(t+s\vee c_T, s)} = 1 \quad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{0\le t\le b_T} \beta(t+b_T, a_T)|Z(t+a_T) - Z(t)| = 1 \quad \text{a.s.}$$

2.2.3 $Z(\cdot)$ 大增量的下极限性质

在定理 2.2.2 中, 我们看到, 当条件 (iii) 满足时, $Z(\cdot)$ 在 $[0,T]$ 中长度为 a_T 的子区间上的增量的上极限与其对应的下极限相等. 但是, 当条件 (iii) 不满足时, 上极限不再与其对应的下极限相等, 那么下极限如何呢? 回答这个问题的下述一般性结果是由 Zhang (1996a) 得到的.

定理 2.2.4 设 a_T $(0 < a_T \le T)$ 为 T 的函数, 满足条件 (i), (ii) 和

(iv) $\lim_{T\to\infty}(T/a_T)/\log\log T = \infty$.

则

$$\liminf_{T\to\infty}\gamma(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$
$$=\liminf_{T\to\infty}\gamma(T)\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=1 \quad \text{a.s.} \tag{2.2.9}$$

其中 $\gamma(T)=\{2\sigma^2(a_T)(\log T/a_T-\log\log\log T)\}^{-1/2}$.

若条件 (iv) 由下述条件代替

(iv′) $\lim_{T\to\infty}(T/a_T)/\log\log T=0$,

则存在两个只依赖于 α 的常数 $c_1=c_1(\alpha)$, $c_2=c_2(\alpha)$ 使得

$$c_1\le\liminf_{T\to\infty}\gamma'(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le c_2 \quad \text{a.s.} \tag{2.2.10}$$

其中

$$\gamma'(T)=\left\{a_T\log\left(1+\frac{T}{a_T\log\log T}\right)\right\}^{-\alpha}.$$

由定理 2.2.4, 我们可以得到下面两个简单的推论.

推论 2.2.1 设 a_T $(0<a_T\le T)$ 为 T 的函数满足条件 (i), (ii) 和

(v) $\lim_{T\to\infty}(\log(T/a_T))/\log\log\log T=\infty$.

则

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$
$$=\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=1 \quad \text{a.s.} \tag{2.2.11}$$

其中 $\gamma_1(T)=\{2\sigma^2(a_T)\log(T/a_T)\}^{-1/2}$.

推论 2.2.2 设 a_T $(0<a_T\le T)T$ 的函数满足条件 (i), (ii) 和

(iii′) $\lim_{T\to\infty}(\log(T/a_T))/\log\log T=r$.

则

$$\begin{aligned}&\liminf_{T\to\infty}\beta_T\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\\ =&\liminf_{T\to\infty}\beta_T\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=\left(\frac{r}{r+1}\right)^{1/2}\quad\text{a.s.}\end{aligned}\tag{2.2.12}$$

其中当 $r=\infty$ 时，$\frac{r}{r+1}=1$.

注 2.2.2 推论 2.2.1 首先是由 Csáki 和 Révész (1979) 对 Wiener 过程得到的. 推论 2.2.2 由洪圣岩 (1990) 得到. 由定理 2.2.4，我们看到 $\{Z(t)\}$ 增量的下极限性质随 $(T/a_T)/\log\log T$ 的极限不同而不同. 但是，定理 2.2.2 告诉我们，它对应的上极限是一致的.

为证定理 2.2.4, 先要叙述一些引理. 在本节的余下部分中，c_α 表示只依赖于 α 的常数，其取值在不同的地方可以不同.

首先，假设 $\{X(t);t\in T\}$ 为一个零均值 Gauss 过程，$x(t)>0$ 为 T 上的实值函数. 如果存在一个 Gauss 过程 $\{U(t);t\in T_c\}$ 和 T_c 上的实值函数 $u(t)>0$, 其中 T_c 为可数集，使得

$$\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}\supset\left\{\sup_{t\in T_c}\frac{|U(t)|}{u(t)}\le 1\right\}\quad\text{a.s.},$$

则由 Kahatri-Šidák 不等式 (定理 1.2.4′) 得

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}\ge\prod_{t\in T_c}P\left\{\frac{|U(t)|}{u(t)}\le 1\right\}=:p_X.$$

如果这样的 $\{U(t);t\in T_c\}$ 和 $u(t)$ 存在，并且进一步假设对每个 $t\in T_c$, $U(t)$ 为某个 $X(s_1),\cdots,X(s_m)$ ($s_1,\cdots,s_m\in T$) 的线性组合, 则下界 p_X 称为 $P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}$ 的一个 KS 下界 (KSLB), 记作

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}\overset{\text{KS}}{\ge}p_X.$$

另外，如果 $\{Y(s); s\in S\}$ 为另一个零均值 Gauss 过程， $y(s)>0$ 为 S 上的实值函数，则不等式

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}\overset{\mathrm{KS}}{\ge} P\left\{\sup_{s\in S}\frac{|Y(s)|}{y(s)}\le 1\right\}$$

意味着 $P\left\{\sup_{s\in S}\frac{|Y(s)|}{y(s)}\le 1\right\}$ 的一个 KS 下界 p_Y 也是 $P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\le 1\right\}$ 的一个 KS 下界.

下述引理是 Khatri-Šidák 不等式 (定理 1.2.4′) 的直接推论.

引理 2.2.2 设 T_i, $i=1,2,\cdots$, 为参数集， $\{Y_i(t); t\in T_i, i=1,2,\cdots\}$ 为联合零均值 Gauss 过程. 假设

$$P\left\{\sup_{t\in T_i}\frac{|Y_i(t)|}{x_i(t)}\le 1\right\}\overset{\mathrm{KS}}{\ge} p_i \quad i=1,2,\cdots.$$

则

$$P\left\{\sup_i\sup_{t\in T_i}\frac{|Y_i(t)|}{x_i(t)}\le 1\right\}\overset{\mathrm{KS}}{\ge}\prod_{i=1}^{\infty}p_i,$$

其中 $x_i(t)>0$, $t\in T_i$, $i=1,2,\cdots$.

下述引理是 Révész (1982) 的引理 2.3 的一个类比. 但 Révész 的证明似乎有误, 因为其引理 2.3 的下界不能像证明其引理 2.2 那样由 Slepian 引理得到.

引理 2.2.3 设 $\{\Gamma(t); -\infty<t<\infty\}$ 为一个几乎处处连续的零均值 Gauss 过程. 假设存在一个 $[0,\infty)$ 上的非降的函数 $u(h)$ 使得

$$E(\Gamma(t+h)-\Gamma(t))^2\le u^2(h) \quad \forall t\ge 0, h\ge 0.$$

则对任何 $x\ge 0.68$, $T>0$, $a>0$ 和 $k\ge 1$ 有

$$P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a}|\Gamma(t+s)-\Gamma(t)|\le x(u(a)+u(a,k)+u^*(a,k))\Big\}$$
$$\overset{\mathrm{KS}}{\ge}\exp\Big(-\frac{76}{\sqrt{2\pi}}\Big(\frac{T}{a}+1\Big)2^{2k}x^{-1}\exp\Big(-\frac{x^2}{2}\Big)\Big),$$

其中
$$u(a,k)=u\Big(\frac{2a}{2^k}\Big)+2\sum_{j=0}^{\infty}u(a2^{-k-j-1}),$$
$$u^*(a,k)=2\sum_{j=0}^{\infty}\sqrt{j}u(a2^{-k-j-1}).$$

证明 对任意的正数 t 和正整数 k, 记 $R=2^k$, $t_j=a[t\frac{2^j}{a}]/2^j$. 则我们有

$$|\Gamma(t+s)-\Gamma(t)|$$
$$\le|\Gamma((t+s)_k)-\Gamma(t_k)|+|\Gamma(t+s)-\Gamma((t+s)_k)|+|\Gamma(t)-\Gamma(t_k)|$$
$$\le|\Gamma((t+s)_k)-\Gamma(t_k)|+\sum_{j=0}^{\infty}|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})|$$
$$+\sum_{j=0}^{\infty}|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})|.$$

显然有

$$\sup_{0\le t\le T}\sup_{0\le s\le a(1-1/R)}|(t+s)_k-t_k|\le a,$$
$$\sup_{0\le t\le T}\sup_{a(1-1/R)\le s\le a}|(t+s)_k-(t+a(1-1/R))_k|\le 2a2^{-k},$$
$$\sup_{0\le t\le T}\sup_{0\le s\le a}|(t+s)_{k+j+1}-(t+s)_{k+j}|\le a2^{-k-j-1},$$
$$\sup_{0\le t\le T}\sup_{0\le s\le a}|\Gamma((t+s)_k)-\Gamma(t_k)|$$
$$\le\sup_{0\le t\le T}\sup_{0\le s\le a(1-1/R)}|\Gamma((t+s)_k)-\Gamma(t_k)|$$
$$+\sup_{0\le t\le T}\sup_{a(1-1/R)\le s\le a}|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)|$$

和

$$\begin{aligned}&\mathrm{Card}\{\Gamma((t+s)_k)-\Gamma(t_k):\quad 0\le t\le T,\\&\qquad 0\le s\le a(1-1/R)\}\le 2R^2\Big(\frac{T}{a}+1\Big),\\&\mathrm{Card}\{\Gamma((t+s)_k)-\Gamma((t+a(1-1/R)_k):\quad 0\le t\le T,\\&\qquad a(1-1/R)\le s\le a\}\le 2R\Big(\frac{T}{a}+1\Big),\\&\mathrm{Card}\{\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j}):\quad 0\le t\le T,\quad 0\le s\le a\}\\&\qquad\le 2^{2+j+1}(\frac{T}{a}+1),\\&\mathrm{Card}\{\Gamma(t_{k+j+1})-\Gamma(t_{k+j+1}):\quad 0\le t\le T\}\le 2^{k+j+1}\Big(\frac{T}{a}+1\Big).\end{aligned}$$

由引理 2.2.2, 我们有

$$\begin{aligned}J:=&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a}|\Gamma(t+s)-\Gamma(t)|\le xu(a)+xu(2a/R)\\&+\sum_{j=0}^{\infty}2x_ju(a2^{-k-j-1})\Big\}\\\overset{\mathrm{KS}}{\ge}&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a(1-1/R)}|\Gamma((t+s)_k)-\Gamma(t_k)|\le xu(a)\Big\}\\&\cdot P\Big\{\sup_{0\le t\le T}\sup_{a(1-1/R)\le s\le a}|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)|\\&\le xu(2a/R)\Big\}\cdot\prod_{j=0}^{\infty}P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a}|\Gamma((t+s)_{k+j+1})\\&-\Gamma((t+s)_{k+j})|\le x_ju(a2^{-k-j-1})\Big\}\\&\cdot\prod_{j=0}^{\infty}P\Big\{\sup_{0\le t\le T}|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})|\le x_ju(a2^{-k-j-1})\Big\}\\\overset{\mathrm{KS}}{\ge}&(\psi(x))^{2R^2(\frac{T}{a}+1)}(\psi(x))^{2R(\frac{T}{a}+1)}\prod_{j=0}^{\infty}(\psi(x_j))^{2^{k+j+1}(\frac{T}{a}+1)}\\&\cdot\prod_{j=0}^{\infty}(\psi(x_j))^{2^{k+j+1}(\frac{T}{a}+1)}=\end{aligned}$$

$$(\psi(x))^{2(R^2+R)(\frac{T}{a}+1)}\cdot\prod_{j=0}^{\infty}(\psi(x_j))^{4R2^j(\frac{T}{a}+1)},$$

其中 $\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t^2}dt\ge 1/2\ (x\ge 0.68)$. 因为当 $0\le y\le 1/2$ 时, $(1-y)e^{2y}\ge 1$, 对 $x\ge 0.68$ 我们有

$$\psi(x)\ge\exp(-2(1-\psi(x)))\ge\exp(-\frac{4}{\sqrt{2\pi}}x^{-1}e^{-x^2/2}).$$

取 $\frac{1}{2}x_j^2=\frac{1}{2}x^2+j$, 则

$$\begin{aligned}
J&\ge\exp\Big(-\Big(\frac{4}{\sqrt{2\pi}}2(R^2+R)\Big(\frac{T}{a}+1\Big)x^{-1}e^{-\frac{x^2}{2}}\\
&\qquad+\frac{4}{\sqrt{2\pi}}4R\Big(\frac{T}{a}+1\Big)\sum_{j=0}^{\infty}2^jx_j^{-1}e^{-\frac{x^2}{2}-j}\Big)\Big)\\
&\ge\exp\Big(-\frac{4}{\sqrt{2\pi}}\Big(\frac{T}{a}+1\Big)x^{-1}e^{-x^2/2}\Big(2R^2+2R+4R\sum_{j=0}^{\infty}2^je^{-j}\Big)\Big)\\
&=\exp\Big(-\frac{4}{\sqrt{2\pi}}\Big(\frac{T}{a}+1\Big)x^{-1}e^{-x^2/2}\Big(2R^2+2R+R\frac{4e}{e-2}\Big)\Big)\\
&\ge\exp\Big(-\frac{76}{\sqrt{2\pi}}x^{-1}e^{-x^2/2}R^2\Big(\frac{T}{a}+1\Big)\Big).
\end{aligned}$$

注意到

$$\begin{aligned}
&xu(2a/R)+2\sum_{j=0}^{\infty}x_ju(a2^{-k-j-1})\\
&\quad\le x(u(2a/R)+2\sum_{j=0}^{\infty}u(a2^{-k-j-1}))+2\sum_{j=0}^{\infty}\sqrt{j}u(a2^{-k-j-1})\\
&\quad=xu(a,k)+u^*(a,k),
\end{aligned}$$

引理得证.

引理 2.2.4 设 $\{\Gamma(t);-\infty<t<\infty\}$, $u(h)$ 如引理 2.2.3 所示, 假设对某个 $\alpha>0$, $u(x)/x^{\alpha}$ 拟增. 则存在只依赖于 α 的常数

$c_\alpha > 0$, $c'_\alpha > 1$ 使得对任何 $x \geq c'_\alpha$ 有

$$P\Big\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}|\Gamma(t+s)-\Gamma(t)| \leq xu(h)\Big\}$$

$$\overset{\mathrm{KS}}{\geq} \exp\big(-c_\alpha(\frac{T}{h}+1)x^{4/\alpha-1}e^{-x^2/2}\big).$$

证明 由于 $u(x)/x^\alpha$ 拟增，存在 $c_0 > 0$ 使得

$$u(ht) \leq c_0 t^\alpha u(h) \quad \forall 0 \leq t \leq 1, h \geq 0.$$

从而

$$u(h,k) \leq c_0 u(h)\Big(\Big(\frac{2}{2^k}\Big)^\alpha + 2\sum_{j=0}^{\alpha} 2^{-\alpha(k+j+1)}\Big)$$

$$= c_0 u(h) 2^{-\alpha k}\Big(2^\alpha + \frac{2^{1+\alpha}}{2^\alpha - 1}\Big),$$

$$u^*(h,k) \leq 2\Big(\sum_{j=0}^{\infty}\sqrt{j}2^{-\alpha(k+j+1)}\Big)c_0 u(h)$$

$$\leq 2c_0 u(h)2^{-\alpha k}(2\alpha\log 2)^{-3/2}.$$

令 $K_\alpha = c_0(2^\alpha + \frac{2^{1+\alpha}}{2^\alpha-1}) + 2c_0(2\alpha\log 2)^{-3/2}$, 则对任何 $x \geq 1$ 有

$$P\Big\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}|\Gamma(t+s)-\Gamma(t)| \leq xu(h)(1+2^{-\alpha k}K_\alpha)\Big\}$$

$$\overset{\mathrm{KS}}{\geq} \exp\Big(-\frac{76}{\sqrt{2\pi}}\Big(\frac{T}{h}+1\Big)2^{2k}x^{-1}e^{-x^2/2}\Big).$$

记 $y = x(1+2^{-\alpha k}K_\alpha)$, 则

$$J := P\Big\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}|\Gamma(t+s)-\Gamma(t)| \leq yu(h)\Big\}$$

$$\overset{\mathrm{KS}}{\geq} \exp\Big(-\frac{76}{\sqrt{2\pi}}\Big(\frac{T}{h}+1\Big)y^{-1}e^{-y^2/2}2^{2k}(1+2^{-\alpha k}K_\alpha)$$

$$\cdot\exp\Big(y^2\frac{2\cdot 2^{-\alpha k}K_\alpha + 2^{-2\alpha k}K_\alpha^2}{(1+2^{-\alpha k}K_\alpha)^2}\Big)\Big)$$

$$\geq \exp\left(-\frac{76}{\sqrt{2\pi}}\Big(\frac{T}{h}+1\Big)y^{-1}e^{-y^2/2}2^{2k}(1+K_\alpha)\exp(3K_\alpha^2 2^{-\alpha k}y^2)\right).$$

现在，对 $y \ge 1+K_\alpha$，取 k 使得

$$2^{k-1} \le y^{2/\alpha} \le 2^k,$$

则

$$2^{2k}\exp(3K_\alpha^2 2^{-\alpha k}y^2) \le 4y^{4/\alpha}\exp(3K_\alpha^2).$$

从而

$$J \ge \exp\Big(-\frac{4\times 76}{\sqrt{2\pi}}\Big(\frac{T}{h}+1\Big)y^{4/\alpha-1}e^{-y^2/2}(1+K_\alpha)\exp(3K_\alpha^2)\Big).$$

引理 2.2.4 得证.

引理 2.2.5 设 $\{Z(t); t \ge 0\}$ 为一个阶为 $\alpha(0<\alpha\le 1/2)$ 的分数 Wiener 过程. 则对任意的 $\varepsilon > 0$ 存在 $u_0 = u_0(\varepsilon) > 0$, $T_0 = T_0(\varepsilon) > 0$ 使得对任何 $u \ge u_0$ 和 $T \ge T_0$ 我们有

$$\begin{aligned} P\Big\{\sup_{0\le t\le T}&(Z(t+1)-Z(t)) \le u\Big\} \\ &\le \exp\Big(-(1-\varepsilon)H_{2\alpha}\frac{T}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\Big). \end{aligned}$$

其中 $H_{2\alpha} = \lim_{T\to\infty} T^{-1}\int_0^\infty e^s P(\sup_{0<t<T} Y(t) > s)ds$, $\{Y(t); 0 \le t \le \infty\}$ 为一非平稳 Gauss 过程, 满足 $Y(0) = 0$ a.s., $EY(t) = -|t|^{2\alpha}$, $\mathrm{Cov}(Y(t_1), Y(t_2)) = |t_1|^{2\alpha} + |t_2|^{2\alpha} - |t_1-t_2|^{2\alpha}$.

证明 对任何整数 $k < T$ 我们有

$$\begin{aligned} \sup_{0\le t\le T}(Z(t+1)-Z(t)) &\ge \max_{0\le i\le l}\sup_{i(k+1)\le t<(i+1)(k+1)}(Z(t+1)-Z(t)) \\ &\ge \max_{0\le i\le l}\sup_{i(k+1)\le t<i(k+1)+k}(Z(t+1)-Z(t)), \end{aligned}$$

其中 $l = \max\{i, (i+1)(k+1)-1 \le T\}$. 注意到 $\alpha \le 1/2$, 由 Slepian

引理我们有

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}(Z(t+1)-Z(t))\le u\Big\}\\
&\quad\le P\Big\{\max_{0\le i\le l}\sup_{i(k+1)\le t<i(k+1)+k}(Z(t+1)-Z(t))\le u\Big\}\\
&\quad\le\Big(P\Big\{\sup_{0\le t\le k}(Z(t+1)-Z(t))\le u\Big\}\Big)^{l+1}.
\end{aligned}
$$

由定理 1.1.2, 得

$$
\lim_{x\to\infty}\frac{P\{\sup_{0\le t\le k}(Z(t+1)-Z(t))>x\}}{\frac{k}{\sqrt{2\pi}}x^{1/\alpha-1}e^{-x^2/2}}=H_{2\alpha}.
$$

从而

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}(Z(t+1)-Z(t))\le u\Big\}\\
&\le\Big(1-\frac{(1+o(1))H_{2\alpha}}{\sqrt{2\pi}}ku^{1/\alpha-1}e^{-u^2/2}\Big)^{l+1}\\
&\le\exp\Big(-(1+o(1))\frac{k(l+1)H_{2\alpha}}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\Big)\\
&\le\exp\Big(-(1-\epsilon)H_{2\alpha}\frac{T}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\Big)(\forall\,T\ge T_0(\epsilon),u\ge u_0(\epsilon)),
\end{aligned}
$$

引理 2.2.5 证毕.

引理 2.2.6 设 $\{Z(t);t\ge 0\}$ 为阶为 α 的分数 Wiener 过程且 $1/2<\alpha<1$. 则对任何 $\delta>0$, 存在 $c_\alpha=c(\alpha,\delta)>0$, 使得对充分大的 T, k ($k\le T$) 和任何 $u>0$ 有

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}|Z(t+1)-Z(t)|\le u\Big\}\\
&\qquad\le\exp\Big(-c_\alpha\frac{T}{k}\frac{1}{u}\exp\Big(-\frac{(1+\delta)u^2}{2}\Big)\Big).
\end{aligned}
$$

证明 设 $Y(t)=Z(t+1)-Z(t)$, 则

$$\begin{aligned}EY(t+h)Y(t)&=E(Z(t+h+1)-Z(t+h))(Z(t+1)-Z(t))\\&=\frac{1}{2}(|h+1|^{2\alpha}+|h-1|^{2\alpha}-2h^{2\alpha})=:\rho(h).\end{aligned}$$

易知 $\rho(h)$ 在 $[0,\infty)$ 上严格递减, 在 $[1,\infty)$ 上为凸函数, 且 $\rho(0)=1$, $\rho(h)\approx h^{2\alpha-2}$ $(h\to\infty)$.

对 $k>1$ (充分大), 令 $t_i=ik$, $i=0,1,2,\cdots$, $Y(t_{-1})=0$,

$$a_{ij}=E(Y(t_i)-Y(t_{i-1}))(Y(t_j)-Y(t_{j-1})),\quad i,j\geq 0.$$

则

$$\begin{aligned}&a_{ij}=2\rho(|i-j|k)-\rho(|i-j+1|k)-\rho(|i-j-1|k),\quad i,j\geq 1;\\&a_{ii}=2(1-\rho(k)),\quad i\geq 1,\quad a_{00}=1;\\&a_{i0}=a_{0i}=\rho(ik)-\rho((i-1)k),\quad i\geq 1.\end{aligned}$$

从而, 当 $i,j\geq 1$, $|i-j|\geq 2$ 时, 由 $\rho(h)$ 在 $[1,\infty)$ 的凸性, 我们有 $a_{ij}<0$; 当 $i,j\geq 1$, $|i-j|=1$ 时, 因为 $\rho(h)\to 0$ $(h\to\infty)$, 我们对充分大的 k 有 $a_{ij}=2\rho(k)-\rho(2k)-1<0$; 当 $i\geq 1$ 时, 由 $\rho(h)$ 的单调性, 我们有 $a_{i0}<0$.

现对任何 $n\geq 2$ 和 $1\leq i\leq n$, 令

$$S_i^n=\sum_{\substack{j=0\\j\neq i}}^{n}|a_{ij}|=a_{ii}-\sum_{j=0}^{n}a_{ij}.$$

则对 $i\geq 1$ 有

$$\begin{aligned}S_i^n=&2(1-\rho(k))-E\big((Y(t_i)-Y(t_{i-1}))Y(t_n)\big)\\=&2(1-\rho(k))-\rho((n-i)k)+\rho((n-i+1)k)<2(1-\rho(k))\\=&a_{ii};\\S_i^i=&1-\rho(k)=\frac{1}{2}a_{ii};\end{aligned}$$

对 $i=0$ 有,

$$S_0^n = \sum_{\substack{j=0\\ j\neq 0}}^{n} |a_{0j}| = \sum_{j=1}^{n} (\rho((j-1)k) - \rho(jk)) = 1 - \rho(nk) < 1 = a_{00};$$

$$S_0^0 = 0 < \frac{1}{2} a_{00}.$$

注意到 $A = (a_{ii})$ 为 $\big(Y(t_0) - Y(T_{-1}), \cdots, Y(t_n) - Y(t_{n-1})\big)$ 的协方差矩阵, 与推论 1.2.6 的证明类似, 我们有

$$\begin{aligned}
&P\Big\{\sup_{0\le i\le n} |Y(t_i)| \le u\Big\} = P\Big\{\sup_{0\le i\le n} |Y(t_i) - Y(t_{-1})| \le u\Big\}\\
&\le \prod_{i=0}^{n} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}u/a_{ii}^{1/2}} e^{-x^2/2} dx \le \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}u/a_{ii}^{1/2}} e^{-x^2/2} dx\\
&= \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{u/\sqrt{1-\rho(k)}} e^{-x^2/2} dx\\
&\le \prod_{i=1}^{n} \exp\Big(-\sqrt{\frac{2}{\pi}} \int_{u/\sqrt{1-\rho(k)}}^{\infty} e^{-x^2/2} dx\Big)\\
&\le \exp\Big(-\frac{\sqrt{1-\rho(k)}}{\sqrt{2\pi}} \cdot n \cdot \frac{1}{u} \exp\Big(-\frac{u^2}{2(1-\rho(k))}\Big)\Big).
\end{aligned}$$

从而

$$\begin{aligned}
P\Big\{\sup_{0\le t\le T} |Z(t+1) - Z(t)| \le u\Big\} &\le P\Big\{\sup_{0\le i\le T/k} |Y(t_i)| \le u\Big\}\\
&\le \exp\Big(-\frac{\sqrt{1-\rho(k)}}{\sqrt{2\pi}} \cdot \frac{T}{k} \cdot \frac{1}{u} \exp\Big(-\frac{u^2}{2(1-\rho(k))}\Big)\Big).
\end{aligned}$$

因此引理 2.2.6 得证.

引理 2.2.7 *存在 $c_\alpha > 0$ 使得对任何 $0 < x < 1$, $T > 0$ 成立*

$$P\Big\{\sup_{0\le t\le T} \sup_{0\le s\le 1} |Z(t+s) - Z(t)| \le x^\alpha\Big\} \le \exp\Big(-c_\alpha \frac{T}{x}\Big).$$

当 $0 < \alpha \le 1/2$ 时，可选取 $c_\alpha = -\log\phi(1) < 0.18$; 当 $1/2 < \alpha < 1$ 时，可选取 $c_\alpha = -\frac{1}{2}\log\phi(1/\sqrt{1-4^{\alpha-1}})$.

证明 令 $\xi_i = Z((i+1)x) - Z(ix)$, $\eta_i = \xi_{2i} - \xi_{2i-1}$. 则

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le 1}|Z(t+s)-Z(t)| \le x^\alpha\Big\}\\
&\quad\le P\Big\{\sup_{0\le i<T/x}\sup_{0\le s\le 1}|Z(ix+s)-Z(ix)| \le x^\alpha\Big\}\\
&\quad\le P\Big\{\sup_{0\le i<T/x}|Z((i+1)x)-Z(ix)| \le x^\alpha\Big\}\\
&\quad\le P\Big\{\sup_{0\le i<T/x}\xi_i \le x^\alpha\Big\}.
\end{aligned}
$$

当 $0 < \alpha \le 1/2$ 时，我们有 $E\xi_i\xi_j \le 0$, $i \ne j$. 由 Slepian 引理 (推论 1.2.1) 得

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le 1}|X(t+s)-X(t)| \le x^\alpha\Big\}\\
&\quad\le (\phi(1))^{T/x} = \exp\Big(\frac{T}{x}\log\phi(1)\Big).
\end{aligned}
$$

当 $1/2 < \alpha < 1$ 时，我们有 $E\eta_i^2 = (4-4^\alpha)x^{2\alpha}$,

$$
\begin{aligned}
E\eta_i\eta_j =& \frac{1}{2}\{4(2|j-i|-1)^{2\alpha} + 4(2|j-i|+1)^{2\alpha} - (2|j-i|-2)^{2\alpha}\\
&- (2|j-i|+2)^{2\alpha} - 6(2|j-i|)^{2\alpha}\}x^{2\alpha} \le 0
\end{aligned}
$$

对 $j \ne i$ 成立. 从而由 Slepian 引理得

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le 1}|Z(t+s)-Z(t)| \le x^\alpha\Big\}\\
&\quad\le P\Big\{\sup_{0\le i<\frac{1}{2}\frac{T}{x}}|\eta_i| \le 2x^\alpha\Big\} \le P\Big\{\sup_{0\le i<\frac{1}{2}\frac{T}{x}}\eta_i \le 2x^\alpha\Big\}\\
&\quad\le \Big(\phi\Big(\frac{2}{\sqrt{4-4^\alpha}}\Big)\Big)^{T/(2x)} = \exp\Big(\frac{1}{2}\log\phi\Big(\frac{1}{\sqrt{1-4^{\alpha-1}}}\Big)\frac{T}{x}\Big).
\end{aligned}
$$

引理 2.2.7 得证.

令 $T_n = e^{n^p}$ $(p>1)$, $d_n = ne^{n^p}$, $A_n = (\frac{d_{n-1}}{T_n}, \frac{d_n}{T_n})$,

$$Y_n(t) = \int_{|x|\notin A_n} \frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\}dW(x). \quad (2.2.13)$$

下述引理给出了 $Y_n(\cdot)$ 的方差的估计.

引理 2.2.8 设 $0<\alpha<1$, $\gamma>0$, $p>1$. 则存在只依赖于 α,γ,p 的常数 $c\in(0,\infty)$ 使得对 $n>2$, $0\le t\le n/2$ 和 $0<h\le 1$ 一致地有,

$$\sigma_n^2(t,h) = E(Y_n(t+h)-Y_n(t))^2 \le ch^{2\alpha}(\log h^{-1})^\gamma n^{-\delta},$$

其中 $\delta = \min((2-2\alpha),(p-1)/2,\gamma(p-1)/2)$.

证明 若 $0<\alpha<1, \alpha\ne 1/2$, 令

$$f^2(y) = \frac{1}{K_\alpha^2}(|y-1/2|^{(2\alpha-1)/2} - |y+1/2|^{(2\alpha-1)/2})^2, -\infty<y<\infty.$$

则 $\int_R f^2(y)dy = 1$ 且在 $[0,\infty)$ 上有 $f^2(y)\le 1/K_\alpha^2$. 交换积分变量得

$$\sigma_n^2(t,h) = h^{2\alpha}\int_{|y+t/h+1/2|\notin A_n/h} f^2(y)dy.$$

由 $|y+t/h+1/2|\notin A_n/h$ 得

$$|y+t/h+1/2|\ge n/h \quad 或 \quad |y+t/h+1/2|\le \frac{d_{n-1}}{T_n}/h.$$

当 $|y+t/h+1/2|\ge n/h$ 时, 注意到 $0\le t\le \frac{1}{2}n$ 我们有

$$y+1/2\ge n/(2h) \text{ 或 } y+1/2\le -n/h.$$

从而对 $n>2$ 有

$$\int_{|y+t/h+1/2|\ge n/h} f^2(y)dy \le \int_{n/(2h)}^\infty f^2(y)dy + \int_{n/(4h)}^\infty f^2(y)dy$$
$$\le 2\int_{n/(4h)}^\infty f^2(y)dy \le c\int_{n/(4h)}^\infty y^{2\alpha-3}dy \le ch^{2-2\alpha}n^{-(2-2\alpha)}.$$

现设

$$|y+t/h+1/2| \leq \frac{d_{n-1}}{T_n}/h.$$

因为 $f^2(y) \leq 1/K_\alpha^2$, 我们有

$$\int_{-1/2-(\frac{d_{n-1}}{T_n}+t)/h}^{-1/2+(\frac{d_{n-1}}{T_n}-t)/h} f^2(y)dy \leq 2\frac{d_{n-1}}{T_n}/(hK_\alpha^2).$$

从而对 $n>2$, 有

$$\sigma_n^2(t,h) \leq h^{2\alpha}\Big(ch^{2-2\alpha}n^{-(2-2\alpha)} + 2\frac{d_{n-1}}{T_n h K_\alpha^2}\Big).$$

如果 $1 \geq h \geq e^{-n^{(p-1)/2}}$, 则有

$$\begin{aligned}\sigma_n^2(t,h) &\leq ch^{2\alpha}(\log h^{-1})^\gamma\Big((\log h^{-1})^{-\gamma}h^{2-2\alpha}n^{-(2-2\alpha)}\\ &\quad + e^{n^{(p-1)/2}}\frac{(n-1)e^{(n-1)^p}}{e^{n^p}}\Big)\\ &\leq ch^{2\alpha}(\log h^{-1})^\gamma(n^{-(2-2\alpha)}+n^{-(2-2\alpha)})\\ &\leq ch^{2\alpha}(\log h^{-1})^\gamma n^{-(2-2\alpha)}.\end{aligned}$$

如果 $0 < h \leq e^{-n^{(p-1)/2}}$, 则有

$$\begin{aligned}\sigma_n^2(t,h) &\leq h^{2\alpha} \leq ch^{2\alpha}(\log h^{-1})^\gamma(\log e^{n^{(p-1)/2)}})^{-\gamma}\\ &= ch^{2\alpha}(\log h^{-1})^\gamma n^{-\gamma(p-1)/2}.\end{aligned}$$

如果 $\alpha = 1/2$, 注意到此时积分核为 $I_{(0,t]}(x)$, 对任何 $h \geq 0$ 和 $n > 2$ 有,

$$\begin{aligned}\sigma_n^2(t,h) &= \begin{cases} 0, & 若 t \geq \dfrac{d_{n-1}}{T_n},\\ \dfrac{d_{n-1}}{T_n} - t, & 若 0 \leq t \leq \dfrac{d_{n-1}}{T_n} < t+h,\\ h, & 若 0 \leq t \leq t+h \leq \dfrac{d_{n-1}}{T_n}\end{cases}\\ &\leq h \wedge \frac{d_{n-1}}{T_n}.\end{aligned}$$

从而当 $h < d_{n-1}/T_n$ 时,

$$\sigma_n^2(t,h) \le h \le h(\log h^{-1})^\gamma \Big(\log \frac{T_n}{d_{n-1}}\Big)^{-\gamma}$$
$$\le ch(\log h^{-1})^\gamma n^{-\gamma(p-1)/2}.$$

当 $h \ge d_{n-1}/T_n$ 时,则对 $0 < \gamma \le 1$ 有 $h(\log h^{-1})^\gamma \ge \frac{d_{n-1}}{T_n}\big(\log \frac{T_n}{d_{n-1}}\big)^\gamma$,因此

$$\sigma_n^2(t,h) \le \frac{d_{n-1}}{T_n} \le h(\log h^{-1})^\gamma \Big(\log \frac{T_n}{d_{n-1}}\Big)^{-\gamma}$$
$$\le ch(\log h^{-1})^\gamma n^{-\gamma(p-1)/2};$$

对 $\gamma > 1$ 我们有 $h(\log h^{-1})^\gamma \ge h \log h^{-1} \ge \frac{d_{n-1}}{T_n} Ln\frac{T_n}{d_{n-1}}$, 所以

$$\sigma_n^2(t,h) \le \frac{d_{n-1}}{T_n} \le h(\log h^{-1})^\gamma \Big(\log \frac{T_n}{d_{n-1}}\Big)^{-1}$$
$$\le ch(\log h^{-1})^\gamma n^{-(p-1)/2}.$$

因此, 在任何情形下都存在一个只依赖于 α, γ, p 的常数 $C \in (0, \infty)$ 使得对 $n > 2$, $0 \le t \le n/2$ 和 $0 < h \le 1$ 一致地有

$$\sigma_n^2(t,h) \le Ch^{2\alpha}(\log h^{-1})^\gamma n^{-\delta},$$

其中 δ 如题设所示.

定理 2.2.4 的证明

证明分四步进行.

第一步 假设条件 (iv) 满足, 则

$$\liminf_{T\to\infty} \gamma(T) \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Z(t+s) - Z(t)| \le 1 \quad \text{a.s.} \tag{2.2.14}$$

证明 如果

$$\limsup_{T\to\infty} (\log(T/a_T))/(\log\log\log T) = \infty, \tag{2.2.15}$$

则存在 $\{T_N\}$ 使得

$$\lim_{N\to\infty}(\log(T_N/a_{T_N}))/(\log\log\log T_N)=\infty. \tag{2.2.16}$$

由引理 2.2.4 得

$$\begin{aligned}
&P\Big\{\Big(2\sigma^2(a_{T_N})\log\frac{T_N}{a_{T_N}}\Big)^{-1/2}\sup_{0\le t\le T_N-a_{T_N}}\sup_{0\le s\le a_{T_N}}\\
&\qquad |Z(t+s)-Z(t)|\ge(1+\varepsilon)^{1/2}\Big\}\\
&\le 1-\exp\Big(-c_\alpha\frac{T_N}{a_{T_N}}\Big(2\log\frac{T_N}{a_{T_N}}\Big)^{\frac12(\frac4\alpha-1)}\exp\Big(-(1+\varepsilon)\log\frac{T_N}{a_{T_N}}\Big)\Big)\\
&\le c_\alpha\Big(\log\frac{T_N}{a_{T_N}}\Big)^{2/\alpha-1/2}\Big(\frac{a_{T_N}}{T_N}\Big)^{\varepsilon}\to 0\quad(N\to\infty).
\end{aligned}$$

从而

$$\begin{aligned}
&\liminf_{N\to\infty}\Big(2\sigma^2(a_{T_N})\log\frac{T_N}{a_{T_N}}\Big)^{-1/2}\\
&\qquad\cdot\sup_{0\le t\le T_N-a_{T_N}}\sup_{0\le s\le a_{T_N}}|Z(t+s)-Z(t)|\le 1\quad\text{a.s.},
\end{aligned}$$

由此和 (2.2.16) 即得证 (2.2.14).

现设

$$\limsup_{T\to\infty}(\log(T/a_T))/(\log\log\log T)<\infty.$$

那么存在常数 $r_0>0$ 使得

$$T/a_T\le(\log\log T)^{r_0}. \tag{2.2.17}$$

令 T_n, d_n, $\{Y_n(t)\}$ $(n=1,2,\cdots)$ 如引理 2.2.8 所示. 对任何 $\varepsilon>0$, 由引理 2.2.4 和条件 (iv), 对充分大的 n 有

$$\begin{aligned}
&P\Big\{\sup_{0\le t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}|Z(t+s)-Z(t)|\le(1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\Big\}\\
&\ge\exp\Big(-c_\alpha\frac{T_n}{a_{T_n}}\frac{\big((1+\varepsilon)^{1/2}(2\log\frac{T_n}{a_{T_n}\log\log T_n})^{1/2}\big)^{4/\alpha-1}}{(T_n/a_{T_n})^{1+\varepsilon}}(\log\log T_n)^{1+\varepsilon}\Big)\\
&\ge n^{-2/3}.
\end{aligned}\tag{2.2.18}$$

令

$$
\begin{aligned}
X_n(t) &= \int_{|x|\in(d_{n-1},d_n)} \frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\}dW(x),\\
\tilde{X}_n(t) &= \int_{|x|\notin(d_{n-1},d_n)} \frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\}dW(x).
\end{aligned} \tag{2.2.19}
$$

则 $\{X_n(t)\}$, $n=1,2,\cdots$, 相互独立, $Z(t)=X_n(t)+\tilde{X}_n(t)$, 且

$$
\begin{aligned}
&\{\tilde{X}_n(t+s)-\tilde{X}_n(t); 0\le t\le T_n-a_{T_n}, 0\le s\le a_{T_n}\}\\
&\quad\overset{\mathcal{D}}{=}\{T_n^\alpha(Y_n(t+s)-Y_n(t)); 0\le t\le 1-\frac{a_{T_n}}{T_n}, 0\le s\le\frac{a_{T_n}}{T_n}\}.
\end{aligned}
$$

从而由引理 2.2.8, 引理 2.2.4 和 (2.2.17), 对充分大的 n 成立

$$
\begin{aligned}
J_n :=& P\left\{\sup_{0\le t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}|\tilde{X}_n(t+s)-\tilde{X}_n(t)|\ge\varepsilon\gamma^{-1}(T_n)\right\}\\
=& P\left\{\sup_{0\le t\le 1-\frac{a_{T_n}}{T_n}}\sup_{0\le s\le\frac{a_{T_n}}{T_n}}|Y_n(t+s)-Y_n(t)|\right.\\
&\left.\ge\left(\left(\frac{a_{T_n}}{T_n}\right)^\alpha\left(\log\frac{T_n}{a_{T_n}}\right)^{\frac{\gamma}{2}}n^{-\delta/2}\right)\varepsilon n^{\delta/2}\frac{(2\log\frac{T_n}{a_{T_n}\log\log T_n})^{1/2}}{(\log T_n/a_{T_n})^{\frac{\gamma}{2}}}\right\}\\
\le& c_\alpha\frac{T_n}{a_{T_n}}\exp\left(-c_{\alpha,\varepsilon}\frac{\log\frac{T_n}{a_{T_n}\log\log T_n}}{(\log T_n/a_{T_n})^\gamma}n^\delta\right)\\
\le& c_\alpha(\log\log T_n)^{r_0}\exp\left(-c_{\alpha,\varepsilon}\frac{\log\frac{T_n}{a_{T_n}\log\log T_n}}{(r_0\log\log\log T_n)^\gamma}n^\delta\right)\\
\le& c_\alpha(\log n)^{r_0}\exp\left(-c_{\alpha,\varepsilon}\frac{n^\delta}{(\log\log n)^\gamma}\right).
\end{aligned} \tag{2.2.20}
$$

因此

$$
\sum_{n=1}^\infty J_n<\infty. \tag{2.2.21}
$$

由 Borel-Cantelli 引理得

$$
\limsup_{n\to\infty}\gamma(T_n)\sup_{0\le t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}|\tilde{X}_n(t+s)-\tilde{X}_n(t)|\le\varepsilon\quad\text{a.s.} \tag{2.2.22}
$$

由 (2.2.18) 和 (2.2.21) 得

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le T_n-a_{T_n}}\ \sup_{0\le s\le a_{T_n}}|X_n(t+s)-X_n(t)|\\
&\le ((1+\varepsilon)^{1/2}+\varepsilon)\gamma^{-1}(T_n)\Big\}\\
&\ge P\left\{\sup_{0\le t\le T_n-a_{T_n}}\ \sup_{0\le s\le a_{T_n}}|Z(t+s)-Z(t)|\le((1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\right\}\\
&\qquad -P\left\{\sup_{0\le t\le T_n-a_{T_n}}\ \sup_{0\le s\le a_{T_n}}|\tilde{X}_n(t+s)-\tilde{X}_n(t)|\ge\varepsilon\gamma^{-1}(T_n)\right\}\\
&\ge n^{-2/3}-J_n. \qquad (2.2.23)
\end{aligned}
$$

注意到 $\{X_n(t)\}$, $n=1,2,\cdots$, 独立, 由 (2.2.21), (2.2.23) 和 Borel-Cantelli 引理得

$$
\begin{aligned}
&\liminf_{n\to\infty}\gamma(T_n)\sup_{0\le t\le T_n-a_{T_n}}\ \sup_{0\le s\le a_{T_n}}|X_n(t+s)-X_n(t)|\\
&\qquad\le(1+\varepsilon)^{1/2}+\varepsilon\quad \text{a.s.} \qquad (2.2.24)
\end{aligned}
$$

综合 (2.2.20) 和 (2.2.24) 得证 (2.2.14).

第二步 假设条件 (i), (ii) 和 (iv) 满足. 则

$$
\liminf_{T\to\infty}\gamma(T)\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|\ge 1\quad \text{a.s.} \qquad (2.2.25)
$$

证明 如果 $0<\alpha\le 1/2$, 由引理 2.2.5 (其中 $\varepsilon=1/2$) 对充分大的 T 有

$$
\begin{aligned}
&P\left\{\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|<\gamma^{-1}(T)\right\}\\
&\le\exp\left(-\frac{1}{2}H_{2\alpha}\frac{T}{a_T\sqrt{2\pi}}\left(2\log\frac{T}{a_T\log\log T}\right)^{1/\alpha-1}\frac{a_T\log\log T}{T}\right)\\
&\le(\log T)^{-4}. \qquad (2.2.26)
\end{aligned}
$$

令 $T_k=k^{\sqrt{k}}$ $(k=1,2,\cdots)$. 由 Borel-Cantelli 引理得

$$
\liminf_{k\to\infty}\gamma(T_k)\sup_{0\le t\le T_k-a_{T_k}}|Z(t+a_{T_k})-Z(t)|\ge 1\quad \text{a.s.} \qquad (2.2.27)
$$

对 $T_k \le T \le T_{k+1}$, 我们有

$$\begin{aligned}
&\gamma(T) \sup_{0\le t\le T-a_T} |Z(t+a_T)-Z(t)| \\
\ge&\Big(2a_{T_{k+1}}^{2\alpha} \log \frac{T_{k+1}}{a_{T_{k+1}} \log\log T_k}\Big)^{-1/2}\Big(\sup_{0\le t\le T_k-a_{T_k}} |Z(t+a_{T_k})-Z(t)| \\
&\quad - \sup_{0\le t\le T_k-a_{T_k}} \sup_{0\le s\le a_{T_{k+1}}-a_{T_k}} |Z(t+s)-Z(t)|\Big) \\
=:&A_k\gamma(T_k)I(T_k)-J_k(T_k),
\end{aligned} \tag{2.2.28}$$

其中

$$A_k = \Big(\frac{a_{T_k}}{a_{T_{k+1}}}\Big)^{\alpha}\Big(\frac{\log((T_k/a_{T_k})/\log\log T_k)}{\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k)}\Big)^{1/2}.$$

注意到 $T_k/T_{k+1} \to 1$ $(k\to\infty)$, 我们有 $a_{T_k}/a_{T_{k+1}} \to 1$ $(k\to\infty)$. 从而

$$A_k \to 1 \quad (k\to\infty). \tag{2.2.29}$$

另一方面，由定理 2.2.2 得

$$\limsup_{k\to\infty} \sup_{0\le t\le T_k-a_{T_k}} \sup_{0\le s\le a_{T_{k+1}}-a_{T_k}} \beta(T_k)|Z(t+s)-Z(t)| \le 1 \quad \text{a.s.} \tag{2.2.30}$$

其中

$$\begin{aligned}
&\beta(T_k) \\
=&\Big(2(a_{T_{k+1}}-a_{T_k})^{2\alpha}\Big(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log(T_k+a_{T_{k+1}})\Big)\Big)^{-1/2}.
\end{aligned}$$

易证

$$a_{T_{k+1}}-a_{T_k} \le a_{T_{k+1}}(1-T_k/T_{k+1}) \le 6a_{T_{k+1}}/k^{1/3},$$

由此得

$$\beta^{-2}(T_k)\Big(2a_{T_{k+1}}^{2\alpha}\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}\Big)^{-1}$$
$$\le\Big(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\Big)^{2\alpha}\frac{\log\frac{2T_{k+1}}{a_{T_{k+1}}\log\log T_k}+\log\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+2\log\log(2T_k)}{\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}}$$
$$\le ck^{2\alpha/3}\log k+c\frac{\big(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\big)^{2\alpha}\log\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}}{\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}}\to 0\quad (k\to\infty).$$

从而我们有

$$\limsup_{k\to\infty}J_k(T_k)=0\quad \text{a.s.}\tag{2.2.31}$$

综合 (2.2.27)—(2.2.31) 得证 (2.2.25).

当 $1/2<\alpha<1$ 时, 用引理 2.2.6 代替引理 2.2.5 且 (2.2.26) 中的 $\gamma^{-1}(T)$ 用 $(1-\varepsilon)\gamma^{-1}(T)$ 代替, 证明类似.

第三步 假设条件 (i), (ii) 和 (iv′) 满足. 则对某个 $c_\alpha>0$ 我们有

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\ge c_\alpha\quad \text{a.s.}\tag{2.2.32}$$

证明 由条件 (iv′) 知 $\gamma'(T)(\frac{T}{\log\log T})^\alpha\to 1$ $(T\to\infty)$. 由引理 2.2.7 对充分大的 T 有

$$P\Big\{(\frac{\log\log T}{T})^\alpha\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le x^\alpha\Big\}$$
$$\le\exp\Big(-\frac{T}{a_T}\frac{a_T\log\log T}{xT}c_\alpha\Big)=(\log T)^{-\frac{c_\alpha}{x}}.$$

令 $T_k=k^{\sqrt{k}}$ $(k=1,2,\cdots)$, 由 Borel-Cantelli 引理得

$$\liminf_{k\to\infty}\Big(\frac{\log\log T_k}{T_k}\Big)^\alpha\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Z(t+s)-Z(t)|\ge\Big(\frac{c_\alpha}{2}\Big)^\alpha.\tag{2.2.33}$$

此外

$$
\begin{aligned}
&\left(\frac{\log\log T_k}{T_{k+1}}\right)^{\alpha}\beta^{-2}(T_k)\\
&=2\left(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\left(\frac{a_{T_{k+1}}\log\log T_k}{T_{k+1}}\right)^{2\alpha}\left(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\right.\\
&\qquad\left.+\log\log(T_k+a_{T_{k+1}})\right)\\
&\le 2\left(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\left(\frac{a_{T_{k+1}}\log\log T_k}{T_{k+1}}\right)^{2\alpha}\left(\log\frac{2T_{k+1}}{a_{T_{k+1}}\log\log T_k}\right.\\
&\qquad\left.+\log\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+2\log\log(2T_k)\right)\\
&\le ck^{-2\alpha/3}(\log k)^{2\alpha+1}+ck^{-2\alpha/3}\log k^{1/3}\to 0\quad(k\to\infty),
\end{aligned}
$$

其中用到了条件 (iv′), 其余证明与 (2.2.25) 的类似.

第四步 假设条件 (iv′) 满足. 则对某个 $c_\alpha>0$ 有

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le c_\alpha\quad \text{a.s.}\tag{2.2.34}$$

证明 由 Chung 型重对数律 (见定理 4.2.2) 和下述不等式则有

$$
\begin{aligned}
&\liminf_{T\to\infty}\gamma'(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\\
&\qquad\le 2\liminf_{T\to\infty}\left(\frac{\log\log T}{T}\right)^{\alpha}\sup_{0\le t\le T}|Z(t)|.
\end{aligned}
$$

注 2.2.3 定理 2.2.4 中的条件 (i) 和 (ii) 可以去掉, 但证明更加复杂 (参见 Zhang 1997b).

注 2.2.4 Zhang (1997a) 还讨论了 $Z(\cdot)$ 的 Hanson-Russo 型增量的下极限性质.

2.2.4 更一般的 Gauss 过程

设 $\{\Gamma(t);t\ge 0\}$ 为零均值 Gauss 过程, 且

$$\sigma^2(h)=E(\Gamma(t+h)-\Gamma(t))^2,$$

其中 $\sigma(s)$ 为非降函数. 在适当的条件下, 这个 Gauss 过程有与分数 Wiener 过程类似的连续模和大增量性质.

定理 2.2.5 设 $\sigma(\cdot)$ 在 0 点以指数 $\alpha>0$ 正则变化. 假设对某 $h_0>0$ 有

$$E(\Gamma(d)-\Gamma(c))(\Gamma(b)-\Gamma(a))\le 0 \quad \forall 0\le a<b\le c<d<h_0, \tag{2.2.35}$$

则

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\frac{|\Gamma(t+s)-\Gamma(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}=1 \quad \text{a.s.} \tag{2.2.36a}$$

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\frac{|\Gamma(t+h)-\Gamma(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}=1 \quad \text{a.s.} \tag{2.2.36b}$$

注 2.2.5 由 (2.1.22) 定义的独立 O-U 过程的无穷级数 $X(\cdot)$ 满足 (2.2.35).

注 2.2.6 若对某个 $\alpha>0$, $\sigma(s)/s^\alpha$ 为 (0, 1) 上的拟增函数且 (2.2.35) 满足, 则 (2.2.36) 仍成立 (见定理 3.3.3).

定理 2.2.6 设 $\sigma(\cdot)$ 在无穷处以一个正指数 α 正则变化. 假设

$$E(\Gamma(d)-\Gamma(c))(\Gamma(b)-\Gamma(a))\le 0 \quad \forall 0<a<b\le c<d<\infty. \tag{2.2.37}$$

设 a_T $(0<a_T\le T)$ 为 T 的函数, 满足定理 2.2.2 中的条件 (i) 和 (ii). 则

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{0\le t\le a_T}\sup_{0\le s\le a_T}\beta_T|\Gamma(t+s)-\Gamma(t)|\\ &\quad=\limsup_{T\to\infty}\beta_T|\Gamma(T+a_T)-\Gamma(t)|=1 \quad \text{a.s.}\end{aligned}$$

进一步，若定理 2.2.2 中的条件 (iii) 满足，则

$$\begin{aligned}&\lim_{T\to\infty}\sup_{0\le t\le a_T}\sup_{0\le s\le a_T}\beta_T|\Gamma(t+s)-\Gamma(t)|\\&\quad=\lim_{T\to\infty}\sup_{0\le t\le a_T}\beta_T|\Gamma(t+a_T)-\Gamma(t)|=1\quad \text{a.s.}\end{aligned}$$

定理 2.2.5 和 2.2.6 的证明分别与定理 2.2.1 和 2.2.2 中的 $\alpha\le 1/2$ 情形类似.

对于下极限，我们也有一个类似于定理 2.2.4 的结果.

定理 2.2.7 *设 (2.2.37) 满足，且 $\sigma(\cdot)$ 在无穷处以一个正指数正则变化. 令 a_T $(0<a_T\le T)$ 为 T 的函数，满足定理 2.2.4 中的条件 (i), (ii) 和 (iv). 则*

$$\begin{aligned}&\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma(T)\big(\Gamma(t+s)-\Gamma(t)\big)\\&\quad=\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\gamma(T)\big(\Gamma(t+a_T)-\Gamma(t)\big)\\&\quad=1\quad \text{a.s.}\end{aligned}\tag{2.2.38}$$

注 2.2.7 值得指出的是：(2.2.38) 是关于增量的单边值的结果，我们不知道单边值用绝对值代替后，(2.2.38) 是否仍然成立.

证明 我们首先证明

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma(T)\big(\Gamma(t+s)-\Gamma(t)\big)\le 1\quad \text{a.s.}\tag{2.2.39}$$

若 (2.2.15) 满足，则 (2.2.39) 的证明与 (2.2.14) 的类似. 若 (2.2.17) 满足，我们令 $T_n=e^{n^p}$ $(p>1)$,

$$A_n=\Big\{\sup_{T_{n-1}<t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}\big(\Gamma(t+s)-\Gamma(t)\big)\le(1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\Big\}.$$

则 $T_{n-1}/T_n\to 0$ $(n\to\infty)$, 与 (2.2.18) 类似，对充分大的 n 我们有

$$P(A_n)\ge n^{-2/3}.$$

从而 $\sum_{n=1}^{\infty} P(A_n) = \infty$. 另一方面，由 Slepian 引理，从 (2.2.37) 可得

$$P(A_j A_k) \le P(A_j)P(A_k), \quad j \ne k.$$

因此由 Borel-Cantelli 引理得 $P(A_n, \text{i.o.}) = 1$, 此即证明了

$$\liminf_{n\to\infty} \gamma(T_n) \sup_{T_{n-1} < t \le T_n - a_{T_n}} \sup_{0\le s\le a_{T_n}} (\Gamma(t+s) - \Gamma(t)) \le 1 \quad \text{a.s.} \tag{2.2.40}$$

显然

$$\begin{aligned}
&\sup_{0\le t\le T_n - a_{T_n}} \sup_{0\le s\le a_{T_n}} (\Gamma(t+s) - \Gamma(t)) \\
&\quad \le \sup_{T_{n-1}\le t\le T_n - a_{T_n}} \sup_{0\le s\le a_{T_n}} (\Gamma(t+s) - \Gamma(t)) \\
&\quad\quad + \sup_{0\le u<v\le T_{n-1}} |\Gamma(u) - \Gamma(v)|.
\end{aligned} \tag{2.2.41}$$

由定理 2.2.6 我们有

$$\limsup_{n\to\infty} (2\sigma^2(T_{n-1}) \log\log T_{n-1})^{-1} \sup_{0\le u<v\le T_{n-1}} |\Gamma(u) - \gamma(v)| \le 1 \text{ a.s.} \tag{2.2.42}$$

由于 $\sigma(x)$ 在无穷处以一个正指数 α 正变, 所以 $\sigma(x)/x^{\alpha/2}$ 在 $[1,\infty)$ 上拟增，由 (2.2.17) 得

$$\begin{aligned}
\frac{\sigma(T_{n-1})}{\sigma(a_{T_{n-1}})} &\le \frac{\sigma(T_{n-1})}{\sigma(T_n/(\log\log T_n)^{\gamma_0})} \\
&\le c\Big(\frac{T_{n-1}}{T_n/(\log\log T_n)^{\gamma_0}}\Big)^{\alpha/2} \le c\exp(-cn^{p-1}),
\end{aligned}$$

由此和 (iv) 得

$$\gamma(T_n)(2\sigma^2(T_{n-1}) \log\log T_{n-1}) \to 0, \quad n \to \infty. \tag{2.2.43}$$

综合 (2.2.40)—(2.2.43) 得证 (2.2.39).

剩下我们只要证明

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \gamma(T)(\Gamma(t+a_T) - \Gamma(t)) \ge 1 \quad \text{a.s.}$$

注意到下述引理，上式的证明与定理 2.2.4 的第二步证明类似.

引理 2.2.9 设 $\{\Gamma(t); t \geq 0\}$ 为如定理 2.2.7 所示的 Gauss 过程. 则

$$
\begin{aligned}
&P\Big\{\sup_{0\leq t\leq T}(\Gamma(t+a)-\Gamma(t)) \leq u\sigma(a)\Big\} \\
&\leq \exp\Big(-\frac{1}{\sqrt{2\pi}}\frac{T}{a}\frac{1}{u}\Big(1-\frac{1}{u^2}\Big)e^{-u^2/2}\Big)
\end{aligned}
$$

对任何 $T > a > 0$ 成立.

证明 由 Slepian 引理，我们有

$$
\begin{aligned}
&P\Big\{\sup_{0\leq t\leq T}(\Gamma(t+a)-\Gamma(t)) \leq u\sigma(a)\Big\} \\
&\leq P\Big\{\sup_{0\leq k\leq [T/a]}(\Gamma(ka+a)-\Gamma(ka)) \leq u\sigma(a)\Big\} \\
&\leq \prod_{k=0}^{[T/a]} P\{N(0,1) \leq u\} \leq \exp\Big(-\frac{1}{\sqrt{2\pi}}\frac{T}{a}\frac{1}{u}\Big(1-\frac{1}{u^2}\Big)e^{-u^2/2}\Big),
\end{aligned}
$$

这就是我们要证的.

§ 2.3 两参数 Wiener 过程的大增量

在本节和本章的余下的章节中，我们研究比单参数情形复杂得多的多参数随机过程的样本轨道性质. 最简单的多参数随机过程就是两参数 Wiener 过程 $\{W(x,y); (x,y) \in \mathcal{R}^2\}$.

一个随机过程 $\{W(x,y); (x,y) \in \mathcal{R}^2\}$ 称为是两参数 Wiener 过程，如果

(1) 对任何矩形 $R = [x_1, x_2) \times [y_1, y_2)$, $W(R) \in N(0, \lambda(R))$, 其中 $\lambda(R) = (x_2 - x_1)(y_2 - y_1)$, $W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1)$,

(2) $W(0, y) = W(x, 0) = 0$ $(x, y \in \mathcal{R}^2)$,

(3) $\{W(x,y)\}$ 是独立增量过程, 即如果 $R_1,\cdots,R_n$ 为互不相交的矩形, 则 $W(R_1)$, $\cdots$, $W(R_n)$ $(n=2,3,\cdots)$ 为相互独立的随机变量,

(4) 以概率 1, 样本函数 $W(x,y;\omega)$ 关于 (x,y) 连续.

$W(\cdot,\cdot)$ 的连续模由 Pruitt 和 Orey (1973) 得到 (参见第 2.5 节). 这里我们只考察当 $T\to\infty$ 时 $W(\cdot,\cdot)$ 在面积为 a_T 的矩形上的增量有多大的问题.

设 $0<a_T\le T$ 和 $b_T\ge T^{1/2}$ 为 T 的两个非降函数, $D_T=D_T(b_T)=\{(x,y):xy\le T,0\le x,y\le b_T\}$, $L_T=L_T(a_T,b_T)$ (对应地 $L_T^*=L_T^*(a_T,b_T)$) 表示满足 $\lambda(R)\le a_T$ (对应地 $\lambda(R)=a_T$) 的矩形 $R=[x_1,x_2)\times[y_1,y_2)\subset D_T(b_T)$ 组成的集合. 定义

$$\begin{aligned}\delta_T=&\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)+\log\log T)\}^{-1/2},\\ \gamma_T=&\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)-\log\log\log T)\}^{-1/2},\\ \lambda_T=&\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))\}^{-1/2}.\end{aligned}$$

我们称一个 T 的函数 $f(T)>0$ 是正则非增的, 如果存在一个非增的函数 $g(T)>0$ 使得

(a) $\lim_{T\to\infty}f(T)/g(T)=1$,

(b) 对任何 $\varepsilon>0$, 存在 $\theta_0=\theta_0(\varepsilon)>1$ 使得对任何 $1<\theta\le\theta_0$ 和 $k\ge1$ 有,

$$\limsup_{T\to\infty}\frac{g(\theta^k)}{g(\theta^{k+1})}\le1+\varepsilon.$$

Csörgő 和 Révész (1978) 研究了两参数 Wiener 过程增量有多大的问题. 他们证明了

定理 2.3.1 设 Ta_T^{-1} 是 T 的非降函数, 且 δ_T 是 T 的正则非增函数. 则

$$\limsup_{T\to\infty}\sup_{R\in L_T^*}\delta_T|W(R)|=\limsup_{T\to\infty}\sup_{R\in L_T}\delta_T|W(R)|=1\quad\text{a.s.}\quad(2.3.1)$$

若我们还有

$$\lim_{T\to\infty} \frac{\log Ta_T^{-1} + \log(1+\log b_T a_T^{-1/2})}{\log\log T} = \infty, \tag{2.3.2}$$

则在 (2.3.1) 中 $\limsup_{T\to\infty}$ 可用 $\lim_{T\to\infty}$ 代替.

详细证明可在 Csörgő 和 Révész (1981) 中找到，我们不再叙述. 我们更为感兴趣的是当条件 (2.3.2) 不满足时 $W(\cdot,\cdot)$ 的下极限性质. 下述结果是由张立新 (1997c) 得到的.

定理 2.3.2 设

(i) Ta_T^{-1} 是 T 的非降函数,

(ii) γ_T 为 T 的正则非增函数,

(iii) $\liminf_{T\to\infty} \dfrac{\log Ta_T^{-1} + \log(\log b_T a_T^{-1/2}+1)}{\log\log\log T} > 1.$

则

$$\liminf_{T\to\infty} \sup_{R\in L_T} \gamma_T|W(R)| = \liminf_{T\to\infty} \sup_{R\in L_T^*} \gamma_T W(R) = 1 \quad \text{a.s.} \tag{2.3.3}$$

在证明定理 2.3.2 之前，我们先给出一个直接的推论:

推论 2.3.1 设定理 2.3.2 中的条件 (i) 满足，且

(ii′) λ_T 为 T 的正则非增函数,

(iii′) $\lim_{T\to\infty} \dfrac{\log Ta_T^{-1} + \log(\log b_T a_T^{-1/2}+1)}{\log\log\log T} = r \quad (1 \le r \le \infty)$.

则

$$\begin{aligned}&\liminf_{T\to\infty} \sup_{R\in L_T} \lambda_T|W(R)| \\ &= \liminf_{T\to\infty} \sup_{R\in L_T^*} \lambda_T W(R) = \left(\frac{r-1}{r}\right)^{1/2} \quad \text{a.s.},\end{aligned} \tag{2.3.4}$$

其中当 $r=\infty$ 时，$(r-1)/r = 1$.

注 2.3.1　推论 2.3.1 首先由林正炎 (1984) 对 $\tau=\infty$ 的情形得到.

定理的证明依赖于下述不等式, 它是引理 2.2.4 的一个类比.

定理 2.3.3　对任何 $\varepsilon>0$, 存在常数 $C=C(\varepsilon)>0$, $u_0=u_0(\varepsilon)>0$ 和 $T_0=T_0(\varepsilon)>0$ 使得

$$
\begin{aligned}
&P\{\sup_{R\in L_T}|W(R)|\le ua_T^{1/2}\}\\
&\ge\exp\left(-CTa_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2})e^{-u^2/(2+\varepsilon)}\right)
\end{aligned}
\tag{2.3.5}
$$

对任何 $u\ge u_0$, $T\ge T_0$ 成立.

为证明这个不等式, 我们首先引入一些记号并给出两个引理.

令 $\mu=\mu(T)$ 为使得下式成立的最小整数:

$$\mu\ge\log b_Ta_T^{-1/2}.$$

对任何正整数 q, 令 $Q=Q(q)=2^q$. 定义

$$z_i=z_i(q)=z_i(q,T)=a_T^{1/2}e^{i/Q}\qquad(i=0,\pm1,\cdots,\pm Qu),$$

$$x_j(i)=x_j(i,T)=jz_iQ^{-1}\qquad(j=0,1,\cdots),$$

$$y_j(i)=y_j(i,T)=ja_Tz_i^{-1}Q^{-1}\qquad(j=0,1,\cdots),$$

$$R_i=R_i(q)=R_i(q,0,0)=[0,z_i]\times[0,a_Tz_i^{-1}],$$

$$
\begin{aligned}
R_i(j,l)=R_i(q,j,l)&=R_i+(x_j(i),y_l(i))\\
&=\{(x,y):(x-x_j(i),y-y_l(i))\in R_i\}.
\end{aligned}
$$

令 $L_T^*(q)$ 表示包含在区域 $D_T(b_T)$ 中的矩形 $R_i(q,j,l)$ 组成的集合. 对任意的 $R=[x_1,x_2]\times[y_1,y_2]\in L_T$ 按如下的方式定义一个矩形 $R(q)\in L_T^*(q)$: 令 $i_0=i_0(R)$ 为使得 $z_{i_0}\ge x_2-x_1$ 成立的最小

整数，$j_0=j_0(R), l_0=l_0(R)$ 分别为使得 $x_{j_0}(i_0)\le x_1, y_{l_0}(i_0)\le y_1$ 成立的最大整数，并令

$$R(q)=R_{i_0}(q,j_0,l_0)=\big(x_{j_0}(i_0),y_{l_0}(i_0)\big)+[0,z_{i_0}]\times[0,a_T z_{i_0}^{-1}].$$

下述引理可在 Csörgő 和 Révész (1981) 中找到.

引理 2.3.1 我们有

$$\text{Card } L_T^*(q)\le 8Q^3Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2}), \quad (2.3.6)$$

$$\text{对每个 } R\in L_T^*,\quad \lambda\big(R\circ R(q)\big)\le 6a_TQ^{-1}, \quad (2.3.7)$$

$$\text{对每个 } R\in L_T^*(q),\quad \lambda(R)=a_T, \quad (2.3.8)$$

其中 λ 为 Lebesgue 测度，符号 $\circ$ 表示集合的对称差.

在定理 2.3.3 的证明中我们还要用到下述结果:

引理 2.3.2 对任何 $\varepsilon>0$, 存在常数 $C=C(\varepsilon)>0$, $u_0=u_0(\varepsilon)\ge 1$ 使得

$$\begin{aligned}P\Bigg\{&\sup_{\substack{x_0\le x\le T_1+x_0\\ y_0\le y\le T_2+y_0}}\ \sup_{\substack{0\le s\le h_1\\ 0\le t\le h_2}}|W([x,x+s]\times[y,y+t])|\le u(h_1h_2)^{1/2}\Bigg\}\\ &\overset{\text{KS}}{\ge}\exp\Big\{-C\Big(\frac{T_1}{h_1}+1\Big)\Big(\frac{T_2}{h_2}+1\Big)e^{-u^2/(2+\varepsilon)}\Big\}\end{aligned} \quad (2.3.9)$$

对任何 $x_0,y_0\ge 0$, $0<h_1\le T_1$, $0<h_2\le T_2$ 和 $u\ge u_0$ 成立.

特别地，对 $u\ge u_0$, $T_1,T_2>0$ 和 $x_0,y_0\ge 0$ 我们有

$$\begin{aligned}P&\Bigg\{\sup_{R\subset[x_0,x_0+T_1]\times[y_0,y_0+T_2]}|W(R)|\le u(T_1T_2)^{1/2}\Bigg\}\\ &\overset{\text{KS}}{\ge}\exp\{-ce^{-u^2/(2+\varepsilon)}\},\end{aligned} \quad (2.3.10)$$

其中 $R=[x_1,x_2]\times[y_1,y_2]$.

证明 不失一般性，不妨设 $x_0=y_0=0$. 对正实数 s 和正整数 r, 令 $s_r=h_1[s\frac{2^r}{h_1}]/2^r$. 记 $R=2^r$. 显然，对固定的 $\omega\in\Omega$ 和 x, y, s, r, t 我们有

$$
\begin{aligned}
&|W([x,x+s]\times[y,y+t])|\\
&\quad\le|W([x_r,(x+s)_r]\times[y,y+t])|\\
&\qquad+\sum_{j=0}^{\infty}|W([(x+s)_{r+j},(x+s)_{r+j+1}]\times[y,y+t])|\\
&\qquad+\sum_{j=0}^{\infty}|W([x_{r+j},x_{r+j+1}]\times[y,y+t])|.
\end{aligned}\tag{2.3.11}
$$

由引理 2.2.4, 对任何固定的 x, s 有

$$
\begin{aligned}
&P\left\{\sup_{0\le y\le T_2}\sup_{0\le t\le h_2}W([x,x+s]\times[y,y+t])|\le us^{1/2}h_2^{1/2}\right\}\\
&\overset{\mathrm{KS}}{\ge}\exp\left\{-C\left(\frac{T_2}{h_2}+1\right)u^7e^{-\frac{u^2}{2}}\right\}\quad(u\ge u_1\ge 1).
\end{aligned}\tag{2.3.12}
$$

从而，由 (2.3.11) 和引理 2.2.2 对任意的 $u\ge u_0$, $x_j\ge u_1$, $0<h_1\le T_1$ 和整数 r, j 我们有

$$
\begin{aligned}
&P\Big\{\sup_{\substack{0\le x\le T_1\\0\le y\le T_2}}\sup_{\substack{0\le s\le h_1\\0\le t\le h_2}}|W([x_r,(x+s)_r]\times[y,y+t])|\le uh_2^{1/2}\sqrt{h_1(1+1/R)}\Big\}\\
&\overset{\mathrm{KS}}{\ge}\exp\left\{-CR^2\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)u^7e^{-\frac{u^2}{2}}\right\},
\end{aligned}\tag{2.3.13}
$$

$$
\begin{aligned}
&P\Big\{\sup_{\substack{0\le x\le T_1\\0\le y\le T_2}}\sup_{\substack{0\le s\le h_1\\0\le t\le h_2}}|W([(x+s)_{r+j},(x+s)_{r+j+1}][y,y+t])|\\
&\qquad\le x_jh_2^{1/2}\frac{h_1^{1/2}}{\sqrt{2^{r+j+1}}}\Big\}\\
&\overset{\mathrm{KS}}{\ge}\exp\left\{-C2^{r+j+1}\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)x_j^7e^{-\frac{x_j^2}{2}}\right\},
\end{aligned}\tag{2.3.14}
$$

和

$$P\left\{\sup_{\substack{0\le x\le T_1\\0\le y\le T_2}}\sup_{\substack{0\le s\le h_1\\0\le t\le h_2}}|W([x_{r+j},x_{r+j+1}]\times[y,y+t])|\le x_j h_2^{1/2}\frac{h_1^{1/2}}{\sqrt{2^{r+j+1}}}\right\}$$

$$\overset{\text{KS}}{\ge}\exp\Big\{-C2^{r+j+1}\Big(\frac{T_1}{h_1}+1\Big)\Big(\frac{T_2}{h_2}+1\Big)x_j^7e^{-\frac{x_j^2}{2}}\Big\}. \qquad (2.3.15)$$

由 (2.3.13), (2.3.14) 和 (2.3.15), 再次运用引理 2.2.2 得

$$P\Big\{\sup_{\substack{0\le x\le T_1\\0\le y\le T_2}}\sup_{\substack{0\le s\le h_1\\0\le t\le h_2}}|W([x,x+s]\times[y,y+t])|$$

$$\le (h_1h_2)^{1/2}(u\sqrt{1+1/R}+2\sum_{j=0}^{\infty}\frac{x_j}{\sqrt{2^{r+j+1}}})\Big\}$$

$$\overset{\text{KS}}{\ge}\exp\Big\{-CR^2\Big(\frac{T_1}{h_1}+1\Big)\Big(\frac{T_2}{h_2}+1\Big)u^7e^{-\frac{u^2}{2}}$$

$$-8CR\Big(\frac{T_1}{h_1}+1\Big)\Big(\frac{T_2}{h_2}+1\Big)\sum_{j=0}^{\infty}2^jx_j^7e^{-\frac{x_j^2}{2}}\Big\}. \qquad (2.3.16)$$

令 $x_j=\sqrt{2j+u^2}$. 取 R 充分大得

$$u\sqrt{1+\frac{1}{R}}+2\sum_{j=0}^{\infty}\frac{x_j^2}{\sqrt{2^{r+j+1}}}$$

$$\le u\Big(1+\Big(\frac{1}{R}\Big)^{1/2}\Big)+2\Big(\frac{1}{R}\Big)^{1/2}\sum_{j=0}^{\infty}\frac{\sqrt{4ju^2}}{\sqrt{2^{j+1}}}$$

$$\le u\Big(1+\Big(\frac{1}{R}\Big)^{1/2}+2\Big(\frac{1}{R}\Big)^{1/2}\sum_{j=0}^{\infty}\frac{\sqrt{2j}}{\sqrt{2^j}}\Big)$$

$$\le u\Big(1+A\Big(\frac{1}{R}\Big)^{1/2}\Big)\le u\Big(1+\frac{\varepsilon}{2}\Big)^{1/2},$$

其中 $A=1+2\sum_{j=0}^{\infty}\sqrt{2j/2^j}$,

$$\sum_{j=0}^{\infty}2^jx_j^7e^{-\frac{x_j^2}{2}}\le e^{-\frac{u^2}{2}}\sum_{j=0}^{\infty}(4ju^2)^{7/2}(2/e)^j=u^7e^{-\frac{u^2}{2}}B,$$

这里 $B=\sum_{j=0}^{\infty}(4j)^{7/2}(2/e)^j$. 从而取 $x=u(1+\varepsilon/2)^{1/2}$ 和 $u_0=2u_1$ 即得 (2.3.9).

定理 2.3.3 的证明

对任意的 $R\in L_T$, 对称差 $R(q)\circ R(q+1)$ 至多是四个矩形的和, 记 $R(q)\circ R(q+1)=R^{(1)}(q)+R^{(2)}(q)+R^{(3)}(q)+R^{(4)}(q)$. 这类矩形 $R^{(i)}(q)$ $(i=1,2,3,4)$ 的全体记为 $\tilde{L}_T^*(q)$. 因为对任何 L_T^* 中的 R, 当 $q\to\infty$ 时 $R(q)\to R$, 我们有

$$\sup_{R\in L_T}|W(R)|\le\sup_{R\in L_T^*(q)}\sup_{S\subset R}|W(S)|+4\sum_{i=0}^{\infty}\sup_{R\in\tilde{L}_T^*}\sup_{S\subset R}|W(S)|, \tag{2.3.17}$$

其中 S 为边平行于坐标轴的矩形.

由引理 2.2.4, 2.3.1 和 2.3.2, 对任何 $\delta>0$, 存在常数 $C_\delta>0$, $x_\delta\ge1$ 使得对任何 $x\ge x_\delta$, $y_i\ge x_\delta$ 成立

$$P\left\{\sup_{R\in L_T^*(q)}\sup_{S\subset R}|W(S)|\le xa_T^{1/2}\right\}\overset{\mathrm{KS}}{\ge}\exp\left\{-C_\delta\mathrm{Card}L_T^*(q)e^{-x^2/(2+\delta)}\right\}, \tag{2.3.18}$$

$$P\left\{\sup_{R\in\tilde{L}_T^*(q+i)}\sup_{S\subset R}|W(S)|\le y_i(6a_TQ^{-1}2^{-i})^{1/2}\right\}\overset{\mathrm{KS}}{\ge}\exp\left\{-C_\delta\mathrm{Card}\tilde{L}_T^*(q+i)e^{-y_i^2/(2+\delta)}\right\}. \tag{2.3.19}$$

注意到 $\mathrm{Card}\tilde{L}_T^*(q+i)\le4\mathrm{Card}L_T^*(q+i)$, 由 (2.3.17), (2.3.18), (2.3.19) 并再次应用引理 2.2.2 得

$$\begin{aligned}P\Big\{\sup_{R\in L_T}|W(R)|\le xa_T^{1/2}+4\sum_{j=0}^{\infty}y_i(6a_TQ^{-1}2^{-i})^{1/2}\Big\}&\\ \overset{\mathrm{KS}}{\ge}\exp\Big\{-C_\delta\Big(\mathrm{Card}L_T^*(q)e^{-x^2/(2+\delta)}&\\ +4\sum_{i=0}^{\infty}\mathrm{Card}L_T^*(q+i)e^{-y_i^2/(2+\delta)}\Big)\Big\}.&\end{aligned} \tag{2.3.20}$$

取 $y_i = (3i(2+\delta)+x^2)^{1/2}$ $(i=0,1,\cdots)$, 对充分大的 Q 和任意的 $x \geq 1$ 我们有

$$\begin{aligned}
&xa_T^{1/2} + 4\sum_{j=0}^{\infty} y_i(6a_T Q^{-1}2^{-i})^{1/2} \\
&\leq xa_T^{1/2}\Big\{1 + 4(6Q^{-1})^{1/2}\sum_{i=0}^{\infty} 2^{-i/2}\Big\} \\
&\quad + 32a_T^{1/2}Q^{-1/2}\sum_{i=0}^{\infty}(i2^{-i})^{1/2} \\
&\leq xa_T^{1/2}(1+Q^{-1/2}A) + a_T^{1/2}Q^{-1/2}B \\
&\leq (1+\delta)xa_T^{1/2},
\end{aligned} \tag{2.3.21}$$

其中 $A = 4\sqrt{6}\sum_{i=0}^{\infty} 2^{-i/2}$, $B = 32\sum_{i=0}^{\infty}(i2^{-i})^{1/2}$; 进一步, 由引理 2.3.1 得

$$\begin{aligned}
&\mathrm{Card}L_T^*(q)e^{-x^2/(2+\delta)} + 4\sum_{i=0}^{\infty}\mathrm{Card}L_T^*(q+i)e^{-y_i^2/(2+\delta)} \\
&\leq CTa_T^{-1}(1+\log Ta_T^{-1})(1+\log b_T a_T^{-1/2})e^{-x^2/(2+\delta)}.
\end{aligned} \tag{2.3.22}$$

给定 $u \geq (1+\delta)x_\delta$, 令 $(1+\delta)x = u$, 由 (2.3.10), (2.3.21) 和 (2.3.22) 得

$$\begin{aligned}
&P\Big\{\sup_{R\in L_T}|W(R)| \leq ua_T^{1/2}\Big\} \\
&\overset{\mathrm{KS}}{\geq} \exp\Big(-C_\delta Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_T a_T^{-1/2})e^{-\frac{u^2}{(2+\delta)(1+\delta)}}\Big).
\end{aligned}$$

从而定理 2.3.3 得证.

定理 2.3.2 的证明

我们分两步进行.

第一步　若

(iv) $\Delta_T = \frac{Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_T a_T^{-1/2})}{\log\log T} \to \infty (T\to\infty)$,

则

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\delta}_T|W(R)| \leq 1 \quad \text{a.s.} \tag{2.3.23}$$

其中 $\bar{\delta}_T = \{2a_T \log \Delta_T\}^{-1/2}$.

证明 若

$$\limsup_{T\to\infty} \frac{\log Ta_T^{-1} + \log(\log Ta_T^{-1}+1) + \log(\log b_T a_T^{-1/2}+1)}{\log\log\log T} = \infty, \tag{2.3.24}$$

则存在序列 $\{T_N\}$ 满足 $T_N \uparrow \infty$ 使得

$$\limsup_{N\to\infty} \frac{\log T_N a_{T_N}^{-1} + \log(\log T_N a_{T_N}^{-1}+1) + \log(\log b_{T_N} a_{T_N}^{-1/2}+1)}{\log\log\log T_N} = \infty. \tag{2.3.25}$$

由定理 2.3.3, 有

$$\begin{aligned} P\Big\{ \sup_{R\in L_{T_N}} \bar{\delta}_{T_N} |W(R)| \geq 1+\varepsilon \Big\} &\leq c T_N a_{T_N}^{-1} (1+\log T_N a_{T_N}^{-1})(\log b_{T_N} a_{T_N}^{-1/2}+1) \\ &\quad \cdot \exp\Big(-\frac{2(1+\varepsilon)^2}{2+\varepsilon} \log \Delta_{T_N} \Big) \\ &\leq c\{T_N a_{T_N}^{-1}(1+\log T_N a_{T_N}^{-1})(\log b_{T_N} a_{T_N}^{-1/2}+1)\}^{-\varepsilon'} \\ &\quad \cdot (\log\log T_N)^{1+\varepsilon'} \\ &\to 0 \quad (N\to\infty), \end{aligned}$$

其中 $\varepsilon' = \frac{2(1+\varepsilon)^2}{2+\varepsilon} - 1 > 0$. 从而

$$\liminf_{T\to\infty} \sup_{R\in L_T} \bar{\delta}_T |W(R)| \leq \liminf_{N\to\infty} \sup_{R\in L_{T_N}} \bar{\delta}_{T_N} |W(R)| \leq 1+\varepsilon \quad \text{a.s.}$$

现设

$$\limsup_{T\to\infty} \frac{\log Ta_T^{-1} + \log(\log Ta_T^{-1}+1) + \log(\log b_T a_T^{-1/2}+1)}{\log\log\log T} < \infty.$$

不妨设对某个 $0 < r_0 < \infty$ 有

$$\begin{aligned} &Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_T a_T^{-1/2}) \\ &\quad \leq (\log\log T)^{r_0} \quad (T>0). \end{aligned} \tag{2.3.26}$$

令 $T_n = e^{n^p}$ $(p>1)$, $D'_{T_{n+1}} = D_{T_{n+1}} \cap D^c_{T_n}$, $D''_{T_{n+1}} = \{(x,y)$:

$0 \le x, y \le b_{T_{n+1}}, xy \le 2T_n\}$. 令 l_n 为介于双曲线 $xy = T_n$ 和 $xy = 2T_n$ 之间的一条折线, 其边平行于坐标轴, 顶点在 $xy = T_n$ 和 $xy = 2T_n$ 上, 其中一个顶点为 $(\sqrt{2T_n}, \sqrt{2T_n})$. 这条折线 l_n 把整个平面分成上、下两部分. 记上部为 U_n, 下部为 V_n. 对任意的 $R = [x_1, x_2] \times [y_1, y_2] \subset D_{T_{n+1}}$, 我们有 $R \cap U_n \subset D'_{T_{n+1}}$, $R \cap V_n \subset D''_{T_{n+1}}$, 并且存在内部互不相交的矩形 $R_1, \cdots, R_k \subset D'_{T_{n+1}}$, $\tilde{R}_1, \cdots, \tilde{R}_{\tilde{k}} \subset D''_{T_{n+1}}$ 使得 $R \cap U_n = \cup_{i=1}^{k} R_i$, $R \cap V_n = \cup_{i=1}^{\tilde{k}} \tilde{R}_i$. 我们定义 $W(R \cap U_n) = \sum_{i=1}^{k} W(R_i)$ 和 $W(R \cap V_n) = \sum_{i=1}^{\tilde{k}} W(\tilde{R}_i)$. 令

$$L'_{T_{n+1}} = \{R \cap U_n : R = [x_1, x_2] \times [y_1, y_2] \subset D_{T_{n+1}}, \lambda(R) \le a_{T_{n+1}}\}.$$

显然, $\{W(S); S \in L'_{T_{n+1}}\}_{n=1}^{\infty}$ 相互独立.

现对任意的 $R = [x_1, x_2] \times [y_1, y_2] \subset D_{T_{n+1}}$, 令 $M_n(R)$ 为 $R \cap V_n$ 的顶点数. 若 $R \subset D'_{T_{n+1}}$ 或 $R \subset D''_{T_{n+1}}$, 则 $M_n(R) \le 6$. 若 $(x_1, y_1) \in D_{T_n}$, 设 $(u_1, v_1), \cdots, (u_k, v_k)$ 为 l_n 的在双曲线 $xy = T_n$ 上且包含在 R 中的所有顶点, 并且 $u_1 < \cdots < u_k$. 则 $v_1 = T_n/u_1$, $u_k = u_1 2^{k-1}$, $v_k = T_n/u_1 2^{-(k-1)}$. 因为 $(u_k - u_1)(v_1 - v_k) \le \lambda(R)$, 即 $2^{k-1}(1 - 2^{-k+1})^2 T_n \le \lambda(R)$, 所以我们有 $k \le \frac{1}{\log 2} \log \frac{\lambda(R)}{T_n} + 2$ (若 $k \ge 3$). 从而 $M_n(R) \le 2(k+4) \le 4\log(1 + \lambda(R)/T_n) + 12$. 因此在任何情形我们都有 $M_n(R) \le 4\log(1 + \lambda(R)/T_n) + 12$. 注意到 $W(R) = W(R \cap U_n) + W(R \cap V_n)$, 我们有

$$\begin{aligned}
&\sup_{R \in L_{T_{n+1}}} |W(R)| \\
&\quad \le \sup_{S \in L'_{T_{n+1}}} |W(S)| + \left\{4\log\left(\frac{a_{T_{n+1}}}{T_n} + 1\right) + 12\right\} \\
&\qquad \cdot \sup_{(x,y) \in D''_{T_{n+1}}} |W(x,y)|, \\
&\sup_{S \in L'_{T_{n+1}}} |W(S)| \\
&\quad \le \sup_{R \in L_{T_{n+1}}} |W(R)| + \left\{4\log\left(\frac{a_{T_{n+1}}}{T_n} + 1\right) + 12\right\} \\
&\qquad \cdot \sup_{(x,y) \in D''_{T_{n+1}}} |W(x,y)|.
\end{aligned} \tag{2.3.27}$$

由定理 2.3.3, 对充分大的 n 有

$$
\begin{aligned}
J''_{n+1} :=& P\Big\{\Big\{4\log\Big(\frac{a_{T_{n+1}}}{T_n}+1\Big)+12\Big\} \\
&\cdot \sup_{(x,y)\in D''_{T_{n+1}}} \tilde{\delta}_{T_{n+1}}|W(x,y)|>\varepsilon\Big\} \\
\le& P\left\{\sup_{(x,y)\in D''_{T_{n+1}}} |W(x,y)|>2(2T_n)^{1/2}a^{1/2}_{T_{n+1}}/(n^p T_n^{1/2})\right\} \\
\le& c\big(1+\log(b_{T_{n+1}}(2T_n)^{-1/2})\big)\exp\big\{-a_{T_{n+1}}/(n^{2p}T_n)\big\} \\
\le& c\big(1+\log b_{T_{n+1}}a^{-1/2}_{T_{n+1}}+\log a^{1/2}_{T_{n+1}}(2T_n)^{-1/2}\big) \\
&\cdot\exp\Big\{-\frac{T_{n+1}}{T_n n^{2p}}\Big/(\log\log T_{n+1})^{r_0}\Big\} \\
\le& c\big((\log\log T_{n+1})^{r_0}+\log T_{n+1}\big)\exp\big\{-e^{(n+1)^p-n^p}/(n^{2p}(n+1)^{r_0})\big\} \\
\le& cn^pe^{-n}.
\end{aligned}
\tag{2.3.28}
$$

从而由 Borel-Cantelli 引理得

$$
\begin{aligned}
&\limsup_{n\to+\infty}\Big\{4\log\Big(\frac{a_{T_{n+1}}}{T_n}+1\Big)+12\Big\}\sup_{(x,y)\in D''_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(x,y)| \\
&\quad\le\varepsilon \quad \text{a.s.}
\end{aligned}
\tag{2.3.29}
$$

由定理 2.3.3 并注意到 (iv), 对充分大的 n 有

$$
\begin{aligned}
J_{n+1} :=& P\left\{\sup_{R\in L_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(R)|\le 1+\varepsilon\right\} \\
&\ge\exp\Big\{-c\Delta_{T_{n+1}}\log\log T_{n+1}\cdot\exp\Big\{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\log\Delta_{T_{n+1}}\Big\}\Big\} \\
&=\exp\{-c\Delta^{-\varepsilon'}_{T_{n+1}}\log\log T_{n+1}\}\ge(n+1)^{1/2},
\end{aligned}
\tag{2.3.30}
$$

其中 $\varepsilon'=\frac{2(1+\varepsilon)^2}{2+\varepsilon}-1>0$. 由 (2.3.28), (2.3.30) 和 (2.3.27), 对充分大的 n 有

$$
\begin{aligned}
J'_{n+1} :=& P\Big\{\sup_{S\in L'_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(S)|\le 1+2\varepsilon\Big\} \\
&\ge J_{n+1}-J''_{n+1}\ge(n+1)^{1/2}-cn^pe^{-n}.
\end{aligned}
\tag{2.3.31}
$$

从而 $\sum_{n=1}^{\infty} J'_{n+1} = \infty$. 由 Borel-Cantelli 引理和 $\{W(S); S \in L'_{T_{n+1}}\}_{n=1}^{\infty}$ 的独立性得

$$\liminf_{n\to\infty} \sup_{S\in L'_{T_{n+1}}} \tilde{\delta}_{T_{n+1}}|W(S)| \le 1+2\varepsilon \quad \text{a.s.} \tag{2.3.32}$$

由 (2.3.27), (2.3.29) 和 (2.3.32), 得

$$\liminf_{T\to\infty} \sup_{R\in L_T} \tilde{\delta}_T|W(R)| \le \liminf_{n\to\infty} \sup_{R\in L_{T_{n+1}}} \tilde{\delta}_{T_{n+1}}|W(R)| \le 1+3\varepsilon \quad \text{a.s.}$$

(2.3.23) 得证.

第二步 设条件 (i) 满足, 且

(ii$''$) $\tilde{\gamma}_T := \{2a_T(\log Ta_T^{-1}+\log(1+\log b_T^2T^{-1})-\log\log\log T)\}^{-1/2}$ 是 T 的正则非增的函数.

令 $\rho = \lim_{T\to\infty} a_T/T$. 若

(iv)$'$ $\tilde{\Delta}_T = \frac{Ta_T^{-1}(\log b_T^2T^{-1}+1)}{\log\log T} \to \infty \quad (T\to\infty)$,

则

$$\liminf_{T\to\infty} \sup_{R\in L_T} \tilde{\gamma}_T W(R) \ge 1 \quad \text{a.s.} \tag{2.3.33}$$

进一步, 若 $\rho<1$ 或 (iii) 成立, 则

$$\liminf_{T\to\infty} \sup_{R\in L_T^*} \tilde{\delta}_T W(R) \ge 1 \quad \text{a.s.} \tag{2.3.34}$$

证明 令 $L = L(T)$ 为使得下式成立的最大整数

$$\frac{T^{L+1}}{(T-a_T)^L b_T} < b_T \quad 若 \quad \rho<1,$$

$$a_T^{1/2}M^{L+1} = T^{1/2}M^{L+1} < b_T \quad 若 \quad \rho=1 \quad (M>1).$$

定义矩形

$$S_i = S_i(T) = [x_1(i), x_2(i)] \times [y_1(i), y_2(i)]$$

$$= \begin{cases} \left[\left(\dfrac{T-a_T}{T}\right)^{i+1} b_T, \left(\dfrac{T-a_T}{T}\right)^{i} b_T\right] \times \left[0, \dfrac{T^{i+1}}{(T-a_T)^i b_T}\right], & 若\rho < 1, \\ [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}], & 若\rho = 1, \end{cases}$$

$i = 0, 1, \cdots, L$. 则 $S_i \subset D_T$ 且 $\lambda(S_i) = a_T$ $(\rho < 1)$, $\lambda(S_i) = a_T(1 - \frac{1}{M}) = T(1 - \frac{1}{M})$ $(\rho = 1)$, $i = 0, 1, \cdots, L$.

若 $\rho < 1$, 则 $L = L(T) \geq (\log b_T^2 T^{-1})/\log \frac{T}{T-a_T} \geq KTa_T^{-1} \log b_T^2 T^{-1}$. 从而

$$\begin{aligned} P\Big\{\sup_{R \in L_T^*} \tilde{\gamma}_T W(R) \leq 1 - \varepsilon\Big\} &\leq P\Big\{\tilde{\gamma}_T \sup_{0 \leq i \leq L} W(S_i(T)) \leq 1 - \varepsilon\Big\} \\ &\leq \{1 - \exp(-(1-\varepsilon)\log \tilde{\Delta}_T\}^{L+1} \leq \exp\{-c\tilde{\Delta}_T^{\varepsilon} \log\log T\}. \end{aligned} \tag{2.3.35}$$

若 $\rho = 1$, 则 $L \geq (\log b_T T^{-1/2})/\log M$. 令 $L_T^{**}(M) = \{R \subset D_T : \lambda(R) = (1 - \frac{1}{M})T\}$, $L_T'(M) = \{R \subset D_T : \lambda(R) \leq \frac{1}{M}T\}$. 对充分大的 M 有

$$\begin{aligned} P\Big\{\sup_{R \in L_T^{**}(M)} \tilde{\gamma}_T W(R) \leq 1 - \varepsilon\Big\} &\leq P\Big\{\tilde{\gamma}_T \sup_{0 \leq i \leq L} W(S_i(T) \leq 1 - \varepsilon\Big\} \\ &\leq \Big\{1 - \Phi\Big(-(1-\varepsilon)\Big(\frac{M}{M-1}\Big)^{1/2}(\log \tilde{\Delta}_T)^{1/2}\Big)\Big\}^{L+1} \\ &\leq \{1 - \exp(-(1-\varepsilon)\log \tilde{\Delta}_T\}^{L+1} \\ &\leq \exp\Big\{-c\frac{1}{\log M}\tilde{\Delta}_T^{\varepsilon} \log\log T\Big\}. \end{aligned} \tag{2.3.36}$$

令 $T_0 = 0$, $T_k = (1 + k^{-1/2})^k$ $(k = 1, 2, \cdots)$, 则 $T_k \uparrow \infty$, 并且对充分大的 k 我们有 $\log T_k = k\log(1 + k^{-1/2}) > k^{1/3}$. 注意到 $\tilde{\Delta}_T \to \infty$, 由 (2.3.35), (2.3.36) 和 Borel-Cantelli 引理得

$$\liminf_{k \to \infty} \sup_{R \in L_{T_k}^*} \tilde{\gamma}_{T_k} W(R) \geq 1 - \varepsilon \quad \text{a.s.} \quad (\rho < 1), \tag{2.3.37}$$

$$\liminf_{k \to \infty} \sup_{R \in L_{T_k}^{**}(M)} \tilde{\gamma}_{T_k} W(R) \geq 1 - \varepsilon \quad \text{a.s.} \quad (\rho = 1). \tag{2.3.38}$$

令

$$L_{T_k}(k)=\{R:R\subset D_{T_{n+1}},\lambda(R)\le a_{T_{k+1}}-a_{T_k}\}.$$

对任意的 $T>0$, 存在 k 使得 $T_k<T\le T_{k+1}$, 则

$$\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\ge \sup_{R\in L_{T_k}^*}\tilde{\gamma}_T W(R)-4\sup_{R\in L_{T_k}(k)}\tilde{\gamma}_T|W(R)|.$$

注意到 $\tilde{\gamma}_T$ 是正则非增的且 $T_k/T_{k+1}\to 1$, 我们有

$$\begin{aligned}&\liminf_{T\to\infty}\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\\ &\quad\ge \liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\tilde{\gamma}_{T_k}W(R)\\ &\qquad-4\limsup_{k\to\infty}\sup_{R\in L_{T_k}(k)}\tilde{\gamma}_{T_{k+1}}|W(R)|.\end{aligned}\tag{2.3.39}$$

注意到若 k 充分大, 则有 $a_{T_{k+1}}/(a_{T_{k+1}}-a_{T_k})>\sqrt{k}$, 我们得

$$\begin{aligned}&(a_{T_{k+1}}-a_{T_k})\Big\{\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}+\log\Big(1+\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\Big)\\ &\cdot\log\Big(1+\frac{b_{T_{k+1}}}{\sqrt{a_{T_{k+1}}-a_{T_k}}}\Big)\Big\}\Big/a_{T_{k+1}}\log\tilde{\Delta}_{T_{k+1}}\le c\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\\ &\times\frac{\log\tilde{\Delta}_{T_{k+1}}+\log\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log T_{k+1}}{\log\tilde{\Delta}_{T_{k+1}}}\to 0\quad(k\to\infty).\end{aligned}$$

由定理 2.3.3, 对充分大的 k 有

$$\begin{aligned}&P\Big\{\sup_{R\in L_{T_k}(k)}\tilde{\gamma}_{T_{k+1}}|W(R)|>\varepsilon\Big\}\\ &\le c\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\Big(1+\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\Big)\Big(1+\log\frac{b_{T_{k+1}}}{\sqrt{a_{T_{k+1}}-a_{T_k}}}\Big)\\ &\quad\cdot\exp\Big\{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\log\tilde{\Delta}_{T_{k+1}}\Big\}\\ &\le\exp\Big\{-\frac{3\varepsilon^2}{2+\epsilon}\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\log\tilde{\Delta}_{T_{k+1}}\Big\}\le\exp(-\sqrt{k}).\end{aligned}\tag{2.3.40}$$

从而由 Borel-Cantelli 引理得

$$\limsup_{k\to\infty}\sup_{R\in L_{T_k}(k)}\tilde{\gamma}_{T_{k+1}}|W(R)|=0\quad\text{a.s.}\tag{2.3.41}$$

综合 (2.3.37), (2.3.39) 和 (2.3.41) 得

$$\liminf_{T\to\infty}\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\geq\liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\tilde{\gamma}_{T_k}W(R)\geq 1-\varepsilon\quad\text{a.s.}\ (\rho<1).\tag{2.3.42}$$

类似地,

$$\begin{aligned}&\liminf_{T\to\infty}\sup_{R\in L_T^{**}(M)}\tilde{\gamma}_T W(R)\\&\quad\geq\liminf_{k\to\infty}\sup_{R\in L_{T_k}^{**}(M)}\tilde{\gamma}_{T_k}W(R)\geq 1-\varepsilon\quad\text{a.s.}\quad(\rho=1).\end{aligned}\tag{2.3.43}$$

若 $\rho<1$, 由 (2.3.42) 得

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\gamma}_T W(R)\geq\liminf_{T\to\infty}\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\geq 1\quad\text{a.s.}\tag{2.3.44}$$

若 $\rho=1$, 由 (2.3.43) 得

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\gamma}_T W(R)\geq\liminf_{T\to\infty}\sup_{R\in L_T^{**}}\tilde{\gamma}_T W(R)\geq 1-\varepsilon\quad\text{a.s.}\tag{2.3.45}$$

从而 (2.3.34) ($\rho<1$) 和 (2.3.33) 得证.

最后, 假设 $\rho=1$ 且 (iii) 成立. 易知

$$\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\geq\sup_{R\in L_T^{**}(M)}\tilde{\gamma}_T W(R)-4\sup_{R\in L_T'(M)}\tilde{\gamma}_T|W(R)|.\tag{2.3.46}$$

在 (2.3.36) 中, 取 $M=M_T=\exp\{\tilde{\Delta}_T^{\varepsilon/2}\}$, 则

$$P\Big\{\sup_{R\in L_T^{**}(M_T)}\tilde{\gamma}_T W(R)\leq 1-\varepsilon\Big\}\leq\exp\{-c\tilde{\Delta}_T^{\varepsilon/2}\log\log T\}.\tag{2.3.47}$$

由 (iii), 存在 $r>1$ 使得对充分大的 T 有

$$\log b_T^2T^{-1}+1\geq(\log\log T)^r.$$

由定理 2.3.3 对充分大的 T 有

$$
\begin{aligned}
&P\Big\{\sup_{R\in L'_T(M_T)}\tilde{\gamma}_T|W(R)|>\varepsilon\Big\}\\
&\quad\le cM_T(\log M_T+1)(\log b_T T^{-1/2}+\log M_T^{1/2}+1)\\
&\qquad\cdot\exp\Big\{-\frac{2\varepsilon^2}{2+\varepsilon}M_T\log\tilde{\Delta}_T\Big\}\\
&\quad\le cM_T(\log M_T+1)(\tilde{\Delta}_T^{\frac{r}{r-1}}+\log M_T)\exp\Big\{-\frac{2\varepsilon^2}{2+\varepsilon}M_T\log\tilde{\Delta}_T\Big\}\\
&\quad\le\exp(-M_T)\le\exp\{-\exp((\log\log T)^{\frac{\varepsilon}{2}(r-1)})\}\\
&\quad\le\exp\{-(\log\log T)^2\}.
\end{aligned}\tag{2.3.48}
$$

综合 (2.3.46)—(2.3.48) 得

$$
\begin{aligned}
&P\Big\{\sup_{R\in L_T^*}\tilde{\gamma}_T W(R)\le 1-5\varepsilon\Big\}\\
&\qquad\le\exp\{-c\tilde{\Delta}_T^{\varepsilon/2}\log\log T\}+\exp\{-(\log\log T)^2\}.
\end{aligned}\tag{2.3.49}
$$

注意到 $\tilde{\Delta}_T\to\infty$ 和 $\log T_k\ge k^{\frac{1}{3}}$, 由 (2.3.49) 和 Borel-Cantelli 引理得

$$
\liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\tilde{\gamma}_{T_k}W(R)\ge 1-5\varepsilon\quad\text{a.s.}\tag{2.3.50}
$$

由 (2.3.42) 和 (2.3.50) 得证当 (iii) 满足时, (2.3.34) 对 $\rho=1$ 也成立.

最后注意到若条件 (iii) 满足, 则

$$
\frac{\log(1+\log Ta_T^{-1})}{\log Ta_T^{-1}+\log(\log b_T a_T^{-1/2}+1)-\log\log\log T}\to 0\quad(T\to\infty),\tag{2.3.51}
$$

并且

$$
\log(1+\log b_T a_T^{-1/2})\le\log(1+\log b_T^2T^{-1})+\log(1+\log Ta_T^{-1}),
$$

我们知 $\gamma_T/\tilde{\gamma}_T\to 1$, $\gamma_T/\tilde{\delta}_T\to 1$, 且条件 (iv) 和 (iv′) 满足. 从而由第一步和第二步即得证定理 2.3.2.

推论 2.3.1 的证明 设条件 (iii′) 成立. 若 $r>1$, 则 (2.3.4) 由 (2.3.3) 即得. 若 $r=1$, 则用 $\log\log T$ 代替 Δ_T, (2.3.4) 的证明与定理 2.3.2 的第一步证明类似, 其中只有 (2.3.30) 要作如下修正.

$$
\begin{aligned}
&P\Big\{\sup_{R\in L_{T_{n+1}}}\{2a_{T_{n+1}}\log\log\log T_{n+1}\}^{1/2}|W(R)|\le\varepsilon\Big\}\\
&\ge\exp\Big\{-c\cdot\exp\Big(-\varepsilon'\log\log\log T_{n+1}\\
&\qquad+\frac{\log(\Delta_{T_{n+1}}\log\log T_{n+1})}{\log\log\log T_{n+1}}\log\log\log T_{n+1}\Big)\Big\}\\
&\ge\exp\Big\{-c\cdot\exp\Big(\Big(1-\frac{\varepsilon'}{2}\Big)\log\log\log T_{n+1}\Big)\Big\}\\
&\ge\exp\Big(-c(\log\log T_{n+1})^{-\varepsilon'/2}\log\log T_{n+1}\Big)\\
&\ge(n+1)^{1/2},
\end{aligned}
\tag{2.3.30′}
$$

其中 $\varepsilon'=\frac{2\varepsilon^2}{2+\varepsilon}>0$.

由定理 2.3.2 的证明我们得下述推论.

推论 2.3.2 设定理 2.3.2 中的条件 (i) 和 (ii) 满足. 若 (2.3.51) 成立, 则

$$\liminf_{T\to\infty}\sup_{R\in L_T}\gamma_T|W(R)|=1\quad\text{a.s.}$$

进一步, $\lim_{T\to\infty}a_T/T<1$ 或 (iii) 成立, 则

$$\liminf_{T\to\infty}\sup_{R\in L_T^*}\gamma_T W(R)=1\quad\text{a.s.}$$

猜测 我们猜测, 若第二步证明中的条件, (i), (ii″) 和 (iv′) 满足, 则

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\gamma}_T|W(R)|=\liminf_{T\to\infty}\sup_{R\in L_T^*}\tilde{\gamma}_T|W(R)|=1\quad\text{a.s.}$$

定理 2.3.1 和 2.3.2 可以推广到阶为 α $(0<\alpha<1)$ 的两参数分数 Wiener 过程, 并且 “a_T 和 T/a_T 是 T 的非降函数” 和 “δ_T 或 γ_T 是 T 的正则非增函数” 这两个条件可以去掉.

设 $\{Z(x,y);x,y\ge 0\}$ 是阶为 α $(0<\alpha<1)$ 的两参数分数 Wiener 过程，即它是一零均值的 Gauss 过程， $Z(0,0)=0$ a.s. 且协方差函数为

$$
\begin{aligned}
&\mathrm{E}Z(x_1,y_1)Z(x_2,y_2)\\
=&\left\{|x_1|^{2\alpha}+|x_2|^{2\alpha}-|x_2-x_1|^{2\alpha}\right\}\left\{|y_1|^{2\alpha}+|y_2|^{2\alpha}-|y_2-y_1|^{2\alpha}\right\}/4.
\end{aligned}
$$

显然，当 $\alpha=1/2$ 时， $Z(\cdot,\cdot)$ 即为两参数 Wiener 过程. 现重新定义 δ_T,γ_T 如下：

$$
\begin{aligned}
\delta_T=&\{2a_T^{2\alpha}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)+\log\log T)\}^{-1/2},\\
\gamma_T=&\{2a_T^{2\alpha}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)-\log\log\log T)\}^{-1/2}.
\end{aligned}
$$

陆传荣，张立新，王尧弘 (2001) 得到了如下大增量结果.

定理 2.3.4 设 $0<a_T\le T$ 和 $b_T\ge\sqrt{T}$ 都为 T 的函数. 假设 b_T 拟增. 则

$$
\limsup_{T\to\infty}\sup_{R\in L_T^*}\delta_TZ(R)=\limsup_{T\to\infty}\sup_{R\in L_T}\delta_T|Z(R)|=1\quad\text{a.s.}
$$

若还有定理 2.3.1 中的条件 (2.3.2) 成立，则

$$
\lim_{T\to\infty}\sup_{R\in L_T^*}\delta_TZ(R)=\lim_{T\to\infty}\sup_{R\in L_T}\delta_T|Z(R)|=1\quad\text{a.s.}
$$

定理 2.3.5 设 $0<a_T\le T$ 和 $b_T\ge\sqrt{T}$ 都为 T 的函数. 假设 b_T 拟增. 若定理 2.3.2 中的条件 (iii) 成立. 则

$$
\liminf_{T\to\infty}\sup_{R\in L_T^*}\gamma_TZ(R)=\liminf_{T\to\infty}\sup_{R\in L_T}\gamma_T|Z(R)|=1\quad\text{a.s.}
$$

定理 2.3.4 和 2.3.5 的证明比较繁琐，证明思路与定理 2.3.1, 2.3.2 和 2.2.4 的有些类似，故从略.

§ 2.4 两参数分数 Lévy-Wiener 过程

令 $\{X(x,y); 0 \le x, y < \infty\}$ 为一个阶为 α $(0 < \alpha < 1)$ 的两参数分数 Lévy-Wiener 过程, 即 $\{X(x,y); 0 \le x, y < \infty\}$ 为几乎处处连续的、零均值平稳实值 Gauss 过程, $X(0,0) = 0$ 且对任意非负的 x_1, y_1, x_2, y_2 成立

$$E\{X(x_1,y_1) - X(x_2,y_2)\}^2 = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{\alpha}. \quad (2.4.1)$$

当 $\alpha = 1/2$ 时, $\{X(x,y); 0 \le x, y < \infty\}$ 是两参数 Lévy-Wiener 过程, 即此时它是满足下述条件的 Gauss 过程:

(a) $X(0,0) = 0$ a.s.,

(b) 对任何 x 和 y, $X(x,y)$ 是零均值的正态变量,

(c) $E\{X(x_1,y_1) - X(x_2,y_2)\}^2 = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{1/2}$,

(d) 样本轨道 $(x,y) \mapsto X(\omega; x, y)$ 关于 (x,y) a.s. 连续.

考察矩形 $R := R(s,t,u,v) = [s, s+t] \times [u, u+v] \subset \mathcal{R}_+^2$ $(s, u \ge 0, t, v > 0)$, 定义两参数分数 Lévy-Wiener 过程在 R 上的增量 $X(R)$ 为

$$\begin{aligned} X(R) :=& X(R(s,t,u,v)) \\ =& X(s+t, u+v) - X(s, u+v) - X(s+t, u) + X(s,u). \end{aligned}$$

由 (2.4.1) 易知 $X(R)$ 的标准差具有关于 s 和 u 的平移不变性. 记

$$S(t,v) = \left\{E\big(X(R(s,t,u,v))\big)^2\right\}^{1/2}.$$

对 $0 < T < \infty$, 令 A_T 和 B_T 为非降连续函数, a_T 和 b_T 为连续函数, 满足 $0 < a_T \le A_T$ 和 $0 < b_T \le B_T$. 定义

$$G_T = \Big(\frac{A_T - a_T}{a_T} \vee 1\Big)\Big(\frac{B_T - b_T}{b_T} \vee 1\Big),$$

$$\beta_T = \left\{2(\log G_T + \log\log A_T + \log\log B_T)\right\}^{1/2},$$

和

$$\begin{aligned} &D_1(A_T, B_T, a_T, b_T) \\ &\quad = \sup_{0\le s\le A_T-a_T}\ \sup_{0\le t\le a_T}\ \sup_{0\le u\le B_T-b_T}\ \sup_{0\le v\le b_T} \frac{|X(R(s,t,u,v))|}{S(a_T,b_T)\beta_T}, \\ &D_2(A_T, B_T, a_T, b_T) = \sup_{0\le s\le A_T-a_T}\ \sup_{0\le u\le B_T-b_T} \frac{|X(R(s,a_T,u,b_T))|}{S(a_T,b_T)\beta_T}. \end{aligned}$$

Lin 和 Choi(1998) 证明了下述结果.

定理 2.4.1 设 $\{X(x,y); 0\le x,y<\infty\}$ 为阶为 α $(0<\alpha<1)$ 的两参数分数 Lévy- Wiener 过程. 对 $0<T<\infty$, 令 A_T 和 B_T 为非降连续函数, a_T 和 b_T 为连续函数, 满足

(i) $0<a_T\le A_T$, $0<b_T\le B_T$,

(ii) 当 A_T 有界时, a_T 趋于零, 否则, $\liminf_{T\to\infty} a_T>0$; B_T 与 b_T 也是如此,

(iii) 对某 $0<c_1\le c_2<\infty$ 成立

$$c_1 \le \liminf_{T\to\infty}\frac{a_T}{b_T} \le \limsup_{T\to\infty}\frac{a_T}{b_T} \le c_2.$$

则我们有

$$\limsup_{T\to\infty} D_1(A_T,B_T,a_T,b_T)\le 1 \quad \text{a.s.} \tag{2.4.2}$$

如果进一步还满足下述条件:

(iv) $\lim_{T\to\infty}\log G_T/(\log\log A_T+\log\log B_T)=\infty$,

则我们有

$$\liminf_{T\to\infty} D_2(A_T,B_T,a_T,b_T)\ge 1 \quad \text{a.s.} \tag{2.4.3}$$

因此, 在条件 (i)—(iv) 下我们有

$$\lim_{T\to\infty} D_1(A_T,B_T,a_T,b_T) = \lim_{T\to\infty} D_2(A_T,B_T,a_T,b_T) = 1 \quad \text{a.s.}$$

注 2.4.1 由上述定理，当 A_T, B_T, a_T, b_T 都趋于无穷时，我们就得到了有关 $\{X(x,y)\}$ 的大增量的结果；当 A_T，B_T 有界且 a_T, b_T 趋于零时，我们就得到了 $\{X(x,y)\}$ 的连续模.

例 2.4.1 设 $\{X(x,y); 0 \le x, y < \infty\}$ 为两参数 Lévy-Wiener 过程，这时 $\alpha = 1/2$. 当 $A_T = T$，$B_T = T^{3/2}$，$a_T = \sqrt{T}$，$b_T = \sqrt{T}/2$ 时，条件 (i)—(iv) 满足，且 $\beta_T \sim (3\log T)^{1/2}$，$S(a_T, b_T) = \sqrt{3-\sqrt{5}}T^{1/4}$ (参见下文的 (2.4.15)). 从而我们有大增量结果：

$$
\begin{aligned}
&\lim_{T\to\infty} \sup_{0\le s\le T-\sqrt{T}} \sup_{0\le t\le\sqrt{T}} \sup_{0\le u\le T^{3/2}-\sqrt{T}/2} \sup_{0\le v\le\sqrt{T}/2} \frac{|X(R(s,t,u,v))|}{T^{1/4}(\log T)^{1/2}} \\
&\quad = \lim_{T\to\infty} \sup_{0\le s\le T-\sqrt{T}} \sup_{0\le u\le T^{3/2}-\sqrt{T}/2} \frac{|X(R(s,\sqrt{T},u,\sqrt{T}/2))|}{T^{1/4}(\log T)^{1/2}} \\
&\quad = \sqrt{3(3-\sqrt{5})} \quad \text{a.s.}
\end{aligned}
$$

当 $A_T = B_T = 1$，$a_T = 1/T$，$b_T = 1/2T$ 时，条件 (i)—(iv) 也满足，且 $\beta_T \sim 2(\log T)^{1/2}$，$S(a_T, b_T) = \sqrt{3-\sqrt{5}}T^{-1/2}$. 从而我们有连续模结果：

$$
\begin{aligned}
&\lim_{T\to\infty} \sup_{0\le s\le 1-1/T} \sup_{0\le t\le 1/T} \sup_{0\le u\le 1-1/2T} \sup_{0\le v\le 1/2T} \frac{|X(R(s,t,u,v))|}{T^{-1/2}(\log T)^{1/2}} \\
&\quad = \lim_{T\to\infty} \sup_{0\le s\le 1-1/T} \sup_{0\le u\le 1-1/2T} \frac{|X(R(s,T^{-1},u,T^{-1}/2))|}{T^{-1/2}(\log T)^{1/2}} \\
&\quad = 2\sqrt{3-\sqrt{5}} \quad \text{a.s.}
\end{aligned}
$$

下述引理 2.4.1—2.4.4 是证明定理 2.4.1 的关键. 令 $\mathcal{D} = \{\mathbf{t}; \mathbf{t} = (t_1, \cdots, t_d), a_i \le t_i \le b_i, i = 1, 2, \cdots, d\}$ 为实值的 d 维时间参数空间，$\|\cdot\|$ 为其上通常意义下的 Euclidean 范数. 设 $\{X(\mathbf{t}); t \in \mathcal{D}\}$ 为零均值的实值可分 Gauss 过程. 假设

$$0 < \sup_{\mathbf{t}\in\mathcal{D}} E\{X(\mathbf{t})\}^2 =: \Gamma^2 < \infty, \tag{2.4.4}$$

和

$$E\{X(\mathbf{t})-X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t}-\mathbf{s}\|), \tag{2.4.5}$$

其中 $\varphi(\cdot)$ 为非降连续函数，满足 $\int_0^\infty \varphi(e^{-y^2})\,dy<\infty$.

下述引理是 Fernique 不等式 (定理 1.1.3) 的一个类比 (参见 Choi 和 Lin 1998).

引理 2.4.1 设 $\{X(\mathbf{t});\mathbf{t}\in\mathcal{D}\}$ 为上述所示. 则对 $\lambda>0$, $x\ge 1$ 和 $A>\sqrt{2d\log 2}$, 有

$$\begin{aligned}P\Big\{\sup_{\mathbf{t}\in\mathcal{D}} X(\mathbf{t}) \ge x\Big(\Gamma+(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{d}\lambda 2^{-y^2})\,dy\Big)\Big\} \\ \le (2^d+B)\Big(\prod_{i=1}^d\big(\frac{b_i-a_i}{\lambda}\vee 1\big)\Big)e^{-x^2/2},\end{aligned}$$

其中 $B=\sum_{n=1}^\infty \exp\big\{-2^{n-1}(A^2-2d\log 2)\big\}<\infty$.

证明 对每个 $n=0,1,2,\cdots$, 记 $\epsilon_n=\lambda 2^{-2^n}$, $\lambda>0$. 对 $\mathbf{k}=(k_1,\cdots,k_d)$, 其中 $k_i=0,1,\cdots,k_{in}:=[(b_i-a_i)/\epsilon_n]$, $i=1,\cdots,d$, 定义 $\mathbf{t}_{\mathbf{k}}^{(n)}=(t_{1k_1}^{(n)},\cdots,t_{dk_d}^{(n)})\in\mathcal{D}$, 其中

$$t_{ik_i}^{(n)}=a_i+k_i\epsilon_n,\ i=1,\cdots,d.$$

令

$$S_n=\{\mathbf{t}_{\mathbf{k}}^{(n)};\mathbf{k}=\mathbf{0},\cdots,\mathbf{k}_n:=(k_{1n},\cdots,k_{dn})\},$$

它包含了 $N_n:=\prod_{i=1}^d k_{in}$ 个点且 $N_n\le 2^{2^n d}\prod_{i=1}^d(b_i-a_i)/\lambda$. 那么，集合 $\bigcup_{n=0}^\infty S_n$ 在 $\mathcal{D}$ 中稠密且 $S_n\subset S_{n+1}$. 对 $x\ge 1$ 和 $A>\sqrt{2d\log 2}$, 令

$$x_m=xA\varphi(\sqrt{d}\epsilon_{m-1})2^{m/2},\ m\ge 1.$$

对 $m\ge 1$, 令 $\delta_m=2^{(m-1)/2}$. 则

$$2^{m/2}=2(\sqrt{2}+1)(\delta_m-\delta_{m-1}).$$

从而

$$
\begin{aligned}
\sum_{m=1}^{\infty} x_m =& xA \sum_{m=1}^{\infty} \varphi\left(\sqrt{d}\lambda 2^{-2^{m-1}}\right) 2^{m/2} \\
=& xA \sum_{m=1}^{\infty} \varphi\left(\sqrt{d}\lambda 2^{-\delta_m^2}\right)(2\sqrt{2}+2)(\delta_m - \delta_{m-1}) \\
\leq& (2\sqrt{2}+2) xA \sum_{m=1}^{\infty} \int_{\delta_{m-1}}^{\delta_m} \varphi\left(\sqrt{d}\lambda 2^{-y^2}\right) dy \\
\leq& (2\sqrt{2}+2) xA \int_0^{\infty} \varphi\left(\sqrt{d}\lambda 2^{-y^2}\right) dy. \qquad (2.4.6)
\end{aligned}
$$

因此，由 (2.4.4) 得

$$
\begin{aligned}
&P\left\{\sup_{\mathbf{t}\in\mathcal{D}} X(\mathbf{t}) > x\left(\Gamma + (2\sqrt{2}+2)A \int_0^{\infty} \varphi\left(\sqrt{d}\lambda 2^{-y^2}\right) dy\right\}\right. \\
&\quad \leq P\left\{\sup_{\mathbf{t}\in\mathcal{D}} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\right\} \\
&\quad = P\left\{\max_{n\geq 0} \sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\right\} \\
&\quad \leq \lim_{n\to\infty} P\left\{\sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\right\}. \qquad (2.4.7)
\end{aligned}
$$

对 $n \geq 0$, 令

$$
A_n = \left\{\sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\right\}.
$$

由归纳法得

$$
\begin{aligned}
P(A_n) =& P(A_n \cap A_{n-1}) + P(A_n \cap A_{n-1}^c) \\
\leq& P(A_{n-1}) + P(A_n \cap A_{n-1}^c) \\
\leq& P(A_{n-2}) + P(A_{n-1} \cap A_{n-2}^c) + P(A_n \cap A_{n-1}^c)
\end{aligned}
$$

$$\le P(A_0)+\sum_{n=1}^{\infty}P(A_n\cap A_{n-1}^c)$$
$$\le P(B_0)+\sum_{n=1}^{\infty}P(B_n\cap B_{n-1}^c), \tag{2.4.8}$$

其中

$$B_0=\left\{\sup_{\mathbf{t}\in S_0}X(\mathbf{t})\ge x\Gamma\right\},\ B_n=\left\{\sup_{\mathbf{t}\in S_n}X(\mathbf{t})\ge\sum_{m=1}^{\infty}x_m\right\},\ n\ge 1.$$

现在对 $n\ge 1$, 我们有

$$\begin{aligned}
&P(B_n\cap B_{n-1}^c)\\
=&P\left\{\left\{\sup_{\mathbf{t}\in S_n}X(\mathbf{t})\ge\sum_{m=1}^{n}X_m\right\}\cap\left\{\sup_{\mathbf{s}\in S_{n-1}}X(\mathbf{s})<\sum_{m=1}^{n-1}X_m\right\}\right\}\\
\le&P\left\{\bigcup_{\mathbf{t}\in S_n}\left\{X(\mathbf{t})\ge\sum_{m=1}^{n}X_m\right\}\cap\bigcap_{\mathbf{s}\in S_{n-1}}\left\{X(\mathbf{s})<\sum_{m=1}^{n-1}x_m\right\}\right\}\\
\le&P\Bigg\{\bigcup_{\mathbf{t}\in S_n-S_{n-1}}\ \bigcup_{\substack{\mathbf{s}\in S_{n-1}\\ \|\mathbf{t}-\mathbf{s}\|\le\sqrt{d}\epsilon_{n-1}}}\{X(\mathbf{t})-X(\mathbf{s})\ge x_n\}\Bigg\}\\
\le&\sum_{\mathbf{t}\in S_n}\ \sum_{\substack{\mathbf{s}\in S_{n-1}\\ \|\mathbf{t}-\mathbf{s}\|\le\sqrt{d}\epsilon_{n-1}}}P\{X(\mathbf{t})-X(\mathbf{s})\ge x_n\}.
\end{aligned}\tag{2.4.9}$$

然而由假设 (2.4.5), 我们得

$$E\{X(\mathbf{t})-X(\mathbf{s})\}^2\le\varphi^2(\|\mathbf{t}-\mathbf{s}\|)\le\varphi^2(\sqrt{d}\epsilon_{n-1}),\ n\ge 1. \tag{2.4.10}$$

从而, 注意到 $A>\sqrt{2d\log 2}$, $x\ge 1$, 并且对任何 $\mathbf{t}\in S_n-S_{n-1}$, 仅有一个点 $\mathbf{s}\in\{\mathbf{s}\in S_{n-1}:\|\mathbf{t}-\mathbf{s}\|\le\sqrt{d}\epsilon_{n-1}\}$, 由 (2.4.10) 知 (2.4.9) 蕴涵了

$$P(B_n\cap B_{n-1}^c)\le\sum_{\mathbf{t}\in S_n}\ \sum_{\substack{\mathbf{s}\in S_{n-1}\\ \|\mathbf{t}-\mathbf{s}\|\le\sqrt{d}\epsilon_{n-1}}}P\left\{N(0,1)\ge\frac{x_n}{\varphi(\sqrt{d}\epsilon_{n-1})}\right\}$$

$$
\begin{aligned}
&\leq\Big(\prod_{i=1}^{d}\frac{b_i-a_i}{\lambda}\Big)(2^{2^n})^dP\left\{N(0,1)\geq\frac{x_n}{\varphi(\sqrt{d}\epsilon_{n-1})}\right\}\\
&=2^{2^nd}\Big(\prod_{i=1}^{d}\frac{b_i-a_i}{\lambda}\Big)P\Big\{N(0,1)\geq Ax2^{n/2}\Big\}\\
&\leq 2^{2^nd}\Big(\prod_{i=1}^{d}\frac{b_i-a_i}{\lambda}\Big)\frac{1}{2\sqrt{\pi}}e^{-A^2x^22^{n-1}}\\
&=\frac{1}{2\sqrt{\pi}}\Big(\prod_{i=1}^{d}\frac{b_i-a_i}{\lambda}\Big)2^{2^nd}e^{-(A^22^{n-1}-1/2)x^2}e^{-x^2/2}\\
&\leq\frac{1}{2\sqrt{\pi}}\Big(\prod_{i=1}^{d}\frac{b_i-a_i}{\lambda}\Big)e^{2^nd\log 2-2^{n-1}A^2+1/2}e^{-x^2/2}\\
&\leq e^{-2^n(A^2/2-d\log 2)}\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2},
\end{aligned}
$$

其中 $N(0,1)$ 表示标准正态变量. 特别地, 如果 $A>0$ 满足

$$
\frac{A^2}{2}-d\log 2>0,
$$

则

$$
\sum_{n=1}^{\infty}P(B_n\cap B_{n-1}^c)\leq B\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2}, \tag{2.4.11}
$$

其中

$$
B=\sum_{n=1}^{\infty}\exp\big\{-2^{n-1}(A^2-2d\log 2)\big\}<\infty.
$$

另一方面,

$$
P(B_0)=P\Big\{\sup_{\mathbf{s}\in S_0}X(\mathbf{t})\geq x\Gamma\Big\}
$$

$$\leq 2^d\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)P\{N(0,1)\geq x\}$$

$$\leq 2^d\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2}. \tag{2.4.12}$$

由 (2.4.8), (2.4.11) 和 (2.4.12) 得：对任何 $n\geq 0$,

$$P(A_n)\leq (2^d+B)\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2}. \tag{2.4.13}$$

从而由 (2.4.13) 和 (2.4.7) 得

$$P\Big\{\sup_{\mathbf{t}\in D}X(\mathbf{t})>x\big(\Gamma+2\sqrt{2}+2\big)A\int_0^\infty \varphi(\sqrt{d}\lambda 2^{-y^2})\,dy\big)\Big\}$$
$$\leq (2^d+B)\Big(\prod_{i=1}^{d}\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2}.$$

引理 2.4.1 得证.

引理 2.4.2 设 $p>0$, N,m 和 a 为非零实数. 则存在常数 $c_0>0$ 使得

$$\left|\int_{\sqrt{a^2+N^2m^2p^2}}^{\sqrt{a^2+(Nm+1)^2p^2}} d(x^{2\alpha})-\int_{\sqrt{a^2+(Nm-1)^2p^2}}^{\sqrt{a^2+N^2m^2p^2}} d(x^{2\alpha})\right|$$
$$\leq c_0\frac{\{a^2+(|Nm|+1)^2p^2\}^\alpha p^2}{a^2+(|Nm|-1)^2p^2}.$$

证明 记 $b=(|Nm|-1)p, c=|Nm|p$ 和 $d=(|Nm|+1)p$. 则

$$\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}} d(x^{2\alpha})-\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}} d(x^{2\alpha})$$
$$=\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} d((x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha})$$

$$
\begin{aligned}
&-\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}} d(x^{2\alpha})\\
=&\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\frac{d((x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha})}{dx}\\
&-\frac{d(x^{2\alpha})}{dx}\Big)\,dx+\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{d(x^{2\alpha})}{dx}\,dx\\
=&\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} \frac{d^2(y^{2\alpha})}{dy^2}dy\Big)\,dx\\
&+\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{d(x^{2\alpha})}{dx}\,dx\\
=:&\,I+J.
\end{aligned}
$$

我们首先估计 I 的上界. 事实上, 存在常数 $c_1>0$ 使得

$$
\begin{aligned}
|I|=&\Big|\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} (2\alpha(2\alpha\\
&-1)\frac{y^{2\alpha}}{y^2})dy\Big)\,dx\Big|\\
\le& c_1\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{(x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha}}{x^2}\\
&\cdot(\sqrt{a^2+c^2}-\sqrt{a^2+b^2})\,dx\\
\le& c_1\frac{(a^2+d^2)^\alpha}{a^2+b^2}(\sqrt{a^2+d^2}-\sqrt{a^2+c^2})(\sqrt{a^2+c^2}-\sqrt{a^2+b^2})\\
=&c_1\frac{(a^2+d^2)^\alpha(d^2-c^2)(c^2-b^2)}{(a^2+b^2)(\sqrt{a^2+d^2}+\sqrt{a^2+c^2})(\sqrt{a^2+c^2}+\sqrt{a^2+b^2})}\\
\le& c_1\frac{(a^2+d^2)^\alpha}{a^2+b^2}(d-c)(c-b)=c_1\frac{(a^2+(|Nm|+1)^2p^2)^\alpha p^2}{a^2+(|Nm|-1)^2p^2}.
\end{aligned}
$$

对 J, 存在常数 $c_2>0$ 使得

$$
J=\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(2\alpha\frac{x^{2\alpha}}{x}\Big)\,dx
$$

$$
\begin{aligned}
&\le c_2\frac{(\sqrt{a^2+d^2})^{2\alpha}}{\sqrt{a^2+c^2}}\left(\sqrt{a^2+d^2}-\sqrt{a^2+c^2}-(\sqrt{a^2+c^2}-\sqrt{a^2+b^2})\right)\\
&\le c_2\frac{(a^2+d^2)^{\alpha}}{\sqrt{a^2+c^2}}\left(\frac{d^2-c^2}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}}-\frac{c^2-b^2}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}}\right)\\
&=c_2\frac{(a^2+d^2)^{\alpha}}{\sqrt{a^2+c^2}}\left(\frac{2|Nm|+1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}}-\frac{2|Nm|-1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}}\right)p^2\\
&=c_2\frac{(a^2+d^2)^{\alpha}}{\sqrt{a^2+c^2}}\left\{\left(\frac{1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}}-\frac{1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}}\right)\right.\\
&\quad\left.\cdot(2|Nm|)p^2+\left(\frac{1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}}+\frac{1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}}\right)p^2\right\}\\
&\le c_2\frac{(a^2+d^2)^{\alpha}}{\sqrt{a^2+c^2}}\frac{2p^2}{\sqrt{a^2+c^2}}=2c_2\frac{(a^2+d^2)^{\alpha}p^2}{a^2+c^2}\\
&=2c_2\frac{\left(a^2+(|Nm|+1)^2p^2\right)^{\alpha}p^2}{a^2+N^2m^2p^2}.
\end{aligned}
$$

综合 I 和 J 的上界，得证引理 2.4.2.

下述引理可在 Leadbetter 等 (1983) 中找到.

引理 2.4.3 设 $\{\xi_{ij}; i,j=1,2,\cdots,n\}$ 为多元标准正态随机变量，其协方差 $\mathrm{Cov}(\xi_{ij},\xi_{i'j'})=\Lambda_{ij}^{i'j'}$ 满足

$$\delta:=\max_{(i,j)\neq(i'j')}|\Lambda_{ij}^{i'j'}|<1.$$

则对任何实数 u 和整数 $1\le l_1<l_2<\cdots<l_f\le n$, $1\le l_1'<l_2'<\cdots<l_g'\le n$, $f,g\le n$, 有

$$
\begin{aligned}
&P\left\{\max_{1\le i\le f}\max_{1\le j\le g}\xi_{l_i l_j'}\le u\right\}\\
&\qquad\le\left\{\Phi(u)\right\}^{fg}+c\sum_{(i,j)\neq(i',j')}|\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right), \quad (2.4.14)
\end{aligned}
$$

其中 $\lambda_{ij}^{i'j'}=\Lambda_{l_i l_j'}^{l_{i'} l_{j'}'}$, 且 $c=c(\delta)$ 为不依赖于 n,u,f 和 g 的常数.

为了估计 (2.4.14) 右边第二项的上界，我们需要下述引理:

引理 2.4.4 设 $\{\xi_{ij}\}$, δ, f, g 和 $\lambda_{ij}^{i'j'}$ 如引理 2.4.3 所示. 假设

$$|\lambda_{ij}^{i'j'}| < \left(|i-i'||j-j'|\right)^{-\nu}, \quad i \neq i', j \neq j',$$

并令 $u = \sqrt{(2-\eta)\log(fg)}$, 其中 ν 和 η 为正常数, 满足 $0 < \eta < (1-\delta)\nu/(1+\nu+\delta)$. 则有

$$\sum := \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \leq c_0(fg)^{-\delta_0},$$

其中 $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$, c_0 为不依赖于 n, f 和 g 的正常数.

证明 令 a 使得 $0 < a = (1+\eta\delta-\delta)/\{(1+\nu)(1+\delta)\} < 1$. 我们把和式 $\sum$ 分成下述四个部分:

$$\begin{aligned}\sum := & \sum_{\substack{1\leq i,i'\leq f\\ 0<|i-i'|\leq[f^a]}} \sum_{\substack{1\leq j,j'\leq g\\ 0<|j-j'|\leq[g^a]}} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right)\\ & + \sum_{\substack{1\leq i,i'\leq f\\ |i-i'|>[f^a]}} \sum_{\substack{1\leq j,j'\leq g\\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right)\\ & + \sum_{\substack{1\leq i,i'\leq f\\ |i-i'|\leq[f^a]}} \sum_{\substack{1\leq j,j'\leq g\\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right)\\ & + \sum_{\substack{1\leq i,i'\leq f\\ |i-i'|>[f^a]}} \sum_{\substack{1\leq j,j'\leq g\\ |j-j'|\leq[g^a]}} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right)\\ =: & \sum^{(1)} + \sum^{(2)} + \sum^{(3)} + \sum^{(4)}.\end{aligned}$$

现在我们来分别估计上述四个和的上界:

$$\begin{aligned}\sum^{(1)} \leq & c(fg)^{1+a} \exp\left(-\frac{2-\eta}{1+\delta}\log(fg)\right) = c(fg)^{1+a-\{(2-\eta)/(1+\delta)\}}\\ = & c(fg)^{\{\eta(1+\delta+\nu)-\nu(1-\delta)\}/\{(1+\nu)(1+\delta)\}} = c(fg)^{-\delta_0},\end{aligned}$$

$$
\begin{aligned}
\sum\nolimits^{(2)} \leq & c(fg)^{2-a\nu}\exp\big(-(1-|\lambda_{ij}^{i'j'}|)u^2\big)\\
\leq & c(fg)^{2-a\nu}\exp\big(-(2-\eta)\log(fg)-(fg)^{-a\nu}(2-\eta)\log(fg)\big)\\
\leq & c(fg)^{\eta-a\nu}=c(fg)^{-\delta_0},\\
\sum\nolimits^{(3)} \leq & cf^{1+a}g^{2-a\nu}\exp\Big(-\frac{(2-\eta)\log g+(2-\eta)\log f}{1+|\lambda_{ij}^{i'j'}|}\Big)\\
\leq & cf^{1+a}\exp\Big(-\frac{2-\eta}{1+\delta}\log f\Big)g^{2-a\nu}\\
& \cdot\exp\big(-(1-g^{-a\nu})(2-\eta)\log g\big)\\
\leq & cf^{1+a-\{(2-\eta)/(1+\delta)\}}g^{\eta-a\nu}=c(fg)^{-\delta_0},\\
\sum\nolimits^{(4)} \leq & cf^{2-2a\nu}g^{1+a}\exp\Big(-\frac{2-\eta}{1+\delta}\log g\Big)\\
& \cdot\exp\big(-(2-\eta)(1-f^{-2a\nu})\log f\big)\\
\leq & cg^{1+a-\{(2-\eta)/(1+\delta)\}}f^{\eta-a\nu}=c(fg)^{-\delta_0}.
\end{aligned}
$$

由此，引理得证.

现在我们来证明定理 2.4.1.

定理 2.4.1 的证明 由关系式 $2ab=a^2+b^2-(a-b)^2$, 对任何 $t,v>0$ 我们有

$$S^2(t,v)=2\{t^{2\alpha}+v^{2\alpha}-(t^2+v^2)^{\alpha}\}>0. \tag{2.4.15}$$

对整数 k,j,l,r 和任何固定的 $\theta>1$, 令

$$
\begin{aligned}
A_{kjlr}=&\{T:\theta^{k-1}\leq A_T<\theta^k,\theta^{j-1}\leq a_T<\theta^j,\theta^{l-1}\leq B_T<\theta^l,\\
&\theta^{r-1}\leq b_T<\theta^r\}.
\end{aligned}
$$

我们总是考察那些使得 A_{kjlr} 非空的 k,j,l 和 r. 由条件 (iii), 对某 $0<c_3\leq c_4<\infty$ 和 $c_5\leq c_6<\infty$, 我们有

$$c_3\theta^r\leq\theta^j\leq c_4\theta^r \quad 等价地 \quad c_5\leq j-r\leq c_6. \tag{2.4.16}$$

通过考察函数 $f(x)=(x^{2\alpha}+1)/(x^2+1)^{\alpha}$ 并利用 (2.4.16), 对某 $0<c_7\leq c_8<\infty$ 和 $0<c_9\leq c_{10}<\infty$ 我们有

$$c_7\theta^{2\alpha(j+1)} \le S^2(\theta^j,\theta^r) \le c_8\theta^{2\alpha(j+1)},$$
$$c_9\theta^{2\alpha(r+1)} \le S^2(\theta^j,\theta^r) \le c_{10}\theta^{2\alpha(r+1)}. \tag{2.4.17}$$

进一步，对充分大的 $k\wedge l$ 有

$$\begin{aligned}
&\inf_{T\in A_{kjlr}} \beta_T \\
&\ge \Big\{2\Big(\log\Big[\Big(\frac{\theta^{k-1}-\theta^j}{\theta^j}\vee 1\Big)\Big(\frac{\theta^{l-1}-\theta^r}{\theta^r}\vee 1\Big)\Big] \\
&\quad + \log\log\theta^{k-1} + \log\log\theta^{l-1}\Big)\Big\}^{1/2} \\
&\ge \theta^{-1}\Big\{2\Big(\log\Big[\Big(\frac{\theta^{k-1}-\theta^j}{\theta^j}\vee 1\Big)\Big(\frac{\theta^{l-1}-\theta^r}{\theta^r}\vee 1\Big)\Big] \\
&\quad + \log\log\theta^{k} + \log\log\theta^{l}\Big)\Big\}^{1/2} \\
&=: \theta^{-1}\beta_{kjlr}.
\end{aligned} \tag{2.4.18}$$

由条件 (ii) 和 (iii), A_T 和 B_T 或者都有界或者都无界. 在有界的情形, a_T 和 b_T 都趋于零, 从而 (注意到条件 (i)) 对某正整数 d_1 和 d_2 有

$$j \le k+1 \le d_1, \qquad r \le l+1 \le d_2. \tag{2.4.19}$$

在无界的情形, a_T 和 b_T 的下极限都大于零, 从而 (也注意到条件 (i)) 对某正整数 d_3 和 d_4 有

$$d_3 \le j \le k+1, \qquad d_4 \le r \le l+1. \tag{2.4.20}$$

下面我们只考察无界的情形. 有界的情形可以同样考虑. 注意到 $S(t,v)$ 关于 t 和 v 单调增加, 利用 (2.4.18), 可写

$$\begin{aligned}
&\limsup_{T\to\infty} D_1(A_T,B_T,a_T,b_T) \\
&\le \limsup_{\substack{k\to\infty\\ l\to\infty}} \sup_{\substack{d_3\le j\le k+1\\ d_4\le r\le l+1\\ c_5\le j-r\le c_6}} \sup_{T\in A_{kjlr}} \sup_{\substack{0\le s\le A_T-a_T\\ 0\le t\le a_T}} \sup_{\substack{0\le u\le B_T-b_T\\ 0\le v\le b_T}} \frac{|X(R(s,t,u,v))|}{S(a_T,b_T)\beta_T}
\end{aligned}$$

$$\leq \limsup_{\substack{k\to\infty\\ l\to\infty}} \sup_{\substack{d_3\leq j\leq k+1\\ d_4\leq r\leq l+1\\ c_5\leq j-r\leq c_6}} \sup_{\substack{0\leq s\leq\theta^k-\theta^{j-1}\\ 0\leq t\leq\theta^j}} \sup_{\substack{0\leq u\leq\theta^l-\theta^{r-1}\\ 0\leq v\leq\theta^r}} \frac{|X(R(s,t,u,v))|\theta^{1+\alpha}}{S(\theta^j,\theta^r)\beta_{kjlr}}. \tag{2.4.21}$$

令 $C_{kjlr}=\{(s,t,u,v): 0\leq s\leq\theta^k-\theta^{j-1}, 0\leq t\leq\theta^j, 0\leq u\leq\theta^l-\theta^{r-1}, 0\leq v\leq\theta^r\}$ 为一个四维集合. 为了应用引理 2.4.1, 记

$$Y_{jr}(s,t,u,v)=\frac{X(R(s,t,u,v))}{S(\theta^j,\theta^r)}, \quad (t,s,u,v)\in C_{kjlr},$$
$$\varphi(z)=\frac{4(\sqrt{2}z)^\alpha}{S(\theta^j,\theta^r)}, \quad z>0.$$

显然, $EY_{jr}(s,t,u,v)=0$, $\Gamma^2:=\sup_{(s,t,u,v)\in C_{kjlr}}E\{Y_{jr}(s,t,u,v)\}^2=1$, 进而

$$\begin{aligned}
&E\{X(R(s_1,t_1,u_1,v_1))-X(R(s_2,t_2,u_2,v_2))\}^2\\
&\quad\leq 2E\{([X(s_1+t_1,u_1+v_1)-X(s_2+t_2,u_2+v_2)]\\
&\qquad-[X(s_1,u_1+v_1)-X(s_2,u_2+v_2)])^2\\
&\qquad+([X(s_2+t_2,u_2)-X(s_1+t_1,u_1)]\\
&\qquad-[X(s_2,u_2)-X(s_1,u_1)])^2\}\\
&\quad\leq 4E\{([X(s_1+t_1,u_1+v_1)-X(s_2+t_2,u_2+v_2)]^2\\
&\qquad+[X(s_1,u_1+v_1)-X(s_2,u_2+v_2)])^2\\
&\qquad+([X(s_2+t_2,u_2)-X(s_1+t_1,u_1)]^2\\
&\qquad+[X(s_2,u_2)-X(s_1,u_1)]^2\}\\
&\quad\leq 16\{(s_1+t_1-s_2-t_2)^2+(u_1+v_1-u_2-v_2)^2\}^\alpha\\
&\quad\leq 16\cdot 2^\alpha\{\sqrt{(s_1-s_2)^2+(t_1-t_2)^2+(u_1-u_2)^2+(v_1-v_2)^2}\}^{2\alpha}.
\end{aligned}$$

从而我们得

$$\begin{aligned}
&E\{Y_{jr}(s_1,t_1,u_1,v_1)-Y_{jr}(s_2,t_2,u_2,v_2)\}^2\\
&\quad\leq\varphi\big(\sqrt{(s_1-s_2)^2+(t_1-t_2)^2+(u_1-u_2)^2+(v_1-v_2)^2}\big).
\end{aligned}$$

另一方面，由 (2.4.17)，对任意给定的 $\varepsilon'>0$，存在一个充分小的常数 $\delta=\delta(\varepsilon')>0$ 使得

$$\begin{aligned}
&(2\sqrt{2}+2)A\int_0^\infty \varphi(2\delta\theta^j 2^{-y^2})dy\\
&\quad=8(\sqrt{2}+1)A\int_0^\infty (2\sqrt{2}\delta\theta^j 2^{-y^2})^\alpha \frac{1}{S(\theta^j,\theta^r)}dy\\
&\quad\le 4(\sqrt{2}+1)A\frac{(2\sqrt{2}\delta\theta^j)^\alpha}{\sqrt{c_7}\theta^{\alpha(j+1)}}\sqrt{\frac{\pi}{\alpha\log 2}}<\frac{\varepsilon'}{8},
\end{aligned}$$

其中 A 为引理 2.4.1 中所定义. 对任意给定的 $\epsilon>0$，取 $0<\varepsilon'<2\varepsilon$. 则由引理 2.4.1 得

$$\begin{aligned}
&P\Bigg\{\sup_{\substack{0\le s\le\theta^k-\theta^{j-1}\\0\le t\le\theta^j}}\ \sup_{\substack{0\le u\le\theta^l-\theta^{r-1}\\0\le v\le\theta^r}}\frac{|X(R(s,t,u,v))|}{S(\theta^j,\theta^r)\beta_{kjlr}}\ge 1+\varepsilon\Bigg\}\\
&\le 2P\Bigg\{\sup_{\substack{0\le s\le\theta^k-\theta^{j-1}\\0\le t\le\theta^j}}\ \sup_{\substack{0\le u\le\theta^l-\theta^{r-1}\\0\le v\le\theta^r}}Y_{jr}(s,t,u,v)\\
&\qquad\ge\sqrt{1+\varepsilon}\beta_{kjlr}\Big(1+\frac{\varepsilon'}{8}\Big)\Bigg\}\\
&\le 2P\Bigg\{\sup_{\substack{0\le s\le\theta^k-\theta^{j-1}\\0\le t\le\theta^j}}\ \sup_{\substack{0\le u\le\theta^l-\theta^{r-1}\\0\le v\le\theta^r}}Y_{jr}(s,t,u,v)\\
&\qquad\ge\sqrt{1+\varepsilon}\beta_{kjlr}\Big[1+(2\sqrt{2}+2)A\int_0^\infty\varphi(2\delta\theta^j2^{-y^2})dy\Big]\Bigg\}\\
&\le c\Big(\frac{\theta^k-\theta^{j-1}}{\delta\theta^j}\vee 1\Big)\frac{1}{\delta}\Big(\frac{\theta^l-\theta^{r-1}}{\delta\theta^j}\vee 1\Big)\Big(\frac{\theta^r}{\delta\theta^j}\vee 1\Big)\\
&\quad\cdot\exp\Big\{-\frac{1+\varepsilon}{2}\beta_{kjlr}^2\Big\}\\
&\le c\delta^{-4}\theta^{k-j+l-j+r-j}(\theta^{k-j+l-r}kl)^{-(1+\epsilon)}\\
&\le c\delta^{-4}c_3^{-2}\theta^{-\varepsilon(k-j+l-r)}(kl)^{-(1+\epsilon)}.
\end{aligned}$$

由此得

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=d_3}^{k+1}\sum_{r=d_4}^{l+1}P\Bigg\{\sup_{\substack{0\le s\le\theta^k-\theta^{j-1}\\0\le t\le\theta^j}}\ \sup_{\substack{0\le u\le\theta^l-\theta^{r-1}\\0\le v\le\theta^r}}|X(R(s,t,u,v))|/$$

$$(S(\theta^j,\theta^r)\beta_{kjlr})\ge 1+\varepsilon\Bigg\}<\infty.$$

从而，注意到 θ 的任意性，由 Borel-Cantelli 引理和 (2.4.21) 得 (2.4.2).

下证 (2.4.3). 我们也只考察 A_T 和 B_T 都无界的情形. 与 (2.4.21) 类似，写

$$\begin{aligned}
&\liminf_{T\to\infty}D_2(A_T,B_T,a_T,b_T)\\
&\ge\liminf_{\substack{k\to\infty\\l\to\infty}}\ \inf_{\substack{d_3\le j\le k+1\\d_4\le r\le l+1\\c_5\le j-r\le c_6}}\ \sup_{0\le s\le\theta^{k-1}-\theta^j}\ \sup_{0\le u\le\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)\theta\beta_{kjlr}}\\
&\quad-\limsup_{\substack{k\to\infty\\l\to\infty}}\ \sup_{\substack{d_3\le j\le k+1\\d_4\le r\le l+1\\c_5\le j-r\le c_6}}\ \sup_{0\le s\le\theta^{k-1}-\theta^j}\ \sup_{0\le u\le\theta^{l-1}-\theta^r}|X(R(s,\theta^j,u,\\
&\qquad\theta^r))|\theta^{1+\alpha}/(S(\theta^j,\theta^r)\beta_{kjlr})\\
&=:J_1-J_2.
\end{aligned}\tag{2.4.22}$$

模仿 (2.4.2) 的证明并比较 (2.4.21) 的右边和 J_2 中 t,v 的范围，我们得对任意的 $\varepsilon>0$, 当 θ 充分接近 1 时，

$$J_2\le\varepsilon\quad\text{a.s.}\tag{2.4.23}$$

考察 J_1. 对任意给定的充分大的 N, 由下式定义整数 f_{kj} 和 g_{lr}:

$$f_{kj}=\left[\frac{\theta^{k-1}-\theta^j}{N\theta^j}\vee 1\right]\quad 和\quad g_{lr}=\left[\frac{\theta^{l-1}-\theta^r}{N\theta^r}\vee 1\right].$$

对 $p=0,1,\cdots,f_{kj}$ 和 $q=0,1,\cdots,g_{lr}$, 定义随机变量

$$X_{jr}(R_{pq}):=X(R(Np\theta^j,\theta^j,Nq\theta^r,\theta^r)).$$

由 (iv) 得：对任意的 $0<\varepsilon'<\varepsilon<1$, 当 $k\wedge l$ 充分大时，

$$P\Big\{\sup_{0\le s\le\theta^{k-1}-\theta^j}\ \sup_{0\le u\le\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)\beta_{kjlr}}<\sqrt{1-\varepsilon}\Big\}$$
$$\le P\Big\{\sup_{0\le s\le\theta^{k-1}-\theta^j}\ \sup_{0\le u\le\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)}$$
$$<\{2(1-\varepsilon')\log(f_{kj}g_{lr})\}^{1/2}\Big\}$$
$$\le P\Big\{\max_{0\le p\le f_{kj}}\ \max_{0\le q\le g_{lr}}\frac{X_{jr}(R_{pq})}{S(\theta^j,\theta^r)}<\{2(1-\varepsilon')\log(f_{kj}g_{lr})\}^{1/2}\Big\}. \tag{2.4.24}$$

定义 $X_{jr}(R_{pq})$ 和 $X_{jr}(R_{p'q'})$ 的相关函数为：

$\lambda_{jr}(p,q,p',q')=\text{Correlation}(X_{jr}(R_{pq}),X_{jr}(R_{p',q'})),\quad p\ne p',q\ne q'.$

令 $h=p-p',m=q-q'$. 由关系式 $2ab=a^2+b^2-(a-b)^2$ 得

$$\begin{aligned}
&|\text{Cov}(X_{jr}(R_{pq}),X_{jr}(R_{p',q'}))|\\
\le&|\{[(Nh\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha-[(Nh\theta^j)^2+(Nm\theta^r)^2]^\alpha\}\\
&-\{[(Nh\theta^j)^2+(Nm\theta^r)^2]^\alpha-[(Nh\theta^j)^2+(Nm\theta^r-\theta^r)^2]^\alpha\}|\\
&+\frac{1}{2}|\{[(Nh\theta^j-\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha-[(Nh\theta^j-\theta^j)^2\\
&+(Nm\theta^r)^2]^\alpha\}-\{[(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2]^\alpha-[(Nh\theta^j-\theta^j)^2\\
&+(Nm\theta^r-\theta^r)^2]^\alpha\}|+\frac{1}{2}|\{[(Nh\theta^j+\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha\\
&-[(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2]^\alpha\}-\{[(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2]^\alpha\\
&-[(Nh\theta^j+\theta^j)^2+(Nm\theta^r-\theta^r)^2]^\alpha\}|\\
=&\Big|\int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^r+\theta^r)^2}}d(x^{2\alpha})-\int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^r)^2}}d(x^{2\alpha})\Big|\\
&+\frac{1}{2}\Big|\int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r+\theta^r)^2}}d(x^{2\alpha})\\
&-\int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2}}d(x^{2\alpha})\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\Big|\int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r+\theta^r)^2}} d(x^{2\alpha}) \\
&-\int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2}} d(x^{2\alpha})\Big|.
\end{aligned}
$$

不失一般性，不妨设 $h,m>0$. 分别应用引理 2.4.2 于 $a=Nh\theta^j$, $Nh\theta^j-\theta^j$, $Nh\theta^j+\theta^j$ 和 $p=\theta^r$, 我们得

$$
\begin{aligned}
&|\mathrm{Cov}(X_{jr}(R_{pq}),X_{jr}(R_{p'q'}))| \\
&\qquad\le c\frac{\{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{2r}\}^{\alpha}\theta^{2r}}{(Nh-1)^2\theta^{2j}+(Nm-1)^2\theta^{2r}}.
\end{aligned}
$$

从而，由 (2.4.16) 和 (2.4.17) 对充分大的 $k\wedge l$ 和 N 我们有

$$
\begin{aligned}
|\lambda_{jr}(p,q,p',q')| &\le c\frac{\{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{2r}\}^{\alpha}\theta^{2r}}{\{(Nh-1)^2\theta^{2j}+(Nm-1)^2\theta^{2r}\}S^2(\theta^j,\theta^r)} \\
&\le c\{(Nh\theta^j)^2+(Nm\theta^r)^2\}^{\alpha-1}\theta^{2r}/S^2(\theta^j,\theta^r) \\
&\le cc_9^{-1}\theta^{-2\alpha}\{(c_3Nh)^2+(Nm)^2\}^{\alpha-1} \\
&\le (h^2+m^2)^{\alpha-1}\le(2hm)^{\alpha-1}<(hm)^{-\nu},
\end{aligned}
$$

其中 $\nu=1-\alpha>0$. 为了估计 (2.4.24) 式右边的上界，我们应用引理 2.4.3 和 2.4.4 于

$$
\begin{aligned}
&\xi_{l_pl_q}=X_{jr}(R_{pq})/S(\theta^j,\theta^r),\quad p=0,1,\cdots,f_{kj};\ q=0,1,\cdots,g_{lr}, \\
&|\lambda_{pq}^{p'q'}|=|\lambda_{jr}(p,q,p',q')|<(|hm|)^{-\nu}, \\
&h=p-p'\ne0, m=q-q'\ne0, \\
&u=u_{kjlr}=\{(2-\eta)\log(f_{kj}g_{lr})\}^{1/2},\quad \eta=2\varepsilon'<\frac{(1-\delta)\nu}{1+\nu+\delta}, \\
&f=f_{kj},\ g=g_{lr}.
\end{aligned}
$$

从而，对某个 $\delta_0>0$ 和充分大的 $k\wedge l$, (2.4.24) 的右边不超过

$$
\{\Phi(u_{kjlr})\}^{(f_{kj}+1)(g_{lr}+1)}+c(f_{kj}g_{lr})^{-\delta_0}.
$$

因而，由 (2.4.24) 得

$$
\begin{aligned}
&P\Big\{\sup_{0\le s\le\theta^{k-1}-\theta^j}\ \sup_{0\le u\le\theta^{l-1}-\theta^r}\frac{X(R(s,\theta^j,u,\theta^r))}{S(\theta^j,\theta^r)\beta_{kjlr}}<\sqrt{1-\varepsilon}\Big\}\\
&\quad\le\exp\{-c_9\big((f_{kj}+1)(g_{lr}+1)\big)^{\varepsilon'}\}+c(f_{kj}g_{lr})^{-\delta_0}\\
&\quad\le c(f_{kj}g_{lr})^{-\delta_0}\le c\theta^{-\delta_0(k-j+l-r)}.
\end{aligned}\tag{2.4.25}
$$

由条件 (iv) 知下述条件之一满足:

(a) 对任意给定的 $M>0$, 当 $k\wedge l$ 充分大时成立 $k-j\ge M\log k$ 和 $l-r\ge M\log l$,

(b) 对任意给定的 $M>0$, 当 $k\wedge l$ 充分大时成立 $k-j\ge M\log k$ 和 $k\ge Ml$,

(c) 对任意给定的 $M>0$, 当 $k\wedge l$ 充分大时成立 $l-r\ge M\log l$ 和 $l\ge Mk$.

若条件 (a) 满足，取 M 充分大，有

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=d_3}^{[k-M\log k]}\sum_{r=d_4}^{[l-M\log l]}\theta^{-\delta_0(k-j+l-r)}\\
&\qquad\le c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\theta^{-\delta_0M(\log k+\log l)}<\infty.
\end{aligned}
$$

若条件 (b) 满足，取 M 充分大，有

$$
\sum_{k=1}^{\infty}\sum_{l=1}^{[k/M]}\sum_{j=d_3}^{[k-M\log k]}\sum_{r=d_4}^{l+1}\theta^{-\delta_0(k-j+l-r)}\le c\sum_{k=1}^{\infty}\frac{k}{M}\theta^{-\delta_0M\log k}<\infty.
$$

当条件 (c) 满足时，我们有同样的收敛性. 从而，由 (2.4.25) 和 Borel-Cantelli 引理得

$$
J_1\le\sqrt{1-\varepsilon}\quad\text{a.s.}\tag{2.4.26}
$$

综合 (2.4.22), (2.4.23) 和 (2.4.26) 即得证 (2.4.3). 定理 2.4.1 证毕.

§2.5 两参数 Ornstein-Uhlenbeck 过程

设 $X(\cdot)$ 是具有系数 $\gamma \geq 0$ 和 $\lambda > 0$ 的 O-U 过程. 在 2.1.5 小节, 我们已指出它是下述随机微分方程的平稳解:

$$d\,X(t) = -\lambda X(t) + (2\gamma)^{1/2} d\,W(t),$$

其中 $\{W(t); -\infty < t < \infty\}$ 是 Wiener 过程. 一般地, 具有边界条件

$$X(0) = X_0$$

的上述随机微分方程有唯一的解:

$$X(t) = e^{-\lambda t}\left\{X_0 + (2\gamma)^{1/2}\int_0^t e^{\lambda s} d\,W(s)\right\},$$

其中 X_0 是与 $W(\cdot)$ 独立的随机变量. 这是一个从 X_0 出发的具有系数 γ 和 λ 的 O-U 过程. 两参数 O-U 过程 (OUP_2) 是上述单参数过程的推广.

两参数 O-U 过程 (OUP_2) $\{X(t,v); t \geq 0, v \geq 0\}$ 定义为

$$X(t,v) = e^{-\alpha t-\beta v}\left\{X_0 + \sigma\int_0^t\int_0^v e^{\alpha x+\beta y}\,dW(x,y)\right\}, \quad t \geq 0, v \geq 0, \tag{2.5.1}$$

其中 $\alpha > 0$ 和 $\beta > 0$ 为两参系数, $W(\cdot,\cdot)$ 为一个两参数 Wiener 过程, X_0 为一个与 $W(\cdot,\cdot)$ 独立的随机变量. 这一定义是由王梓坤 (1983) 引入的. 若 X_0 是 Gauss 变量, 则 $X(\cdot,\cdot)$ 是一个 Gauss 过程. 若记

$$J(t,v) = e^{-\alpha t-\beta v}\int_0^t\int_0^v e^{\alpha x+\beta y}\,dW(x,y),$$

则易知

$$\begin{aligned} EJ(t_1,v_1)J(t_2,v_2) =& e^{-\alpha(t_1+t_2)-\beta(v_1+v_2)} \\ & \cdot \big(e^{2\alpha(t_1\wedge t_2)}-1\big)\big(e^{2\beta(v_1\wedge v_2)}-1\big)/(4\alpha\beta). \end{aligned}$$

可以证明 OUP_2 既不是独立增量过程，也不是平稳增量过程. 另一方面，对固定的 $c>0$, $\{X(t,c);t\geq 0\}$ 是一个从 $X(0,c)$ 出发的单参数 O-U 过程. 事实上,

$$X(t,c)=e^{-\alpha t}X(0,c)+\sigma J(t,c),$$

且 $EJ(t,c)=0$,

$$E[J(t_1,c)J(t_2,c)]=E[J(t_1)J(t_2)],$$

其中

$$J(t)=\sqrt{\frac{1-e^{-2\beta c}}{2\beta}}e^{-\alpha t}\int_0^t e^{\alpha s}d\,W(s).$$

因此， $\{X(t,c);t\geq 0\}$ 与下述单参数 O-U 过程同分布:

$$\left\{\widetilde{X}(t):=e^{-\alpha t}X(0,c)+\sigma\sqrt{\frac{1-e^{-2\beta c}}{2\beta}}e^{-\alpha t}\int_0^t e^{\alpha s}d\,W(s);\ \ t\geq 0\right\}.$$

同样，对固定的 $c>0$, $\{X(c,s);s\geq 0\}$ 是一个从 $X(c,0)$ 出发的单参数 O-U 过程.

王梓坤 (1983) 研究了 OUP_2 的一些 Markov 性质. 陈雄 (1989) 通过给出这一过程的像集和图集的 Hausdorff 维数研究了它的样本轨道性质. Lin (1995a, b) 通过建立它的连续模和大增量的极限定理给出了其样本轨道性质的直接描述.

为了简单起见,我们设 $\sigma=1$, $EX_0=0$, $EX_0^2=1$, $E\exp(tX_0^2)<\infty$ 对任何 $0<t<\frac{1}{2}$ 成立. 考察增量:

$$\begin{aligned}X(t+s,v)-X(t,v)&=e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})X_0\\&\quad+e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})\int_0^{t+s}\int_0^v e^{\alpha x+\beta y}\,dW(x,y)\\&\quad+e^{-\alpha t-\beta v}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}\,dW(x,y)\\&=:\xi_1(t,s,v)+\xi_2(t,s,v)+\xi_3(t,s,v)\end{aligned}\tag{2.5.2}$$

和

$$
\begin{aligned}
X(R(t,s,v,u)) :=& X(t+s,v+u)-X(t+s,v)\\
&-X(t,v+u)+X(t,v)\\
=&e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})(1-e^{\beta u})X_0\\
&+e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})\int_0^t\int_v^{v+u}e^{\alpha x+\beta y}\,dW(x,y)\\
&+e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})(1-e^{\beta u})\int_0^t\int_0^v e^{\alpha x+\beta y}\,dW(x,y)\\
&+e^{-\alpha(t+s)-\beta(v+u)}\int_t^{t+s}\int_v^{v+u}e^{\alpha x+\beta y}\,dW(x,y)\\
&+e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\beta u})\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}\,dW(x,y),
\end{aligned}\tag{2.5.3}
$$

其中 $R(t,s,v,u)=[t,t+s]\times[v,v+u]$. 从而我们有

$$
\begin{aligned}
&E\big(X(t+s,v)-X(t,v)\big)^2\\
&\quad=e^{-2\alpha t-2\beta v}(1-e^{-\alpha s})^2+\frac{1}{4\alpha\beta}(1-e^{-2\beta v})\\
&\qquad\cdot\{(1-e^{-2\alpha t})(1-e^{-\alpha s})^2+1-e^{-2\alpha s}\}
\end{aligned}\tag{2.5.4}
$$

和

$$
\begin{aligned}
EX^2(R(t,s,v,u)) =& e^{-2\alpha t-2\beta v}(1-e^{-\alpha s})^2(1-e^{-\beta u})^2\\
&+\frac{1}{4\alpha\beta}(1-e^{-\alpha s})^2(1-e^{-2\alpha t})(1-e^{-2\beta u})\\
&+\frac{1}{4\alpha\beta}(1-e^{-\alpha s})^2(1-e^{-\beta u})^2(1-e^{-2\alpha t})(1-e^{-2\beta v})\\
&+\frac{1}{4\alpha\beta}(1-e^{-2\alpha s})(1-e^{-2\beta u})\\
&+\frac{1}{4\alpha\beta}(1-e^{-\beta u})^2(1-e^{-2\alpha s})(1-e^{-2\beta v}).
\end{aligned}\tag{2.5.5}
$$

易知

$$
\begin{aligned}
&E(X(t+s,v)-X(t,v))^2\\
&=\begin{cases} \alpha^2 s^2 e^{-2\alpha(t+s)-2\beta v}+\dfrac{s}{2\beta}(1-e^{-2\beta v})+O(s^2), & s\to 0,\\ e^{-2\alpha t-2\beta v}+\dfrac{1}{4\alpha\beta}(1-e^{-2\beta v})(2-e^{-2\alpha t})+O(e^{-\alpha s}), & s\to\infty \end{cases}\\
&:=\begin{cases} \sigma^2(t,s,v)+\sigma^2(s,v)+O(s^2), & s\to 0,\\ \overline{\sigma}_1^2(t,v)+O(e^{-\alpha s}), & s\to\infty, \end{cases}
\end{aligned}
\tag{2.5.6}
$$

$$
\begin{aligned}
&EX^2(R(t,s,v,u))\\
&=\begin{cases} \dfrac{1}{4\alpha\beta}(e^{2\alpha s}-1)(1-e^{-2\beta u})+o(su), & s\to 0, u\to 0,\\ \overline{\sigma}_2^2(t,v)+O(e^{-\alpha s}+e^{-\beta v}), & s\to\infty, u\to\infty, \end{cases}
\end{aligned}
\tag{2.5.7}
$$

其中 $\overline{\sigma}_2^2(t,v)=e^{-2\alpha t-2\beta v}+\frac{1}{4\alpha\beta}(2-e^{-2\alpha t})(2-e^{-2\beta v})$.

另外，由 (2.5.4), 对充分小的 s 有

$$
E(X(t+s,v)-X(t,v))^2\le\Big(\alpha^2+\frac{\alpha}{4\beta}\Big)s^2+\frac{1}{2\beta}s,
$$

由 $X(t,v)$ 关于 t 和 v 的对称性，对充分小的 u 有

$$
E(X(t,v+u)-X(t,v))^2\le\Big(\alpha^2+\frac{\alpha}{4\beta}\Big)u^2+\frac{1}{2\beta}u.
$$

从而

$$
\begin{aligned}
&E(X(t+s,v+u)-X(t,v))^2\\
&\le 2E(X(t+s,v+u)-X(t+s,v))^2+2E(X(t+s,v)-X(t,v))^2\\
&\le 2\Big\{\Big(\alpha^2+\frac{\alpha}{4\beta}\Big)(s^2+u^2)+\frac{1}{2\beta}(s+u)\Big\}.
\end{aligned}
$$

由此和定理 2.1.3 知 $X(t,v)$ 关于 (t,v) 几乎处处连续.

2.5.1 OUP$_2$ 的连续模

记 $\sigma_1(t,s,v)=\sigma(t,s,v)+\sigma(s,v)$, $\sigma_2(t,s,v)=\sigma(t,s,v)\wedge\sigma(s,v)$. 对任意固定的 $t\ge 0$ 和 $s>0$ 有

$$
\sigma(s,v)=o\big(\sigma(t,s,v)\big),\quad v\to 0.
$$

考察 $X(t,v)$ 对每个参数变量的连续模，我们有

定理 2.5.1 设 a_h 为 h 的函数，当 $h\to 0$ 时，$a_h=o(h^{-\delta})$ 对任何 $\delta>0$ 成立且 $\liminf_{h\to 0}a_h>0$. 则

$$\lim_{h\to 0}\sup_{v>0}\sup_{0\le t\le a_h}\sup_{0\le s\le h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\left\{2(\log\frac{1}{h}+\log\log\sigma_2^{-1}(t,h,v))\right\}^{1/2}}=1 \text{ a.s.} \tag{2.5.8}$$

且对任何固定的 $v>0$ 成立

$$\lim_{h\to 0}\sup_{0\le t\le a_h}\frac{|X(t+h,v)-X(t,v)|}{\sigma_1(t,h,v)\left\{2(\log\frac{1}{h}+\log\log\sigma_2^{-1}(t,h,v))\right\}^{1/2}}=1 \text{ a.s.} \tag{2.5.9}$$

注 2.5.1 由 $X(t,v)$ 关于 t 和 v 的对称性，我们也有

$$\lim_{h\to 0}\sup_{t>0}\sup_{0\le v\le a_h}\sup_{0\le u\le h}\frac{|X(t,v+u)-X(t,v)|}{\nu(t,v,h)}=1 \text{ a.s.}$$

$$\lim_{h\to 0}\sup_{0\le v\le a_h}\frac{|X(t,v+h)-X(t,v)|}{\nu(t,v,h)}=1 \text{ a.s.}$$

其中 $\nu(t,v,h)$ 与 (2.5.8) 和 (2.5.9) 中的正则化因子类似.

考察 $X(t,v)$ 同时关于两参数变量的连续模，我们有下述定理.

定理 2.5.2 设 a_h 和 b_h 为 h 的函数，满足 $\liminf_{h\to 0}a_hb_h>0$，且 c_h 为 h 的连续非增函数，当 $h\to 0$ 时，$c_h\to 0$, $a_hb_h=o\big((hc_h)^{-\delta}\big)$ 对任何 $\delta>0$ 成立. 则

$$\lim_{h\to 0}\sup_{0\le t\le a_h}\sup_{0<s\le h}\sup_{0\le v\le b_h}\sup_{0<u\le c_h}\frac{|X(R(t,s,v,u))|}{\left(2hc_h\log(hc_h)^{-1}\right)^{\frac{1}{2}}}=1 \text{ a.s.} \tag{2.5.10}$$

$$\lim_{h\to 0}\sup_{0\le t\le a_h}\sup_{0\le v\le b_h}\frac{|X(R(t,h,v,c_h))|}{\left(2hc_h\log(hc_h)^{-1}\right)^{\frac{1}{2}}}=1 \text{ a.s.} \tag{2.5.11}$$

为了证明定理 2.5.1 和 2.5.2, 我们需要一些指数不等式.

引理 2.5.1 对任意的 $0 < \varepsilon < 1/2$, 存在 $h = h(\varepsilon) > 0$ 和 $C = C(\varepsilon) > 0$ 使得对任何固定的 $t \geq 0$ 和 $0 < s \leq h$ 有

$$P\left\{\sup_{v>0} \frac{|X(t+s,v) - X(t,v)|}{\sigma_1(t,s,v)\{x^2 + 2\log\log\sigma_2^{-1}(t,s,v)\}^{1/2}} \geq 1 + 2\varepsilon\right\}$$
$$\leq C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}. \tag{2.5.12}$$

证明 令 $0 < \theta < 1$, $\delta > 0$ 待定. 由以下两式分别定义 v_k 和 v_k':

$$\sigma^2(s, v_k) = \theta^k, \quad k = k_0, k_0+1, \cdots,$$

其中 $k_0 = [\log(\delta s/2\beta)/\log\theta]$, 和

$$\sigma^2(t, s, v_k') = \theta^k, \ \ k = k_1, k_1+1, \cdots,$$

其中 $k_1 = [\log\sigma^2(t,s,v_{k_0})/\log\theta]$. 由定义易知

$$v_k \to 0 \ \text{且} v_k' \to \infty \quad (k \to \infty), \tag{2.5.13}$$

$$v_{k_1}' \leq v_{k_0}, \tag{2.5.14}$$

$$1 - e^{-2\beta v_{k_0}+1} \leq \delta \leq 1 - e^{-2\beta v_{k_0}} \tag{2.5.15}$$

$$\theta(1 - e^{-2\beta v_k}) = 1 - e^{-2\beta v_{k+1}}. \tag{2.5.16}$$

由 (2.5.15) 和 (2.5.16), 对任何 $k \geq k_0$ 有

$$\begin{aligned} e^{-2\beta(v_k - v_{k+1})} &= 1 - (1-\theta)(1 - e^{-2\beta v_k})e^{2\beta v_{k+1}} \\ &\geq 1 - \frac{1-\theta}{1-\delta} = \frac{\theta - \delta}{1-\delta}. \end{aligned} \tag{2.5.17}$$

此外, 显然有

$$e^{-2\beta(v_{k+1}' - v_k')} = \theta. \tag{2.5.17'}$$

由于

$$1 - e^{-2\beta v_{k+1}'} \geq 1 - e^{-2\beta v_{k_1+1}'} \geq 1 - e^{-2\beta v_{k_0}} \geq \delta,$$

只要 θ 充分接近于 1, 对 $k \geq k_1$ 我们有

$$
\begin{aligned}
1-e^{-2\beta v_k'} &= 1-\frac{1}{\theta}e^{-2\beta v_{k+1}'} \\
&= \frac{1}{\theta}(1-e^{-2\beta v_{k+1}'})-\left(\frac{1}{\theta}-1\right)\geq \frac{1}{\sqrt{\theta}}(1-e^{-2\beta v_{k+1}'}). \qquad (2.5.16')
\end{aligned}
$$

由 (2.5.2) 得

$$
\begin{aligned}
&P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,s,v)\left\{x^2+2\log\log\sigma_2^{-1}(t,s,v)\right\}^{1/2}}\geq 1+2\varepsilon\right\}\\
\leq&\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geq(1+2\varepsilon)\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
&+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'\leq v<v_{k+1}'}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geq(1+2\varepsilon)\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
&+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geq\frac{\varepsilon}{2}\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
&+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'\leq v<v_{k+1}'}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geq\frac{\varepsilon}{2}\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
&+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geq\left(1+\frac{3\varepsilon}{2}\right)\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
&+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'\leq v<v_{k+1}'}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geq\left(1+\frac{3\varepsilon}{2}\right)\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
:=&\sum_{j=1}^{6}p_j. \qquad (2.5.18)
\end{aligned}
$$

首先估计 p_1. 由关于 X_0 的假设，对充分小的 s 我们有

$$
\begin{aligned}
p_1&=\sum_{k=k_0}^{\infty}P\left\{(e^{\alpha s}-1)|X_0|\geq(1+2\varepsilon)\alpha s(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}\\
&\leq\sum_{k=k_0}^{\infty}P\left\{|X_0|\geq(1+\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}
\end{aligned}
$$

$$\le \sum_{k=k_0}^{\infty} E\exp\left(\frac{1-\varepsilon/2}{2}X_0^2\right)\exp\left\{-\frac{1}{2}(1+\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\le c\exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\}\sum_{k=k_0}^{\infty}k^{-(1+\varepsilon)} \le c\exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\}. \tag{2.5.19}$$

对 p_2 我们有同样的估计.

现考察 p_3. 令

$$Y(v)=\int_0^{t+s}\int_0^{v} e^{\alpha s+\beta y}\, dW(x,y),$$

这是一个具有独立增量的 Gauss 过程且

$$EY^2(v)=\frac{1}{4\alpha\beta}\left(e^{2\alpha(t+s)}-1\right)\left(e^{2\beta v}-1\right).$$

注意到 (2.5.16) 和 (2.5.17), 对充分小的 s 我们有

$$\begin{aligned}
p_3 \le& \sum_{k=k_0}^{\infty} P\Big\{\sup_{v_{k+1}<v\le v_k}|Y(v)| \ge \frac{\varepsilon}{2}e^{\alpha(t+s)+\beta v_{k+1}}(e^{\alpha s}-1)^{-1}\\
&\qquad\cdot\sigma(s,v_{k+1})\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\Big\}\\
\le& 2\sum_{k=k_0}^{\infty} P\Big\{|Y(v_k)|/\left(EY^2(v_k)\right)^{\frac{1}{2}} \ge \frac{\varepsilon}{2}\left(EY^2(v_k)\right)^{-\frac{1}{2}}e^{\alpha(t+s)+\beta v_{k+1}}\\
&\qquad\cdot(e^{\alpha s}-1)^{-1}\sigma(s,v_{k+1})\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\Big\}\\
\le& c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\varepsilon^2\theta}{8\alpha s}e^{-2\beta(v_k-v_{k+1})}\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\}\\
\le& c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\varepsilon^2\theta(\theta-\delta)}{8\alpha s(1-\delta)}\left(x^2+2\log\log\theta^{-k}\right)^{\frac{1}{2}}\right\} \le c\exp(-x^2).
\end{aligned} \tag{2.5.20}$$

用 (2.5.16′) 和 (2.5.17′) 代替 (2.5.16) 和 (2.5.17), 对 p_4 我们有同样的估计.

现在来估计 p_5. 令

$$Z(v)=\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}\,dW(x,y),$$

这也是一个具有独立增量的 Gauss 过程且

$$EZ^2(v)=\frac{1}{4\alpha\beta}e^{2\alpha t}(e^{2\alpha s}-1)(e^{2\beta v}-1).$$

与 (2.5.20) 类似, 对充分接近于 1 的 θ 和充分小的 δ 有

$$\begin{aligned}
p_5\le&\sum_{k=k_0}^{\infty}P\Big\{\sup_{v_{k+1}<v\le v_k}|Z(v)|\ge\Big(1+\frac{3\varepsilon}{2}\Big)e^{\alpha t+\beta v_{k+1}}\\
&\cdot\sigma(s,v_{k+1})\big(x^2+2\log\log\theta^{-k}\big)^{\frac12}\Big\}\\
\le&c\sum_{k=k_0}^{\infty}\exp\Big\{-\frac12\Big(1+\frac{3\varepsilon}{2}\Big)\theta e^{-2\beta(v_k-v_{k+1})}\big(x^2+2\log\log\theta^{-k}\big)^{\frac12}\Big\}\\
\le&c\sum_{k=k_0}^{\infty}\exp\Big\{-\frac{\theta}{2}\Big(1+\frac{3\varepsilon}{2}\Big)\frac{\theta-\delta}{1-\delta}\big(x^2+2\log\log\theta^{-k}\big)^{\frac12}\Big\}\\
\le&c\exp\Big\{-\frac{1+\varepsilon}{2}x^2\Big\}.
\end{aligned}\tag{2.5.21}$$

对 p_6 我们有同样的估计.

把这些不等式代入 (2.5.18) 即得证 (2.5.12), 引理 2.5.1 得证.

引理 2.5.2 设 $a>0$, $0<\epsilon<1/2$. 则存在 $h=h(\varepsilon)>0$ 和 $C_1=C_1(\varepsilon)>0$ 使得

$$\begin{aligned}
&P\Big\{\sup_{v>0}\sup_{0\le t\le a}\sup_{0<s\le h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\}^{1/2}}\ge1+4\varepsilon\Big\}\\
&\le\frac{C_1a}{h}\exp\Big\{-\frac{1+\varepsilon}{2}x^2\Big\}.
\end{aligned}\tag{2.5.22}$$

证明 不妨设 $x^2\ge2$. 令 k 为一待定正整数, 对任意的 $t\ge0$, 令

$$t_j=[t2^j/h]h/2^j,\quad j=k,k+1,\cdots.$$

由于 $X(t,v)$ 关于 (t,v) 几乎处处连续, 我们可写

$$
\begin{aligned}
&|X(t+s,v)-X(t,v)| \\
\le &|X\big((t+s)_k,v\big)-X(t_k,v)|+\sum_{j=0}^{\infty}|X\big((t+s)_{k+j+1},v\big) \\
&-X\big((t+s)_{k+j},v\big)|+\sum_{j=0}^{\infty}|X(t_{k+j+1},v) \\
&-X(t_{k+j},v)|.
\end{aligned} \tag{2.5.23}
$$

由定义，对充分小的 h 和充分大的 k 及 $0<s\le h$ 有

$$
\begin{aligned}
\sigma^2\big(t_k,(t+s)_k-t_k,v\big) &\le \alpha^2(1+2^{-k})^2h^2e^{-2\alpha(t-2^{-k}h)-2\beta v} \\
&\le (1+\varepsilon/2)\sigma^2(t,h,v), \\
\sigma^2\big((t+s)_k-t_k,v\big) &\le (1-2^{-k})\frac{h}{2\beta}(1-e^{-2\beta v}) \\
&\le (1+\varepsilon/2)\sigma^2(h,v),
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2\big((t+s)_{k+j},h/2^{k+j+1},v\big) &\le \alpha^2 2^{-2(k+j+1)}h^2e^{-2\alpha t-2\beta v} \\
&\le 2^{-(k+j+1)}\sigma^2(t,h,v),
\end{aligned}
$$

$$
\sigma^2(h/2^{k+j+1},v)\le 2^{-(k+j+1)}\frac{h}{2\beta}(1-e^{-2\beta v})\le 2^{-(k+j+1)}\sigma^2(h,v).
$$

由这些不等式和引理 2.5.1 对充分大的 k 我们有

$$
\begin{aligned}
&P\left\{\sup_{v>0}\sup_{0\le t\le a}\sup_{0<s\le h}\frac{|X\big((t+s)_k,v\big)-X(t_k,v)|}{\sigma_1(t,h,v)\big\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\big\}^{1/2}}\ge 1+3\varepsilon\right\} \\
&\quad\le c2^{2k}\frac{a}{h}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
&P\left\{\sup_{v>0}\sup_{0\le t\le a}\sup_{0<s\le h}\sum_{j=0}^{\infty}\frac{|X\big((t+s)_{k+j+1},v\big)-X\big((t+s)_{k+j},v\big)|}{ma_1(t,h,v)\big\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\big\}^{1/2}}\ge\frac{\varepsilon}{2}\right\} \\
&\quad\le c\sum_{j=0}^{\infty}2^{2(k+j+1)}\frac{a}{h}\exp\left\{-\frac{(1+\varepsilon)\varepsilon^2}{8(1+2\varepsilon)^2}2^{k+j+1}x^2\right\} \\
&\quad\le c\frac{a}{h}e^{-x^2}\sum_{j=0}^{\infty}2^{2(k+j+1)}\exp(-\varepsilon^2 2^{k+j+4})\le c\frac{a}{h}e^{-x^2},
\end{aligned}
$$

其中用到了初等不等式 $bd \geq b+d$ ($\forall b \geq 2,\ d \geq 2$). 对 (2.5.23) 中的第二个级数我们有同样的估计. 综合这些不等式和 (2.5.23) 即得证 (2.5.22).

引理 2.5.3 设 $a>0, b>0, 0<\varepsilon<1/2$. 则存在 $h=h(\varepsilon)>0$, $d=d(silon)>0$, $C_2=C_2(\varepsilon)>0$ 使得

$$
\begin{aligned}
P&\left\{\sup_{0\leq t\leq a}\sup_{0<s\leq h}\sup_{0\leq v\leq b}\sup_{0<u\leq d}\frac{|X(R(t,s,v,u))|}{(su)^{1/2}}\geq(1+2\varepsilon)x\right\}\\
&\leq C_2\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}
\end{aligned}\tag{2.5.24}
$$

对任何 $x>0$ 成立

证明 不妨设 $x\geq\sqrt{2}$. 令 k 为待定正整数, 对任意的 $t\geq 0$, $v\geq 0$ 令

$$
t_j=[t2^j/h]h/2^j,\quad v_j'=[v2^j/d]d/2^j,\quad j=k,k+1,\cdots.
$$

与 (2.5.23) 类似我们有

$$
\begin{aligned}
|X(R(t,s,v,u))|\leq&|X\big(R(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k')\big)|\\
&+|X\big(R((t+s)_k,(t+s)-(t+s)_k,v_k',(v+u)_k'-v_k')\big)|\\
&+|X\big(R(t_k,t-t_k,v_k',(v+u)_k'-v_k')\big)|\\
&+|X\big(R(t,s,v_k',v-v_k')\big)|\\
&+|X\big(R(t,s,(v+u)_k',(v+u)-(v+u)_k')\big)|\\
\leq&|X\big(R(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k')\big)|\\
&+\sum_{j=0}^{\infty}|X\big(R((t+s)_{k+j},(t+s)_{k+j+1}\\
&\qquad-(t+s)_{k+j},v_k',(v+u)_k'-v_k')\big)|\\
&+\sum_{j=0}^{\infty}|X\big(R(t_{k+j},t_{k+j+1}-t_{k+j},v_k',(v+u)_k'-v_k')\big)|\\
&+|X\big(R(t,s,v_k',v-v_k')\big)|\\
&+|X\big(R(t,s,(v+u)_k',(v+u)-(v+u)_k')\big)|.
\end{aligned}\tag{2.5.25}
$$

进一步，注意到 (2.5.7), 当 $s\to 0$ 和 $u\to 0$ 时有

$$EX^2\left(R\left(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k'\right)\right)\le(1+2^{-k})^2su+o(su),$$
$$EX^2\left(R\left((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v_k',(v+u)_k'-v_k'\right)\right)$$
$$=2^{-2(k+j+1)}su+o\left(2^{-2(k+j+1)}su\right),$$

$$EX^2(R(t,s,v_k',v-v_k'))=2^{-k}su+o(su).$$

从而，对充分大的 k, 充分小的 s 和 u 有

$$P\Big\{\sup_{0\le t\le a}\sup_{0<s\le h}\sup_{0\le v\le b}\sup_{0<u\le d}\left|X\left(R\left(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k'\right)\right)\right|\Big/(su)^{1/2}\ge(1+\varepsilon)x\Big\}$$
$$\le 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},$$

$$P\left\{\sup_{0\le t\le a}\sup_{0<s\le h}\sup_{0\le v\le b}\sup_{0<u\le d}\sum_{j=0}^{\infty}\frac{\left|X\left(R\left((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v_k',(v+u)_k'-v_k'\right)\right)\right|}{\sum_{j=0}^{\infty}(2^{-(j+1)}su)^{1/2}}\right.$$
$$\left.\ge\frac{\sqrt{2}-1}{4}\varepsilon x\right\}$$
$$\le\sum_{j=0}^{\infty}2^{4(k+j+1)}\frac{ab}{hd}\sup_{0\le t\le a}\sup_{0<s\le h}\sup_{0\le v\le b}\sup_{0<u\le d}$$
$$P\left\{\left|X\left(R\left((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v_k',(v+u)_k'-v_k'\right)\right)\right|\right.$$
$$\left.\ge\frac{\sqrt{2}-1}{4}\varepsilon x2^{k+(j+1)/2}(2^{-2(k+j+1)}su)^{\frac12}\right\}$$
$$\le\frac{ab}{hd}\sum_{j=0}^{\infty}2^{4(k+j+1)}\exp\left\{-\frac{\varepsilon^2}{200}2^{2k+j+1}x^2\right\}\le c\frac{ab}{hd}e^{-x^2}.$$

对 (2.5.25) 右边的第二项我们有同样的估计. 对 $X\left(R(t,s,v_k',v-v_k')\right)$, 我们有

$$P\left\{\sup_{0\le t\le a}\sup_{0<s\le h}\sup_{0\le v\le b}\sup_{0<u\le d}\frac{\left|X\left(R(t,s,v_k',v-v_k')\right)\right|}{(su)^{1/2}}\ge\frac{\varepsilon}{4}x\right\}$$
$$\le 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{\varepsilon}{40}2^k x^2\right\}\le c\frac{ab}{hd}e^{-x^2}.$$

对 (2.5.25) 右边的最后一项我们也有同样的估计. 综合这些不等式和 (2.5.25) 即得 (2.5.24).

定理 2.5.1 的证明 首先, 我们证明

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\le t\le a_h}\sup_{0<s\le h}|X(t+s,v)-X(t,v)|/$$
$$\cdot\sigma_1(t,h,v)\left\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\right\}^{1/2}\le 1\quad\text{a.s.}\ (2.5.26)$$

不妨设 a_h 关于 $0\le h\le 1$ 单调增加, 否则可用 $a_h^*=\sup_{h\le s\le 1}a_s$ 代替 a_h.

设 $0<\varepsilon<1/2$, $\theta=1-\varepsilon$. 令 $h_j=\theta^j$. 对充分大的 j, 由引理 2.5.2 得

$$P\left\{\sup_{v>0}\sup_{0\le t\le a_{h_{j+1}}}\sup_{0<s\le h_j}|X(t+s,v)-X(t,v)|/\right.$$
$$\left.\cdot\,\sigma_1(t,h_j,v)\left\{2(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\right\}^{1/2}\ge 1+\varepsilon\right\}$$
$$\le C_1\frac{a_{h_j+1}}{h_j}\exp\left\{-\left(1+\frac{\varepsilon}{4}\right)\log h_j^{-1}\right\}$$
$$\le C_1\frac{(h_{j+1})^{-\varepsilon/8}}{h_j}h_j^{1+\varepsilon/4}\le C_1\theta^{(j-1)\varepsilon/8}.$$

由此和 Borel-Cantelli 引理得

$$\limsup_{j\to\infty}\sup_{v>0}\sup_{0\le t\le a_{h_{j+1}}}\sup_{0<s\le h_j}|X(t+s,v)-X(t,v)|/$$
$$\cdot\,\sigma_1(t,h_j,v)\left\{2(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\right\}^{1/2}$$
$$\le 1+\varepsilon\quad\text{a.s.}$$

进而

$$\begin{aligned}
&\limsup_{h\to 0}\sup_{v>0}\sup_{0\le t\le a_h}\sup_{0<s\le h}|X(t+s,v)-X(t,v)|/\\
&\quad\cdot\sigma_1(t,h,v)\big\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\big\}^{1/2}\\
\le&\limsup_{j\to\infty}\sup_{v>0}\sup_{0\le t\le a_{h_{j+1}}}\sup_{0<s\le h_j}|X(t+s,v)-X(t,v)|/\\
&\quad\cdot\theta\sigma_1(t,h_j,v)\big\{2(\log h_j{}^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\big\}^{1/2}\\
\le&(1-\varepsilon)^{-1}(1+\varepsilon)\quad \text{a.s.}
\end{aligned}$$

由 ε 的任意性，(2.5.26) 得证.

下面证明对固定的 $v>0$ 我们有

$$\begin{aligned}
&\liminf_{h\to 0}\sup_{0\le t\le a_h}\frac{|X(t+h,v)-X(t,v)|}{\sigma_1(t,h,v)\big\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\big\}^{1/2}}\\
&\ge 1\quad \text{a.s.}
\end{aligned}\tag{2.5.27}$$

注意到对固定的 $v>0$ 和 $t\ge 0$ 有

$$\sigma(t,h,v)=o\big(\sigma(h,v)\big),\quad h\to 0,$$

并回顾引理 2.5.1 的证明，我们看到 (2.5.27) 等价于

$$\liminf_{h\to 0}\sup_{0\le t\le a_h}\frac{|\xi_3(t,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}\ge 1\quad \text{a.s.}\tag{2.5.28}$$

记 $t_i=ih,\ i=0,1,\cdots,i_h:=[a_h/h]$. 因为 $\xi_3(t_i,h,v),\ i=0,1,\cdots,i_h$, 相互独立，对任意的 $\varepsilon>0$ 有

$$\begin{aligned}
&P\left\{\max_{0\le i\le i_h}\frac{|\xi_3(t_i,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}\le 1-\varepsilon\right\}\\
&=\prod_{i=0}^{i_h}\left\{1-P\Big\{\frac{|\xi_3(t_i,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}>1-\varepsilon\Big\}\right\}\\
&\le\prod_{i=0}^{i_h}\big\{1-\exp\{-(1-\varepsilon)\log h^{-1}\}\big\}\\
&\le\exp\big(-i_h h^{1-\varepsilon}\big)\le\exp\big(-h^{-\varepsilon/2}\big).
\end{aligned}\tag{2.5.29}$$

取 $h_k = k^{-1}$. 由 (2.5.29) 得

$$\begin{aligned}
&\liminf_{h\to 0} \sup_{0\le t\le c_h} \frac{|\xi_3(t,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}} \\
&\quad \ge \liminf_{k\to\infty} \sup_{0\le i\le i_{h_k}} \frac{|\xi_3(t_i,h_k,v)|}{\sigma(h_k,v)(2\log h_k^{-1})^{1/2}} \ge 1-\varepsilon \quad \text{a.s.}
\end{aligned}$$

从而 (2.5.27) 得证. 综合 (2.5.26) 和 (2.5.27) 即得定理 2.5.1 的结论.

定理 2.5.2 的证明 首先，我们证明

$$\limsup_{h\to 0} \sup_{0\le t\le a_h} \sup_{0<s\le h} \sup_{0\le v\le b_h} \sup_{0<u\le c_h} \frac{|X(R(t,s,v,u))|}{\left(2hc_h\log(hc_h)^{-1}\right)^{\frac{1}{2}}} \le 1 \ \text{a.s.} \tag{2.5.30}$$

不妨设 a_h 和 b_h 非增，否则可考察 $a_h^* = \sup_{h\le s\le 1} a_s$ 和 $b_h^* = \sup_{h\le s\le 1} b_s$. 设 $0<\varepsilon<1/2$, $\theta=1-\varepsilon$. 定义 h_j 使得 $h_jc_{h_j}=\theta^j$, $j=0,1,\cdots$. 由引理 2.5.3 得

$$\begin{aligned}
&P\Big\{\sup_{0\le t\le a_{h_{j+1}}} \sup_{0<s\le h_j} \sup_{0\le v\le b_{h_{j+1}}} \sup_{0<u\le c_{h_j}} |X(R(t,s,v,u))|/ \\
&\quad \cdot \left(2h_jc_{h_j}\log(h_jc_{h_j})^{-1}\right)^{\frac{1}{2}} \ge 1+2\varepsilon\Big\} \\
&\quad \le C_2\frac{a_{h_{j+1}}b_{h_{j+1}}}{h_jc_{h_j}}\exp\left\{-(1+\varepsilon)\log(h_jc_{h_j})^{-1}\right\} \\
&\quad \le C_2\frac{(h_{j+1}c_{h_{j+1}})^{-\varepsilon/2}}{h_jc_{h_j}}(h_jc_{h_j})^{1+\varepsilon} = C_2\theta^{(j-1)\varepsilon/2}.
\end{aligned}$$

由此得

$$\begin{aligned}
&\limsup_{j\to\infty} \sup_{0\le t\le a_{h_{j+1}}} \sup_{0<s\le h_j} \sup_{0\le v\le b_{h_{j+1}}} \sup_{0<u\le c_{h_j}} \frac{|X(R(t,s,v,u))|}{\left(2h_jc_{h_j}\log(h_jc_{h_j})^{-1}\right)^{\frac{1}{2}}} \\
&\quad \le 1+2\varepsilon \quad \text{a.s.}
\end{aligned}$$

进而

$$\limsup_{h\to0}\sup_{0\le t\le a_h}\sup_{0<s\le h}\sup_{0\le v\le b_h}\sup_{0<u\le c_h}\frac{|X(R(t,s,v,u))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}$$

$$\le\limsup_{j\to\infty}\sup_{0\le t\le a_{h_{j+1}}}\sup_{0<s\le h_j}\sup_{0\le v\le b_{h_{j+1}}}\sup_{0<u\le c_{h_j}}\frac{|X(R(t,s,v,u))|}{(\theta 2h_jc_{h_j}\log(h_jc_{h_j})^{-1})^{\frac{1}{2}}}$$

$$\le(1-\varepsilon)^{-\frac{1}{2}}(1+2\varepsilon)\quad\text{a.s.}$$

由 ε 的任意性即得 (2.5.30).

下面证明

$$\liminf_{h\to0}\sup_{0\le t\le a_h}\sup_{0\le v\le b_h}\frac{|X(R(t,h,v,c_h))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}\ge1\quad\text{a.s.}\tag{2.5.31}$$

记 $t_i=ih,i=0,1,\cdots,i_h:=[a_h/h],v_j=jc_h,j=0,1,\cdots,j_h:=[b_h/c_h]$. 则对任意给定的 $\varepsilon>0$

$$P\left\{\max_{0\le i\le i_h}\max_{0\le j\le j_h}\frac{|X(R(t_i,h,v_j,c_h))|}{(2hc_h\log(hc_h)^{-1})^{1/2}}\le1-\varepsilon\right\}$$

$$\le\prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\left\{1-P\left\{\frac{|X(R(t_i,h,v_j,c_h))|}{(2hc_h\log(hc_h)^{-1})^{1/2}}>1-\varepsilon\right\}\right\}$$

$$\le\prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\{1-\exp(-(1-\varepsilon)\log(hc_h)^{-1})\}$$

$$\le\exp\{-i_hj_h(hc_h)^{1-\varepsilon}\}\le\exp\{-a_hb_h(hc_h)^{-\varepsilon}/2\}$$

$$\le\exp\{-c(hc_h)^{-\varepsilon}\}\tag{2.5.32}$$

对充分小的 h 成立. 定义 h_k 使得 $h_kc_{h_k}=k^{-1}$. 则由 (2.5.32) 得

$$\liminf_{h\to0}\sup_{0\le t\le a_h}\sup_{0\le v\le b_h}\frac{|X(R(t,h,v,c_h))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}$$

$$\ge\liminf_{k\to\infty}\max_{0\le i\le i_{h_k}}\max_{0\le j\le j_{h_k}}\frac{|X(R(t_i,h_k,v_j,c_{h_k}))|}{(2h_kc_{h_k}\log(h_kc_{h_k})^{-1})^{\frac{1}{2}}}\ge1-\varepsilon\quad\text{a.s.}$$

即 (2.5.31) 成立. 由 (2.5.30) 和 (2.5.31) 得证定理 2.5.2 的结论.

2.5.2 OUP_2 的大增量

在上一节中我们看到，当一个或两个参数变量具有小的增量时，过程 OUP_2 的增量有多大. 现在我们要研究当一个或两个参数变量具有大的增量时，过程 OUP_2 的增量有多大.

定理 2.5.3 设 a_T 为 T 的函数，满足 $0 < a_T \le T$, $a_T \to \infty$ $(T \to \infty)$. 则

$$\limsup_{T\to\infty} \sup_{v>0} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2[\log((T-a_T)a_T)+\log\hat{v}]\}^{1/2}} \le 1 \quad \text{a.s.}, \tag{2.5.33}$$

其中 $\hat{v} = v \vee \log v^{-1}$, $\log((T-a_T)a_T)$ 意味着 $\log(T-a_T)+\log a_T$. 进一步，如果还存在 $0 \le b < 1$ 使得

$$a_T = o(T^{b+\varepsilon}), \quad T \to \infty \tag{2.5.34}$$

对任何 $\varepsilon > 0$ 成立，则对固定的 $v > 0$ 有

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|X(t+a_T,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2\log((T-a_T)a_T)\}^{1/2}} \ge \left(\frac{1-b}{1+b}\right)^{1/2} \quad \text{a.s.}, \tag{2.5.35}$$

$$\limsup_{T\to\infty} \frac{|X(T,v)-X(T-a_T,v)|}{\bar{\sigma}_1(T,v)\{2\log((T-a_T)a_T)\}^{1/2}} \ge \left(\frac{1-b}{1+b}\right)^{1/2} \quad \text{a.s.} \tag{2.5.36}$$

在 (2.5.34) 中令 $b = 0$, 得到定理 2.5.3 的一个直接推论.

推论 2.5.1 设当 $T \to \infty$ 时，$a_T = o(T^{\varepsilon})$ 对任何 $\varepsilon > 0$ 成立. 则

$$\lim_{T\to\infty} \sup_{v>0} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2(\log T+\log\hat{v})\}^{1/2}} = 1 \quad \text{a.s.},$$

且对任何固定的 $v>0$ 有

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\overline{\sigma}_1(t,v)\{2\log T\}^{1/2}}=1\quad \text{a.s.},$$

$$\limsup_{T\to\infty}\frac{|X(T,v)-X(T-a_T,v)|}{\overline{\sigma}_1(T,v)\{2\log T\}^{1/2}}=1\quad \text{a.s.}$$

注 2.5.2 由 $X(t,v)$ 关于 t 和 v 的对称性, 我们可以像注 2.5.1 一样给出关于 $X(t,v+u)-X(t,v)$ 对应的结果.

下述定理讨论的是 $X(t,v)$ 同时对两个参数变量的大增量的极限性质.

定理 2.5.4 设 a_T, b_T 和 V_T 为 T 的函数, 满足 $0<a_T\le T$, $0<b_T\le V_T$ 且 $a_T\to\infty$, $b_T\to\infty$ $(T\to\infty)$. 则

$$\limsup_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\sup_{0\le v\le V_T}\sup_{0\le u\le b_T}\frac{|X(R(t,s,v,u))|}{\overline{\sigma}_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\le 1\quad \text{a.s.}\tag{2.5.37}$$

如果, 进一步还存在 $0\le b<1$ 使得对任意的 $\varepsilon>0$ 成立

$$a_Tb_T=o\big((TV_T)^{b+\varepsilon}\big),\quad T\to\infty,\tag{2.5.38}$$

则

$$\liminf_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le v\le V_T}\frac{|X(R(t,a_T,v,b_T))|}{\overline{\sigma}_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\ge\left(\frac{1-b}{1+b}\right)^{1/2}\quad \text{a.s.},\tag{2.5.39}$$

$$\limsup_{T\to\infty}\frac{|X(R(T,a_T,V_T,b_T))|}{\overline{\sigma}_2(T,V_T)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\ge\left(\frac{1-b}{1+b}\right)^{1/2}\quad \text{a.s.}\tag{2.5.40}$$

与推论 2.5.1 类似, 我们有

推论 2.5.2 若在 (2.5.38) 中取 $b=0$, 则有

$$\lim_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\sup_{0\le v\le V_T}\sup_{0\le u\le b_T}\frac{|X(R(t,s,v,u))|}{\overline{\sigma}_2(t,v)\{2\log(TV_T)\}^{1/2}}=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le v\le V_T}\frac{|X(R(t,a_T,v,b_T))|}{\overline{\sigma}_2(t,v)\{2\log(TV_T)\}^{1/2}}=1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty}\frac{|X(R(T,a_T,V_T,b_T))|}{\overline{\sigma}_2(T,V_T)\{2\log(TV_T)\}^{1/2}}=1 \quad \text{a.s.}$$

为证明定理 2.5.3 和 2.5.4, 我们还需要下述有关连续模的结果, 其证明可沿着 (2.5.26) 和 (2.5.30) 的证明路线得到.

引理 2.5.4 对任何满足 $V_T \ge w_T$, $h_T \to 0$ 和 $w_T \to 0$ $(T\to\infty)$ 的正函数 h_T, w_T 和 V_T 我们有

$$\limsup_{T\to\infty}\sup_{v>0}\sup_{0\le t\le T}\sup_{0\le s\le h_T}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h_T,v)\left\{2[\log\frac{T}{h_T}+\log\log\frac{1}{\sigma_2(t,h_T,v)}]\right\}}$$
$$\le 1 \quad \text{a.s,} \tag{2.5.41}$$

$$\limsup_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le h_T}\sup_{0\le v\le V_T}\sup_{0\le u\le w_T}\frac{|X(R(t,s,v,u))|}{\{2h_Tw_T\log(TV_T/h_Tw_T)\}^{1/2}}$$
$$\le 1 \quad \text{a.s.} \tag{2.5.42}$$

下述引理与引理 2.5.1 类似.

引理 2.5.5 对任意的 $0<\varepsilon<1/2$, 存在 $b=b(\varepsilon)>0$ 和 $C=C(\varepsilon)>0$ 使得对任意固定的 $t\ge 0$ 和 $s\ge b$ 有,

$$P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\ge 1+2\varepsilon\right\}\le C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}.$$

证明 注意到 $\overline{\sigma}_1(t,v)$ 为 (2.5.6) 所定义, 当 $(2-e^{-2\alpha t})/2\alpha > 2\beta e^{-2\alpha t}$ $((2-e^{-2\alpha t})/2\alpha \le 2\beta e^{-2\alpha t})$ 时, 它关于 v 单调增加 (单调减少). 不妨设 $\overline{\sigma}_1(t,v)$ 关于 v 单调增加. 对任意的 $\varepsilon>0$, 令 $\theta=\theta(\varepsilon)>1$. 由

$$\overline{\sigma}_1(t,v_k)=\theta^{-k}$$

定义 v_k. 因为当 v 从 0 变到 ∞ 时, $\overline{\sigma}_1(t,v)$ 从 $e^{-2\alpha t}$ 单调增加到 $(2-e^{-2\alpha t})/4\alpha\beta$, 所以 v_k 的个数有限 (比如说 k 是从 k_0 到 k_1). 进一步, 由 $e^{-2\beta v_k'}=\theta^{-k}$ 定义 v_k' $(k=0,1,\cdots)$. 我们把 $\{v_k;k=k_0,\cdots,k_1\}$ 和 $\{v_k';k=0,1,\cdots\}$ 放在一起形成一个新的单调增加的序列 $\{v_k'';k=0,1,\cdots\}$. 由定义, $v_k''\le\frac{k}{2\beta}\log\theta$. 令 $K=[2\beta/\log\theta]+1$,

$$\begin{aligned}Y_{t,s}(v):=&e^{-\alpha(t+s)}(1-e^{\alpha s})X_0+e^{-\alpha(t+s)}(1-e^{\alpha s})\\&\cdot\int_0^t\int_0^v e^{\alpha x+\beta y}dW(x,y)+e^{-\alpha(t+s)}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y).\end{aligned}$$

对固定的 t 和 s, 这是一个具有独立增量的 Gauss 过程. 显然

$$e^{-2\beta v_{k-1}''}EY_{t,s}^2(v_k'')\le\theta\overline{\sigma}_1^2(t,v_k'')\le\theta^2\overline{\sigma}_1{}^2(t,v_{k-1}'').$$

从而, 注意到 $\log x=\log(x\vee e)$, 对充分接近 1 的 $\theta>1$ 成立

$$\begin{aligned}&P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\ge1+2\varepsilon\right\}\\&\le\sum_{k=1}^\infty P\left\{\sup_{v_{k-1}''<v\le v_k''}\frac{|X(t+s,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\ge1+2\varepsilon\right\}\\&\le\sum_{k=1}^\infty P\left\{\sup_{v_{k-1}''<v\le v_k''}\frac{e^{-\beta v_{k-1}''}|Y_{t,s}(v)|}{\overline{\sigma}_1(t,v_{k-1}'')(x^2+2\log v_k'')^{1/2}}\ge1+2\varepsilon\right\}\\&\le\sum_{k=1}^\infty\exp\left\{-\frac12(1+2\varepsilon)\theta^{-2}(x^2+2\log v_{k-1}'')\right\}\\&\le\exp\left\{-\frac12(1+\varepsilon)x^2\right\}\left(K+\sum_{k=K}^\infty\exp\left\{-(1+\varepsilon)\log(\frac{k}{2\beta}\log\theta)\right\}\right)\\&\le c\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}.\end{aligned}$$

定理 2.5.3 的证明 首先, 我们证明 (2.5.33). 对任意给定的充分小的 $h>0$ 有

$$\sup_{v>0}\sup_{0\le t\le T}\frac{\sigma_1(t,h,v)}{\overline{\sigma}_1(t,v)}\le\alpha h+(2\alpha h)^{1/2}\le ch^{1/2},$$

进一步,

$$\sup_{v>0}\sup_{0\le t\le T}\frac{\sigma_1(t,h,v)\{\log\frac{T}{h}+\log\log\frac{1}{\sigma_2(t,h,v)}\}^{1/2}}{\overline{\sigma}_1(t,v)\{\log((T-a_T)a_T)+\log\hat{v}\}^{1/2}}\le ch^{1/3}. \tag{2.5.43}$$

从而, 由引理 2.5.4 的 (2.5.41) 得

$$\limsup_{T\to\infty}\sup_{v>0}\sup_{0\le t\le T}\sup_{0\le s\le h}\frac{|X(t+s,v)-X(t,v)|}{\overline{\sigma}_1(t,v)\{2(\log((T-a_T)a_T)+\log\hat{v})\}^{1/2}}$$
$$\le ch^{1/3}\quad \text{a.s.} \tag{2.5.44}$$

因此, 为证 (2.5.33), 只要证明对任意的 $\epsilon>0$ 有

$$\limsup_{T\to\infty}\sup_{v>0}\max_{0\le j\le j_T}\max_{0\le i\le i_T}\frac{|X((j+i)h,v)-X(jh,v)|}{\overline{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log\hat{v})\}^{1/2}}$$
$$\le 1+\epsilon\quad \text{a.s.}, \tag{2.5.45}$$

其中 $j_T=[(T-a_T)/h]$, $i_T=[a_T/h]$. 对某 $\theta>1$, 记 $A_k=\{T:\theta^k\le a_T\le\theta^{k+1}\}$, $\mathcal{A}=\{k:A_k\ne\emptyset\}$. 设 T_k 为使得 $T_k-a_{T_k}=\max\{T-a_T:T\in A_k\}$ 成立的 T. 则

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{v>0}\max_{0\le j\le j_T}\max_{0\le i\le i_T}\frac{|X((j+i)h,v)-X(jh,v)|}{\overline{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}}\\
&\quad\le\limsup_{\substack{k\to\infty\\ k\in\mathcal{A}}}\sup_{v>0}\sup_{T\in A_k}\max_{0\le j\le j_T}\max_{0\le i\le i_T}|X((j+i)h,v)-X(jh,v)|/\\
&\qquad\cdot\overline{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}\\
&\quad\le\limsup_{\substack{k\to\infty\\ k\in\mathcal{A}}}\sup_{v>0}\max_{0\le j\le j_{T_k}}\max_{0\le i\le\theta^{k+1}/h}|X((j+i)h,v)-X(jh,v)|/\\
&\qquad\cdot\overline{\sigma}_1(jh,v)\{2(\log((j+1)h\theta^k)+\log v)\}^{1/2}.
\end{aligned}\tag{2.5.46}$$

由引理 2.5.5, 并注意到 $E(X(t+s,v)-X(t,v))^2\le\overline{\sigma}_1^2(t,v)$, 对充分大的 k 有

$$P\Big\{\sup_{v>0}\max_{0\le j\le j_{T_k}}\max_{0\le i\le\theta^{k+1}/h}|X((j+i)h,v)-X(jh,v)|/$$

$$\cdot\,\overline{\sigma}_1(jh,v)\big\{2(\log((j+1)h\theta^k)+\log v)\big\}^{1/2}\ge 1+2\varepsilon\Big\}$$

$$\le\sum_{j=0}^{j_{T_k}}\sum_{i=0}^{[\theta^{k+1}/h]}P\left\{\sup_{v>0}\frac{|X((j+i)h,v)-X(jh,v)|}{\overline{\sigma}_1(jh,v)\big\{2(\log((j+1)h\theta^k)+\log v)\big\}^{1/2}}\right.$$

$$\left.\ge 1+2\varepsilon\right\}\le c\theta^{k+1}h^{-1}\sum_{j=0}^{j_{T_k}}\exp\{-(1+\varepsilon)\log((j+1)h\theta^k)\}$$

$$\le ch^{-2-\varepsilon}\theta^{1-\varepsilon k}\sum_{j=0}^{\infty}(j+1)^{-1-\varepsilon},\tag{2.5.47}$$

由此和 (2.5.46) 即得 (2.5.45). (2.5.43) 得证.

下面证明 (2.5.35). 由条件 (2.5.34) 得

$$\limsup_{T\to\infty}\frac{\log((T-a_T)a_T)}{\log(T/a_T)}\le\frac{1+b}{1-b}.\tag{2.5.48}$$

因此, 为证 (2.5.35), 只要证明对任意的 $0<\epsilon<1/4$

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log(T/a_T))^{1/2}}\ge 1-\epsilon\quad\text{a.s.}\tag{2.5.49}$$

令 $B_{nk}=\{T:kh\le a_T<(k+1)h,n-1\le T<n\}$, $a_n'=\inf\{a_T:n-1\le T<n\}$, $a_n^*=\sup\{a_T:n-1\le T<n\}$. 则

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log(T/a_T))^{1/2}}$$

$$\ge\liminf_{n\to\infty}\min_{a_n'/h-1\le k<a_n^*/h}\inf_{T\in B_{nk}}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log(T/a_T))^{1/2}}$$

$$\ge\liminf_{n\to\infty}\min_{a_n'/h-1\le k<a_n^*/h}\sup_{0\le t\le n/2}\frac{|X(t+kh,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log(n/kh))^{1/2}}$$

$$-\limsup_{n\to\infty}\sup_{0\le t\le n}\sup_{0\le s\le h}\frac{|X(t+s,v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log((n-1)/a_n^*))^{1/2}}$$

$$=:I_1-I_2.\tag{2.5.50}$$

由 (2.5.44) 和条件 (2.5.34), 我们得

$$I_2 \leq ch^{1/3} \quad \text{a.s.} \tag{2.5.51}$$

进一步,

$$I_1 \geq \liminf_{n\to\infty} \min_{a_n'/h-1\leq k<a_n^*/h} \max_{0\leq j\leq n/2kh} \frac{X((j+1)kh,v)-X(jkh,v)}{\overline{\sigma}_1(jkh,v)(2\log(n/kh))^{1/2}}. \tag{2.5.52}$$

容易证明对充分大的 k,

$$E\{X((j+1)kh,v)-X(jkh,v)\}\{X((i+1)kh,v)-X(ikh,v)\} \leq 0 \tag{2.5.53}$$

对 $j\neq i$ 成立. 因此, 令 $G_j, j=0,1,\cdots$ 为独立的标准正态变量, 由 Slepian 引理得

$$\begin{aligned}
&P\left\{\min_{a_n'/h-1\leq k<a_n^*/h}\max_{0\leq j\leq n/2kh}\frac{X((j+1)kh,v)-X(jkh,v)}{\overline{\sigma}_1(jkh,v)(2\log(n/kh))^{1/2}}\leq 1-\varepsilon\right\}\\
&\quad\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} P\left\{\max_{0\leq j\leq n/2kh} G_j\leq (1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\right\}\\
&\quad\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \left(P\left\{G_1\leq (1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\right\}\right)^{[n/2kh]+1}\\
&\quad\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \left(1-\exp\left\{-(1-\varepsilon)\log\frac{n}{kh}\right\}\right)^{n/2kh}\\
&\quad\leq n^{b+\delta}h^{-1}\exp\left\{-\frac{1}{2}\Big(\frac{n}{a_n^*}\Big)^{\varepsilon}\right\}\leq n^{b+\delta}h^{-1}\exp\{-n^{(1-b-\delta)\varepsilon}\}
\end{aligned} \tag{2.5.54}$$

对 $0<\delta<1-b$ 成立, 由此和 (2.5.52) 即得 $I_1\geq 1-\varepsilon$. 把 I_1 和 I_2 的估计代入 (2.5.50), 并注意到 h 的任意性, 我们得 (2.5.49), 因此 (2.5.35) 得证.

最后, 我们证明 (2.5.36). 令 $t_0=1$. 由 $t_k=t_{k-1}+a_{t_{k-1}}$ 定义 $t_k, k=1,2,\cdots$. 记 $D_n=\{k:\frac{1}{2}n\leq t_k\leq n-1\}$. 显然, 由条件 (2.5.34), 对 $k\in D_n$ 和任何 $0<\delta<1-b$, 我们有 $a_{t_k}=o(n^{b+\delta})$, 进而有

$$\sum_{k\in D_n} a_{t_k} \ge n-1-\frac{n}{2}-\max_{k\in D_{n-1}} a_{t_k} \ge \frac{1}{3}n$$

对充分大的 n 成立. 此外, 我们有一个与 (2.5.53) 类似的关系式. 因此, 若令 $t'=t'(t)$ 为方程 $t'-a_{t'}=t$ 的解, 则对 $n-1<T\le n$ 有

$$\begin{aligned}
&P\left\{\sup_{T/2\le t\le T}\frac{|X(t+a_{t'},v)-X(t,v)|}{\overline{\sigma}_1(t,v)(2\log T)^{1/2}}\le(1-\varepsilon)(1-b)^{1/2}\right\}\\
&\le P\left\{\max_{k\in D_n}\frac{X(t_k+a_{t'_k},v)-X(t_k,v)}{\overline{\sigma}_1(t_k,v)(2\log(n/a_{t_k}))^{1/2}}\le 1-\varepsilon/2\right\}\\
&\le \prod_{k\in D_n}\Big(1-\exp\Big\{-(1-\varepsilon/2)\log\frac{n}{a_{t_k}}\Big\}\Big)\\
&\le \exp\Big\{-\sum_{k\in D_n}(a_{t_k}/n)^{1-\varepsilon/2}\Big\}\\
&\le \exp\Big\{-c\Big(n/\max_{k\in D_n}a_{t_k}\Big)^{\varepsilon/2}\Big\}\\
&\le \exp(-cn^{(1-b-\delta)\varepsilon/2})\to 0,\quad n\to\infty.
\end{aligned}\tag{2.5.55}$$

从而, 注意到

$$\overline{\sigma}_1^2(t,v)\to(1-e^{-2\beta v})/2\alpha\beta=:\overline{\sigma}_1^2(v),\quad t\to\infty,$$

我们得

$$\begin{aligned}
&P\left\{\sup_{\frac{1}{2}T\le t-a_t\le T}\frac{|X(t,v)-X(t-a_t,v)|}{\overline{\sigma}_1(t,v)\{2\log((t-a_t)a_t)\}^{1/2}}\ge(1-\varepsilon)\Big(\frac{1-b}{1+b}\Big)^{1/2}\right\}\\
&\quad\ge P\left\{\sup_{\frac{1}{2}T\le t\le T}\frac{|X(t+a_{t'},v)-X(t,v)|}{\overline{\sigma}_1(v)(2\log T)^{1/2}}\ge\Big(1-\frac{\varepsilon}{2}\Big)(1-b)^{1/2}\right\}\\
&\quad\to 1,\quad T\to\infty.
\end{aligned}$$

(2.5.36) 得证.

定理 2.5.4 的证明 证明与定理 2.5.3 的类似, 我们只给出不同之处.

由引理 2.5.4 的 (2.5.42) 并注意到 $\inf_{t,v}\overline{\sigma}_2(t,v)>0$, 我们得

$$\limsup_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le h}\sup_{0\le v\le V_T}\sup_{0\le u\le h}\frac{|X(R(t,s,v,u))|}{\overline{\sigma}_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}$$
$$\le ch^{2/3}\quad \text{a.s.}\tag{2.5.56}$$

从而 (2.5.37) 可由下式得到

$$\limsup_{T\to\infty}\max_{0\le j\le j_T}\max_{0\le i\le i_T}\max_{0\le r\le r_T}\max_{0\le l\le l_T}\frac{|X(R(jh,ih,rh,lh))|}{\overline{\sigma}_2(jh,rh)\{2\log(Ta_TV_Tb_T)\}^{1/2}}$$
$$\le 1+\varepsilon\quad \text{a.s.}\tag{2.5.57}$$

对任意的 $\varepsilon>0$ 成立，其中 $j_T=[T/h]$, $i_T=[a_T/h]$, $r_T=[V_T/h]$, $l_T=[b_T/h]$. 对某 $\delta>1$ 令 $A_{kl}=\{T:\theta^k\le a_T<\theta^{k+1},\theta^l\le b_T<\theta^{l+1}\}$. 记 $\mathcal{A}=\{(k,l):A_{kl}\ne\emptyset\}$, $T_{kl}=\sup\{T:T\in A_{kl}\}$, $V_{T'_{kl}}=\sup\{V_T:T\in A_{kl}\}$. 则 (2.5.57) 的左边不超过

$$\limsup_{\substack{k,l\to\infty\\(k,l)\in\mathcal{A}}}\max_{0\le j\le j_{T_{kl}}}\max_{0\le i\le\theta^{k+1}/h}\max_{0\le r\le r_{T'_{kl}}}\max_{0\le w\le\theta^{l+1}/h}$$
$$\frac{|X(R(jh,ih,rh,wh))|}{\overline{\sigma}_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}}.$$

注意到 $EX^2(R(t,s,v,u))\le\overline{\sigma}_2^2(t,v)$, 我们得

$$P\Big\{\max_{0\le j\le j_{T_{kl}}}\max_{0\le i\le\theta^{k+1}/h}\max_{0\le r\le r_{T'_{kl}}}\max_{0\le w\le\theta^{l+1}/h}|X(R(jh,ih,rh,wh))|/$$
$$\overline{\sigma}_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}\ge1+\varepsilon\Big\}$$
$$\le\sum_{j=0}^{j_{T_{kl}}}\sum_{i=0}^{[\theta^{k+1}/h]}\sum_{r=0}^{r_{T'_{kl}}}\sum_{w=0}^{[\theta^{l+1}/h]}P\Big\{|X(R(jh,ih,rh,wh))|/$$
$$\overline{\sigma}_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}\ge1+\varepsilon\Big\}$$
$$\le c\theta^{k+l+2}h^{-2}\sum_{j=0}^{j_{T_{kl}}}\sum_{r=0}^{r_{T'_{kl}}}\exp\Big\{-(1+\varepsilon)\log((j+1)(r+1)h^2\theta^{k+l})\Big\}$$
$$\le ch^{-4-2\varepsilon}\theta^{2-\varepsilon(k+l)}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}(j+1)^{-(1+\varepsilon)}(r+1)^{-(1+\varepsilon)}.$$

由 Borel-Cantelli 引理 (其在两指标情形的推广是显然的) 有 (2.5.57), 从而 (2.5.37) 得证.

现在证明 (2.5.39). 由条件 (2.5.38), (2.5.39) 可由下式得到

$$\liminf_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le v\le V_T}\frac{|X(R(t,a_T,v,b_T))|}{\overline{\sigma}_2(t,v)\{2\log(TV_T/(a_Tb_T))\}^{1/2}} \\ \ge 1-\varepsilon \quad \text{a.s.}$$

令 $B_{mnkl}=\{T: kh\le a_T\le (k+1)h, lh\le b_T\le (l+1)h, m-1\le T<m, n-1\le V_T<n\}$, $a_m^*=\sup\{a_T: m-1\le T<m\}$, $a_m'=\inf\{a_T: m-1\le T<m\}$, $b_n^*=\sup\{b_T: n-1\le V_T<n\}$, $b_n'=\inf\{b_T: n-1\le V_T<n\}$. 则

$$\liminf_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le v\le V_T}\frac{|X(R(t,a_T,v,b_T))|}{\overline{\sigma}_2(t,v)\{2\log(TV_T/(a_Tb_T))\}^{1/2}}$$
$$\ge \liminf_{m,n\to\infty}\min_{\substack{a_m'/h-1\le k<a_m^*/h\\ b_n'/h-1\le l<b_n^*/h}}\inf_{T\in B_{mnkl}}\sup_{\substack{0\le t\le T\\ 0\le v\le V_T}}|X(R(t,kh,v,lh))|/$$
$$\cdot\,\overline{\sigma}_2(t,v)\{2\log(mn/(klh^2))\}^{1/2}$$
$$-\limsup_{m,n\to\infty}\sup_{\substack{0\le t\le m\\ 0\le s\le h}}\sup_{\substack{0\le v\le n\\ 0\le u\le h}}\frac{|X(R(t,s,v,u))|}{\overline{\sigma}_2(t,v)\{2\log((m-1)(n-1)/(a_m^*b_n^*))\}^{1/2}}$$
$$:= J_1-J_2.$$

由 (2.5.56) 和 (2.5.38) 得

$$J_2\le ch^{2/3}\quad \text{a.s.}$$

对 J_1, 我们有

$$J_1\ge \liminf_{m,n\to\infty}\min_{\substack{a_m'/h-1\le k<a_m^*/h\\ b_n'/h-1\le l<b_n^*/h}}\max_{\substack{0\le j\le m/2kh\\ 0\le i\le n/2lh}}X(R(jkh,kh,ilh,lh))/$$
$$\cdot\,\overline{\sigma}_2(jkh,ilh)\{2\log(mn/(klh^2))\}^{1/2}.$$

通过一些初等计算, 可以验证: 对充分大的 k 和 / 或 l, $j\ne p$和 / 或 $i\ne q$, 有

$$EX(R(jkh,kh,ilh,lh))X(R(pkh,kh,qlh,lh))\le 0. \qquad (2.5.58)$$

余下的证明与定理 2.5.3 的类似 (见 (2.5.54)), 因此从略.

模仿 (2.5.36) 的证明, 利用与 (2.5.58) 类似的非正相关性, 我们可证 (2.5.40), 细节从略.

§2.6 带核的两参数 Gauss 过程

在前一节中, 我们研究了由下式定义的两参数 O-U 过程 (OUP_2) $X(t,v)$:

$$X(t,v)=e^{-\alpha t-\beta v}\left\{X_0+\sigma\int_0^t\int_0^v e^{\alpha x+\beta y}\,dW(x,y)\right\}. \tag{2.6.1}$$

若 $X_0=0$, 则 OUP_2 $X(t,v)$ 可以写成

$$X(t,v)=\int_0^t\int_0^v \sigma e^{-\alpha t-\beta v}e^{\alpha x+\beta y}\,dW(x,y). \tag{2.6.2}$$

在第 2.1.5 节, 我们研究了由 (2.1.22) 定义的独立 O-U 过程的无穷级数, 它是过程 $X(\cdot,n)=\sum_{k=1}^n X_k(\cdot)$ 当 $n\to\infty$ 时的极限. 对方程 (2.1.22) 从 $-\infty$ 到 t 进行积分, 那么 O-U 过程 $X_i(\cdot)$ 可以写成

$$X_i(t)=\int_{-\infty}^t \exp(-\lambda_i|t-s|)(2\gamma_i)^{1/2}\,dW_i(s),\quad i=1,2,\cdots, \tag{2.6.3}$$

其中 $\{W_i(t);-\infty<t<\infty\}$ 为独立的 Wiener 过程, 从而我们也有

$$X(t,n)=\sum_{k=1}^n X_k(t)=\sum_{k=1}^n\int_{-\infty}^t \exp(-\lambda_k|t-s|)(2\gamma_k)^{1/2}\,dW_k(s). \tag{2.6.4}$$

后者导致我们去研究下述两参数 Gauss 过程

$$X(t,v)=\int_0^v\int_{-\infty}^t \exp\big(-\lambda(y)(t-x)\big)(2\gamma(y))^{1/2}\,dW(x,y), \tag{2.6.5}$$

其中 $\gamma(y)$ 和 $\lambda(y)$ 假设为 $[0,\infty)$ 上的正的连续函数, $\{W(x,y);-\infty<x,y<\infty\}$ 为标准两参数 Wiener 过程 (见 2.3 节).

综合上述一些过程的表达形式就使得我们去进一步研究如下形式更加一般的两参数 Gauss 过程 $\{X(t,v); t\in\mathcal{R}, v\in\mathcal{R}_+\}$:

$$X(t,v)=\int_0^\infty\int_{-\infty}^\infty \Gamma(t,v,x,y)\,dW(x,y), \tag{2.6.6}$$

其中核函数 $\Gamma(t,v,x,y)$ 假设为在 $\mathcal{R}_+\times\mathcal{R}$ 上关于 (x,y) 平方可积, $W(x,y)$ 为标准两参数 Wiener 过程. 从而 $X(t,v)$ 为零均值 Gauss 过程, 具有协方差函数

$$\operatorname{Cov}\big(X(t,v),X(s,u)\big)=\int_0^\infty\int_{-\infty}^\infty \Gamma(t,v,x,y)\Gamma(s,u,x,y)\,dx\,dy. \tag{2.6.7}$$

令

$$H_1^2(t,s,v)=E\{X(t+s,v)-X(t,v)\}^2, \tag{2.6.8}$$

$$X\big(R(t,s,v,u)\big)=X(t+s,v+u)-X(t,v+u)-X(t+s,v)+X(t,v),$$

$$H_2^2(t,s,v,u)=EX^2\big(R(t,s,v,u)\big), \tag{2.6.9}$$

其中 $R(t,s,v,u)=[t,t+s]\times[v,v+u]$. 易知

$$H_1^2(t,s,v)=\int_0^\infty\int_{-\infty}^\infty \big(\Gamma(t+s,v,x,y)-\Gamma(t,v,x,y)\big)^2\,dxdy, \tag{2.6.10}$$

$$\begin{aligned}H_2^2(t,s,v,u)=\int_0^\infty\int_{-\infty}^\infty \big(&\Gamma(t+s,v+u,x,y)-\Gamma(t,v+u,x,y)\\&-\Gamma(t+s,v,x,y)+\Gamma(t,v,x,y)\big)^2\,dxdy.\end{aligned} \tag{2.6.11}$$

下面是一些例子.

例 2.6.1 若 $\Gamma(t,v,x,y)=I_{(-\infty,t]\times[0,v]}(x,y)$, $-\infty<t<\infty$, $0\le v<\infty$, 则

$$X(t,v)=W(t,v),$$

$$H_1^2(t,s,v)=sv,\ 0\le s<\infty,$$

$$H_2^2(t,s,v,u)=su,\ 0\le s,u<\infty.$$

例 2.6.2 若 $\Gamma(t,v,x,y) = I_{[0,t]\times[0,v]}(x,y) - tI_{[0,1]\times[0,v]}(x,y)$, $0 \le t \le 1$, $0 \le v < \infty$, 则 $X(t,v) = W(t,v) - tW(1,v)$ 为一个 Kiefer 过程 (见 Csörgő 和 Révész (1981) 的 1.15 节), 且

$$H_1^2(t,s,v) = s(1-s)v, \quad 0 \le s \le 1, \quad 0 \le v < \infty,$$

$$H_2^2(t,s,v,u) = s(1-s)u, \quad 0 \le s \le 1, \quad 0 \le u < \infty.$$

例 2.6.3 对 $-\infty < t < \infty, 0 < v < \infty$ 记

$$\Gamma(t,v,x,y) = I_{(-\infty,t]\times(0,v]}(x,y) \exp\big(-\lambda(y)(t-x)\big)\big(2\gamma(y)\big)^{1/2},$$

其中 $\lambda(y)$ 和 $\gamma(y)$ 为在 $[0,\infty)$ 上的正连续函数, 则 $X(t,v)$ 为由 (2.6.5) 所示的 Gauss 过程, 且

$$H_1^2(t,s,v) = 2\int_0^v \frac{\gamma(x)}{\lambda(x)}\Big(1-\exp\big(-\lambda(x)s\big)\Big)\,dx,$$

$$H_2^2(t,s,v,u) = 2\int_v^{v+u} \frac{\gamma(x)}{\lambda(x)}\Big(1-\exp\big(-\lambda(x)s\big)\Big)\,dx.$$

例 2.6.4 对 $-\infty < t < \infty, 0 < v < \infty$ 记

$$\Gamma(t,v,x,y) = \sum_{k=0}^{\infty} \phi_k(v) I_{(-\infty,t]\times(k,k+1]}(x,y) \exp\big(-\lambda_k(t-x)\big)\big(2\gamma_k\big)^{1/2},$$

则

$$H_1^2(t,s,v) = 2\sum_{k=0}^{\infty} \phi_k^2(v)(1-e^{-\lambda_k s})\Big(\frac{\gamma_k}{\lambda_k}\Big),$$

$$H_2^2(t,s,v,u) = 2\sum_{k=0}^{\infty} \big(\phi_k(v+u) - \phi_k(v)\big)^2(1-e^{-\lambda_k s})\frac{\gamma_k}{\lambda_k},$$

$$X(t,v) = \sum_{k=0}^{\infty} \phi_k(v) X_k(t),$$

其中 $\{X_k(t); -\infty < t < \infty\}$ 为独立的 O-U 过程, 它们的系数为 $\gamma_k \ge 0$, $\lambda_k > 0$.

Csörgő 和 Lin (1991), 林正炎 (1991) 及 Csörgő, Lin 和 Shao (1994b) 研究了 (2.6.6) 所定义的 $X(t,v)$ 的样本轨道性质. 在这一节里我们给出关于 $X(t,v)$ 增量的大偏差的一些结果. 利用这些大偏差我们建立了有关 $X(t,v)$ 轨道性质的定理. 在这方面的早期研究读者可参看 Csörgő 和 Lin (1991) 及林正炎 (1991). 较一般的结果可参看 Csörgő, Lin 和 Shao (1994b).

记

$$
\begin{aligned}
H^2(t,s,v,u) &= E\big(X(t+s,v+u)-X(t,v)\big)^2 \\
&= \int_0^\infty \int_{-\infty}^\infty \big(\Gamma(t+s,v+u,x,y)-\Gamma(t,v,x,y)\big)^2\,dxdy. \quad (2.6.12)
\end{aligned}
$$

令

$$\phi(h,B)=\sup_{0\le s,u\le h,|t|\le B,|v|\le B} H(t,s,v,u).$$

由定理 2.1.3, 我们有: 若

$$\int_0^\infty \phi(e^{-y^2},B)dy<\infty \quad \forall B>0, \qquad (2.6.13)$$

则 $X(t,v)$ 几乎处处连续. 由于我们只对 $X(t,v)$ 的连续模和其增量的一些轨道性质感兴趣, 为了叙述方便, 在这一节中我们总设 $X(t,v)$ 几乎处处连续. 我们还设 $H_1(t,s,v)$ 对 s 非降, $H_2(t,s,v,u)$ 对 s 和 u 非降, a_T、b_T, c_T 和 d_T 为 T 的连续函数, $H_1(t,s,T)$ 和 $H_2(t,s,v,u)$ 为 T、s, u 的连续函数.

2.6.1 大偏差

命题 2.6.1 设 $A\subset\mathcal{R}_+$, $s_0>0$, $b_{1,T}\le b_{2,T}$. 假设

$$
\begin{aligned}
&E\big(X(t+s,v)-X(t,v)\big)\big(X(t+s,u)-X(t,u)\big) \\
&\quad \ge E\big(X(t+s,u)-X(t,u)\big)^2 \qquad (2.6.14)
\end{aligned}
$$

对任何 $v\ge u$ 和 t,s 成立, 且存在正数 c_0 和 α 使得

$$\frac{H_1(t,s,T)}{s^\alpha}\le c_0\frac{H_1(t,s_1,T)}{s_1^\alpha} \qquad (2.6.15)$$

对任何 $T \in A$, $b_{1,T} \le t \le b_{2,T}$, $0 \le s \le s_1 \le s_0$ 成立. 则对任意的 $0 < \varepsilon < 1/(1+c_0^{1+\alpha})$, 存在正只依赖于 α, c_0, ε 的常数 $C(\varepsilon)$ 使得对任何 $x \ge 1$ 有

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\le t\le b_{2,T}}\sup_{0\le s\le s_0}\frac{|X(t+s,T)-X(t,T)|}{x\{H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\}}\right.$$
$$\left.\ge 1+\varepsilon\right\}\le C(\varepsilon)\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp\left(-\frac{x^2}{2}\right), \quad (2.6.16)$$

其中 $T^* = \sup\{T : T \in A\}$.

证明 在证明 (2.6.16) 前, 我们先证明: 对任意固定的 t 和 s

$$P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T^*)y}\ge 1\right\}\le 8\exp(-y^2/2) \quad (2.6.17)$$

对任何 $y > 0$ 和 $H_1^*(t,s,T^*) \ge H_1(t,s,T^*)$ 成立.

令 $Y(T)$ 为一个独立增量的 Gauss 过程, 满足 $Y(T) \overset{\mathcal{D}}{=} X(t+s,T) - X(t,T)$. 则 $EY^2(T) = H_1^2(t,s,T)$ 且由 (2.6.14),

$$\begin{aligned}EY(T)Y(T') &= H_1^2(t,s,T')\\ &\le E\big(X(t+s,T)-X(t,T)\big)\big(X(t+s,T')-X(t,T')\big)\end{aligned}$$

对任何 $T > T'$ 成立. 由 (2.6.14) 还有 $H_1(t,s,T)$ 对 T 非降. 由 Slepian 引理得

$$\begin{aligned}&P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T^*)y}\ge 1\right\}\\
&\le P\left\{\sup_{T\in A}\frac{X(t+s,T)-X(t,T)}{H_1^*(t,s,T^*)y}\ge 1\right\}\\
&\quad + P\left\{\sup_{T\in A}\frac{-\big(X(t+s,T)-X(t,T)\big)}{H_1^*(t,s,T^*)y}\ge 1\right\}\\
&\le P\left\{\sup_{T\in A}\frac{Y(T)}{H_1^*(t,s,T^*)y}\ge 1\right\}+P\left\{\sup_{T\in A}\frac{-Y(T)}{H_1^*(t,s,T^*)y}\ge 1\right\}\\
&\le 2P\left\{\sup_{T\in A}\frac{|Y(T)|}{H_1^*(t,s,T^*)y}\ge 1\right\}\le 8\exp\left(-y^2/2\right),\end{aligned}$$

(2.6.17) 得证. 我们现在证明 (2.6.16). 令 $K=2^{2^k}$,

$$t_{j+k}=\left(\left[\frac{t2^{2^{j+k}}}{s_0}\right]+1\right)s_0/2^{2^{j+k}},\quad j=0,1,2,\cdots.$$

由于我们已经假设 $X(\cdot,\cdot)$ 几乎处处连续, 可写

$$\begin{aligned}|X(t+s,T)-X(t,T)|\le{}&|X\big((t+s)_k,T\big)-X(t_k,T)|\\&+\sum_{j=0}^{\infty}|X\big((t+s)_{k+j+1},T\big)-X\big((t+s)_{k+j},T\big)|\\&+\sum_{j=0}^{\infty}|X\big(t_{k+j+1},T\big)-X\big(t_{k+j},T\big)|.\end{aligned}\tag{2.6.18}$$

由 $H_1(t,s,T)$ 和 t_{k+j} 的定义, 显然有

$$\begin{aligned}&H_1(t_k,(t+s)_k-t_k,T)\le H_1(t,t_k-t,T)+H_1(t,(t+s)_k-t,T)\\&\quad\le H_1(t,s_0,T)+H_1(t,s_0/K,T)+H_1(t+s,s_0/K,T),\end{aligned}\tag{2.6.19}$$

$$\begin{aligned}&H_1\big((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},T\big)\\&\qquad\le 2H_1(t+s,s_0/2^{2^{k+j}},T),\end{aligned}\tag{2.6.20}$$

$$H_1\big(t_{k+j+1},t_{k+j}-t_{k+j+1},T\big)\le 2H_1(t,s_0/2^{2^{k+j}},T).\tag{2.6.21}$$

由 (2.6.19) 和 (2.6.17) 得

$$\begin{aligned}&P\Big\{\sup_{T\in A}\sup_{b_{1,T}\le T\le b_{2,T}}\sup_{0\le s\le s_0}|X\big((t+s)_k,T\big)-X(t_k,T)|/\\&\qquad\Big\{\big(H_1(t,s_0,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*)\big)x\Big\}\ge 1\Big\}\\&\quad\le 8\cdot 2^{2^{k+1}}\sup_{T\in A}\Big(\frac{b_{2,T}-b_{1,T}}{s_0}+1\Big)\exp(-x^2/2).\end{aligned}\tag{2.6.22}$$

同理, 由 (2.6.20), (2.6.21) 和 (2.6.17), 对任意的 $x_j>0$ 有

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\le T\le b_{2,T}}\sup_{0\le s\le s_0}|X((t+s)_{k+j+1},T)-X((t+s)_{k+j},T)|/\right.$$

$$\left.\{2H_1(t+s,s_0/2^{2^{k+j}},T^*)x_j\}\ge 1\right\}$$

$$\le 82^{2^{k+j+1}}\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x_j^2/2), \tag{2.6.23}$$

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\le T\le b_{2,T}}\sup_{0\le s\le s_0}|X(t_{k+j+1},T)-X(t_{k+j},T)|/\right.$$

$$\left.\{2H_1(t,s_0/2^{2^{k+j}},T^*)x_j\}\ge 1\right\}$$

$$\le 82^{2^{k+j+1}}\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x_j^2/2). \tag{2.6.24}$$

由 (2.6.22)—(2.6.24) 得

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\le T\le b_{2,T}}\sup_{0\le s\le s_0}|X(t+s,T)-X(t,T)|/\right.$$

$$\left\{(H_1(t,s_0,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*))x\right.$$

$$\left.\left.+2\sum_{j=0}^{\infty}(H_1(t+s,s_0/2^{2^{k+j}},T^*)+H_1(t,s_0/2^{2^{k+j}},T^*))x_j\right\}\ge 1\right\}$$

$$\le 8\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\left(2^{2^{k+1}}\exp(-x^2/2)\right.$$

$$\left.+\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(-x_j^2/2)\right). \tag{2.6.25}$$

令 $x_j^2=x^2+2^{k+j+2}$. 由 (2.6.15), 对充分大的 k 和任何 $x\ge 1$ 有

$$((H_1(t,s,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*))x$$

$$+2\sum_{j=0}^{\infty}(H_1(t+s,s_0/2^{2^{k+j}},T^*)+H_1(t,s_0/2^{2^{k+j}},T^*))x_j$$

$$\le \left(H_1(t,s_0,T^*)+c_0\left(\frac{1}{K}\right)^{\alpha}H_1(t,s_0,T^*)x\right.$$

$$
\begin{aligned}
&+\left(\frac{1}{\varepsilon K}\right)^{\alpha} c_0 H_1(t+s,\varepsilon s_0,T^*)\Big)x\\
&+2c_0\sum_{j=0}^{\infty}\left(\frac{1}{\varepsilon 2^{2^{k+j}}}\right)^{\alpha} H_1(t+s,\varepsilon s_0,T^*)\left(x+2^{\frac{k+j+1}{2}}\right)\\
&+2c_0\sum_{j=0}^{\infty}\left(\frac{1}{2^{2^{k+j}}}\right)^{\alpha} H_1(t,s_0,T^*)\left(x+2^{\frac{k+j+1}{2}}\right)\\
\leq &\left(H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\right)\\
&\cdot\left(x\left(1+3c_0\left(1+\frac{1}{\varepsilon^{\alpha}}\right)\right)\sum_{j=0}^{\infty}2^{-\alpha 2^{k+j}}\right)+\\
&2c_0\sum_{j=0}^{\infty}\left(\frac{1}{\varepsilon^{\alpha}}+1\right)2^{-\alpha 2^{k+j}+\frac{k+j+1}{2}}\Big)\\
\leq &\left(H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\right)(1+\varepsilon)x, \qquad (2.6.26)
\end{aligned}
$$

和

$$
\begin{aligned}
&\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp\left(-x_j^2/2\right)\\
&\qquad=\exp\left(-\frac{x^2}{2}\right)\sum_{j=0}^{\infty}(2/e)^{2^{k+j+1}}\leq C(\varepsilon)\exp\left(-\frac{x^2}{2}\right). \quad (2.6.27)
\end{aligned}
$$

综合 (2.6.25)—(2.6.27) 得

$$
\begin{aligned}
&P\left\{\sup_{T\in A}\sup_{b_{1,T}\leq t\leq b_{2,T}}\sup_{0\leq s\leq s_0}\frac{|X(t+s,T)-X(t,T)|}{x\{H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\}}\right.\\
&\qquad\left.\geq 1+\varepsilon\right\}\leq C(\varepsilon)\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp\left(-x^2/2\right),
\end{aligned}
$$

这就是所要证的.

为了研究 $X(t,v)$ 同时关于 t 和 v 的增量，我们给出另一个关于 $X(t,v)$ 的大偏差结果. 记

$$
\begin{aligned}
&H_{21}(t,s,v,u,K)\\
=&2H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big)+2H_2\Big(t,s,v+u-\frac{c_T}{K},\frac{2c_T}{K}\Big)\\
&+H_2\Big(t,\frac{a_T}{K},v,c_T\Big(1+\frac{2}{K}\Big)\Big)+2H_2\Big(t+s,\frac{a_T}{K},v,c_T\Big(1+\frac{2}{K}\Big)\Big).
\end{aligned}
$$

这里以及本节的余下部分中，若 $v<0$，记

$$
H_2(t,s,v,u)=H_2(t,s,0,u).
$$

命题 2.6.2 设 $H_2(t,s,v,u)$ 对 s 和 u 非降. 假设对任何 $t,s,a\ge v\ge v'>0$ 有

$$
\begin{aligned}
&EX\big(R(t,s,v',a-v')\big)X\big(R(t,s,v,a-v)\big)\\
&\quad\ge EX^2\big(R(t,s,v,a-v)\big),
\end{aligned}
\tag{2.6.28}
$$

且存在正数 c_0 和 α，使得对任何 $0\le s\le s_1\le a_T$，$0\le v\le d_T+c_T$，$0\le u\le 2c_T$，$|t|\le b_T$ 成立

$$
\frac{H_2(t,s,v,u)}{s^\alpha}\le c_0\frac{H_2(t,s_1,v,u)}{s_1^\alpha}.
\tag{2.6.29}
$$

则对任意的 $0<\varepsilon<1$，存在只依赖于 ε，c_0，α 的常数 $C(\varepsilon)$，使得

$$
\begin{aligned}
&P\Big\{\sup_{0\le v\le d_T}\ \sup_{0\le u\le c_T}\ \sup_{|t|\le b_T}\ \sup_{0\le s\le a_T}|X\big(R(t,s,v,u)\big)|/ \qquad (2.6.30)\\
&\quad\big\{H_2(t,a_T,v,c_T(1+\varepsilon))+3H_2(t,a_T,v-\varepsilon c_T,2\varepsilon c_T)\\
&\quad+3H_2(t,a_T,v+u-\varepsilon c_T,2\varepsilon c_T)+H_2(t,\varepsilon a_T,v-\varepsilon c_T,c_T(1+\varepsilon))\\
&\quad+H_2(t,\varepsilon a_T,v+u-\varepsilon c_T,c_T(1+\varepsilon))\\
&\quad+H_2(t+s,\varepsilon a_T,v-\varepsilon a_T,c_T(1+\varepsilon))\\
&\quad+H_2(t+s,\varepsilon a_T,v+u-\varepsilon a_T,c_T(1+\varepsilon))\big\}\ge(1+\varepsilon)x\Big\}\\
&\le C(\varepsilon)\Big(\frac{d_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)e^{-x^2/2}
\end{aligned}
$$

对任何 $x\ge 1$ 成立.

证明 令

$$t_{k+j}=\Big(\Big[\frac{t2^{2^{k+j}}}{a_T}\Big]+1\Big)a_T/2^{2^{k+j}},$$

$$v'_{k+j}=\Big(\Big[\frac{v2^{2^{k+j}}}{c_T}\Big]+1\Big)c_T/2^{2^{k+j}}.$$

我们有

$$\begin{aligned}
&\big|X\big(R(t,s,v,u)\big)\big|\\
\le&\big|X\big(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\big|X\big(R(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\big|X\big(R(t,t_k-t,v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\big|X\big(R(t,s,v,v'_k-v)\big)\big|+\big|X\big(R(t,s,v+u,(v+u)'_k-(v+u))\big)\big|\\
\le&\big|X\big(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\sum_{j=0}^{\infty}\big|X\big(R((t+s)_{k+j+1},(t+s)_{k+j}\\
&\qquad-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\sum_{j=0}^{\infty}\big|X\big(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k)\big)\big|\\
&+\big|X\big(R(t,s,v,v'_k-v)\big)\big|+\big|X\big(R(t,s,v+u,(v+u)'_k-(v+u))\big)\big|.
\end{aligned}\tag{2.6.31}$$

由 (2.6.28), 对任意的 $v'\ge v$, $v+u\ge v'+u'$ 有

$$H_2(t,s,v,u)\ge H_2(t,s,v',u').\tag{2.6.32}$$

由 (2.6.32) 得

$$\begin{aligned}
&H_2(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\\
\le&H_2(t,s,v,u)+H_2(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k)\\
&+H_2(t,t_k-t,v'_k,(v+u)'_k-v'_k)\\
&+H_2\big(t,s,v,v'_k-v\big)+H_2\big(t,s,v+u,(v+u)'_k-(v+u)\big)\\
\le&H_2(t,s,v,u)+H_{21}(t,s,v,u,K).
\end{aligned}$$

同样，我们有

$$
\begin{aligned}
&H_2\big((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v_k',(v+u)_k'-v_k'\big)\\
&\qquad\le 2H_2\Big(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big),\\
&H_2\big(t_{k+j+1},t_{k+j}-t_{k+j+1},v_k',(v+u)_k'-v_k'\big)\\
&\qquad\le 2H_2\Big(t,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big).
\end{aligned}
$$

从而，对任何 $x>0$ 和 $x_j>0$ 有

$$
\begin{aligned}
&P\Bigg\{\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\\
&\qquad\frac{\Big|X\Big(R\big(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k'\big)\Big)\Big|}{x\big(H_2(t,s,v,u)+H_{21}(t,s,v,u,K)\big)}\ge 1\Bigg\}\\
&\qquad\le 4\cdot 2^{2^{k+2}}\Big(\frac{d_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)e^{-x^2/2}, \qquad (2.6.33)
\end{aligned}
$$

$$
\begin{aligned}
&P\Bigg\{\Bigg\{\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\Bigg[\sum_{j=0}^{\infty}\Big|X\Big(R\big((t+s)_{k+j+1},(t+s)_{k+j}-\\
&(t+s)_{k+j+1},v_k',(v+u)_k'-v_k'\big)\Big)\Big|\Bigg]\Big/\Bigg[2\sum_{j=0}^{\infty}x_jH_2(t+s,a_T/2^{2^{k+j}},v,\\
&c_T(1+1/K))\Bigg]\Bigg\}\ge 1\Bigg\}\le 4\Big(\frac{d_T}{c_T}+1\Big)\Big(\frac{b_T}{c_T}+1\Big)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2},
\end{aligned}
$$

(2.6.34)

$$
\begin{aligned}
&P\Bigg\{\Bigg\{\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\Bigg[\sum_{j=0}^{\infty}\Big|X\Big(R\big(t_{k+j+1},t_{k+j}-t_{k+j+1},\\
&v_k',(v+u)_k'-v_k'\big)\Big)\Big|\Bigg]\Big/\Bigg[2\sum_{j=0}^{\infty}x_jH_2(t,a_T/2^{2^{k+j}},v,c_T(1+1/K))\Bigg]\Bigg\}\\
&\ge 1\Bigg\}\le 4\Big(\frac{d_T}{c_T}+1\Big)\Big(\frac{b_T}{a_T}+1\Big)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2}. \qquad (2.6.35)
\end{aligned}
$$

令 $x_j^2 = x^2 + 2^{k+j-2}$, 得

$$\sum_{j=0}^{\infty} 2^{2^{k+1}+2^{k+j+1}} e^{-x_j^2/2} \le 2K^2 e^{-x^2/2} \tag{2.6.36}$$

考察 $X\big(R(t,s,v,v_k'-v)\big)$ 和 $X\big(R(t,s,v+u,(v+u)_k'-(v+u))\big)$. 对任何 $y>0$ 有

$$P\left\{\sup_{0\le v\le d_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\left|X\big(R(t,s,v,v_k'-v)\big)\right|\ge y\right\}$$

$$\le P\left\{\max_{0\le i\le \frac{d_T}{c_T}K}\ \sup_{\frac{ic_T}{K}\le v\le\frac{(i+1)c_T}{K}}\ \sup_{|t|\le b_T}\ \sup_{0\le s\le a_T}\left|X\big(R(t,s,v,v_k'-v)\big)\right|\ge y\right\}$$

$$\le \sum_{i=0}^{[\frac{d_T}{c_T}K]} \mathrm{P}\left\{\sup_{\frac{ic_T}{K}\le v\le\frac{(i+1)c_T}{K}}\ \sup_{|t|\le b_T}\ \sup_{0\le s\le a_T}\left|X\Big(R\Big(t,s,v,\frac{(i+1)c_T}{K}-vg\Big)\Big)\right|\right.$$

$$\left.\ge y\right\}. \tag{2.6.37}$$

令 $d_i=(i+1)c_T/K$. 我们下面证明, 对任何固定的 t,s,

$$P\left\{\sup_{\frac{ic_T}{K}\le v\le\frac{(i+1)c_T}{K}}\frac{\left|X\big(R(t,s,v,d_i-v)\big)\right|}{H_2(t,s,v-\frac{c_T}{K},\frac{2c_T}{K})}\ge y\right\}\le 4\exp(-y^2/2) \tag{2.6.38}$$

对任何 $y>0$ 成立.

设 $Y(v)$ 为一个独立增量过程, 满足 $Y(d_i-v)\overset{\mathcal{D}}{=}X\big(R(t,s,v,d_i-v)\big)$ 对 $ic_T/K\le v\le(i+1)c_T/K$ 成立. 则对任何 $v>v'$ 有

$$\begin{aligned}&EY(d_i-v)Y(d_i-v')\\&\quad=EY^2(d_i-v)=EX^2\big(R(t,s,v,d_i-v)\big)\\&\quad\le EX\big(R(t,s,v,d_i-v)\big)X\big(R(t,s,v',d_i-v')\big),\end{aligned}$$

其中最后一个不等式由 (2.6.28) 得到. 由 (2.6.32) 有

$$H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big)\ge H_2\Big(t,s,\frac{ic_T}{K},\frac{c_T}{K}\Big) \tag{2.6.39}$$

对任何 $ic_T/K\le v\le(i+1)c_T/K$ 成立. 由 Slepian 引理和 (2.6.39)

得

$$P\left\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\frac{\left|X\left(R(t,s,v,d_i-v)\right)\right|}{H_2(t,s,v-c_T/K,2c_T/K)}\ge y\right\}$$

$$\le P\left\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\frac{X\left(R(t,s,v,d_i-v)\right)}{H_2(t,s,v-c_T/K,2c_T/K)}\ge y\right\}$$

$$+P\left\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\frac{-X\left(R(t,s,v,d_i-v)\right)}{H_2(t,s,v-c_T/K,2c_T/K)}\ge y\right\}$$

$$\le P\left\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\frac{Y(d_i-v)}{H_2(t,s,v-c_T/K,2c_T/K)}\ge y\right\}$$

$$+P\left\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\frac{-Y(d_i-v)}{H_2(t,s,v-c_T/K,2c_T/K)}\ge y\right\}$$

$$\le 2P\left\{\frac{|Y(d_i-ic_T/K)|}{H_2(t,s,ic_T/K,c_T/K)}\ge y\right\}$$

$$=2P\left\{\frac{X\left(R(t,s,ic_T/K,d_i-ic_T/K)\right)}{H_2(t,s,ic_T/K,c_T/K)}\ge y\right\}$$

$$\le 4\exp(-y^2/2),$$

由此，(2.6.38) 得证.

沿着 (2.6.25)—(2.6.26) 的证明路线，由 (2.6.38) 我们可得

$$P\Bigg\{\sup_{ic_T/K\le v\le (i+1)c_T/K}\ \sup_{|t|\le b_T}\ \sup_{0\le s\le a_T}\left|X\left(R(t,s,v,d_i-v)\right)\right|/$$

$$\left[xI_1(t,s,v,K)+I_2(t,s,v,K)\right]\ge 1\Bigg\}\le 8\cdot 2^{2^{k+1}}\left(\frac{b_T}{a_T}+1\right)\exp(-x^2/2), \tag{2.6.40}$$

其中

$$\begin{aligned}I_1(t,s,v,K)=&H_2\Big(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\Big)\\&+16\sum_{j=0}^{\infty}H_2\Big(t,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T(1+\frac{1}{K})\Big)\\&+16\sum_{j=0}^{\infty}H_2\Big(t+s,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T(1+\frac{1}{K})\Big),\end{aligned}$$

$$
\begin{aligned}
I_2(t,s,v,K) =& 40\sum_{j=0}^{\infty} H_2\Big(t, \frac{a_T}{2^{2^{k+j}}}, v-\frac{c_T}{K}, c_T(1+\frac{1}{K})\Big)2^{\frac{k+j+1}{2}} \\
&+\sum_{j=0}^{\infty} H_2\Big(t+s, \frac{a_T}{2^{2^{k+j}}}, v-\frac{c_T}{K}, c_T(1+\frac{1}{K})\Big)2^{\frac{k+j+1}{2}}.
\end{aligned}
$$

综合 (2.6.37) 和 (2.6.40) 得

$$
\begin{aligned}
&P\left\{\sup_{0\le v\le d_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{\big|X\big(R(t,s,v,v_k'-v)\big)\big|}{xI_1(t,s,v,K)+I_2(t,s,v,K)}\ge 1\right\} \\
&\quad\le 16\cdot 2^{2^{k+2}}\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{d_T}{c_T}+1\Big)\exp(-x^2/2). \qquad (2.6.41)
\end{aligned}
$$

同理，我们有

$$
\begin{aligned}
&P\Big\{\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v+u,(v+u)_k' \\
&\quad -(v+u))\big)\big|\Big/\Big[xI_1(t,s,v+u,K)+I_2(t,s,v+u,K)\Big]\ge 1\Big\} \\
&\quad\le 16\cdot 2^{2^{k+2}}\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{d_t}{c_T}+1\Big)\exp(-x^2/2). \qquad (2.6.42)
\end{aligned}
$$

综合 (2.6.33)—(2.6.36) 和 (2.6.41)—(2.6.42) 得

$$
\begin{aligned}
&P\Big\{\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,v)\big)\big|\Big/\Big\{x\big(H_2(t,s,v,u) \\
&\quad + H_{21}(t,s,v,u,K)\big)+\sum_{j=0}^{\infty}2x_jH_2\Big(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big) \\
&\quad +\sum_{j=0}^{\infty}2x_jH_2\Big(t,\frac{a_T}{2^{2^{k+j}}},v,c_T\Big(1+\frac{2}{K}\Big)\Big)+xI_1(t,s,v,K) \\
&\quad\quad +I_2(t,s,v,K)+xI_1(t,s,v+u,K)+I_2(t,s,v+u,K)\Big\}\ge 1\Big\} \\
&\le 52\cdot 2^{2^{k+2}}\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{d_T}{c_T}+1\Big)\exp(-x^2/2). \qquad (2.6.43)
\end{aligned}
$$

沿着 (2.6.26) 的证明路线，由 (2.6.29) 我们可以证明对充分大的 k 和任何 $x \geq 1$ 有

$$
\begin{aligned}
& x\big(H_2(t,s,v,u)+H_{21}(t,s,v,u,K)\big) \\
& \quad + \sum_{j=0}^{\infty} 2x_j H_2\Big(t+s, \frac{a_T}{2^{2^{k+j}}}, v, c_T\Big(1+\frac{2}{K}\Big)\Big) \\
& \quad + \sum_{j=0}^{\infty} 2x_j H_2\Big(t, \frac{a_T}{2^{2^{k+j}}}, v, c_T\Big(1+\frac{2}{K}\Big)\Big) + xI_1(t,s,v,K) \\
& \quad + I_2(t,s,v,K) + xI_1(t,s,v+u,K) + I_2(t,s,v+u,K) \\
& \leq (1+\varepsilon)x\Big\{H_2(t,a_T,v,c_T(1+\varepsilon)) + 3H_2(t,a_T,v-\varepsilon c_T,2\varepsilon c_T) \\
& \quad + 3H_2(t,a_T,v+u-\varepsilon c_T,2\varepsilon c_T) + H_2(t,\varepsilon a_T,v-\varepsilon c_T,c_T(1+\varepsilon)) \\
& \quad + H_2(t,\varepsilon a_T,v+u-\varepsilon c_T,c_T(1+\varepsilon)) \\
& \quad + H_2(t+s,\varepsilon a_T,v-\varepsilon a_T,c_T(1+\varepsilon)) \\
& \quad + H_2(t+s,\varepsilon a_T,v+u-\varepsilon a_T,c_T(1+\varepsilon)),
\end{aligned} \tag{2.6.44}
$$

从而由 (2.6.43) 和 (2.6.44), (2.6.30) 得证.

类似地，可以证明

命题 2.6.3 设 $A \in \mathcal{R}_+$, $a_0 > 0$, $c_0 > 0$. 假设 $H_2(t,s,v,u)$ 对 s 和 u 非降，对任何 $t, s, a \geq v \geq v' > 0$,

$$
\begin{aligned}
& EX\big(R(t,s,v',a-v')\big)X\big(R(t,s,v,a-v)\big) \\
& \qquad \geq EX^2\big(R(t,s,v,a-v)\big)
\end{aligned}
$$

成立，并且存在正数 c_1 和 α, 使得对任何 $T \in A$, $0 \leq s \leq s_1 \leq a_0$, $0 \leq v \leq d_T + c_0$, $0 \leq u \leq 2c_0$, $|t| \leq b_T$ 成立

$$
\frac{H_2(t,s,v,u)}{s^\alpha} \leq c_1 \frac{H_2(t,s_1,v,u)}{s_1^\alpha}.
$$

则对任意的 $0 < \varepsilon < 1$, 存在只依赖于 ε, c_1, α 的常数 $C(\varepsilon)$, 使得

对任何 $x \geq 1$ 有

$$
\begin{aligned}
P\Big\{ & \sup_{T\in A}\sup_{0\le v\le d_T}\sup_{0\le u\le c_0}\sup_{0\le s\le b_T}\sup_{|t|\le a_0}\big|X\big(R(t,s,v,u)\big)\big|/ \\
& \big\{H_2(t,a_0,v,c_0(1+\varepsilon))+3H_2(t,a_0,v-\varepsilon c_0,2\varepsilon c_0) \\
& +3H_2(t,a_0,v+u-\varepsilon c_0,2\varepsilon c_0)+H_2(t,\varepsilon a_0,v-\varepsilon c_0,c_0(1+\varepsilon)) \\
& +H_2(t,\varepsilon a_0,v+u-\varepsilon c_0,c_0(1+\varepsilon)) \\
& +H_2(t+s,\varepsilon a_0,v-\varepsilon a_0,c_0(1+\varepsilon)) \\
& +H_2(t+s,\varepsilon a_0,v+u-\varepsilon a_0,c_0(1+\varepsilon))\big\}\geq(1+\varepsilon)x\Big\} \\
\leq{} & C(\varepsilon)\sup_{T\in A}\Big(\frac{d_T}{c_0}+1\Big)\Big(\frac{b_T}{a_0}+1\Big)e^{-x^2/2}.
\end{aligned}
$$

2.6.2 样本轨道性质

我们现在利用关于 (2.6.6) 所定义的过程 $X(t,v)$ 的大偏差结果，证明它的一些样本轨道性质.

定理 2.6.1 设 $H_1(t,s,T)=H_1(0,s,T)=:H_0(s,T)$ ($\forall s>0,|t|\leq b_T+a_T$), 对任何 $v\geq u$ 和 t,s 成立

$$
\begin{aligned}
& E\big(X(t+s,v)-X(t,v)\big)\big(X(t+s,u)-X(t,u)\big) \\
& \quad\geq E\big(X(t+s,u)-X(t,u)\big)^2, \qquad (2.6.45)
\end{aligned}
$$

并且存在正数 c_0 和 α 使得对任何 $0\leq s\leq s_1\leq a_T$ 成立

$$\frac{H_0(s,T)}{s^\alpha}\leq c_0\frac{H_0(s_1,T)}{s_1^\alpha}. \qquad (2.6.46)$$

进一步假设

$$\log\log\Big(a_T+\frac{1}{b_T}\Big)=o\Big(\log\frac{b_T}{a_T}\Big),\quad T\to\infty, \qquad (2.6.47)$$

$$\log\log\Big(H_0(a_T,T)+\frac{1}{H_0(a_T,T)}\Big)=o\Big(\log\frac{b_T}{a_T}\Big),\quad T\to\infty \quad (2.6.48)$$

且对任何 $j \neq l$, $s>0$ 有

$$E\big(X((j+1)s,v)-X(js,v)\big)\big(X((l+1)s,v)-X(ls,v)\big) \leq 0. \quad (2.6.49)$$

则有

$$\lim_{T\to\infty} \sup_{0\leq t\leq b_T} \frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}} = 1 \quad \text{a.s.}, \qquad (2.6.50)$$

$$\lim_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}} = 1 \quad \text{a.s.} \quad (2.6.51)$$

证明 首先证明

$$\limsup_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}} \leq 1 \quad \text{a.s.} \qquad (2.6.52)$$

对任意的 $0<\varepsilon<1/2$, 由 (2.6.46) 存在常数 N 使得

$$\frac{H_0(a_T/N,T)}{H_0(a_T,T)} \leq \varepsilon. \qquad (2.6.53)$$

令 $1<\theta<\min(1+\frac{1}{N},1+\frac{\varepsilon^2}{38})$. 记

$$A_i=\{T:\theta^i<a_T\leq\theta^{i+1}\},\quad -\infty<i<\infty,$$

$$B_j=\{T:\theta^j<1+b_T/a_T\leq\theta^{j+1}\},\quad j=0,1,2,\cdots,$$

$$C_{ki}=\{T:\theta^k<H_0(\theta^{i+1},T)\leq\theta^{k+1}\},\quad -\infty<k<\infty.$$

显然, 由 (2.6.47) 有: 当 $T\to\infty$ 时, $b_T/a_T\to\infty$, 且对充分大的 j, 当 $|i|\geq\theta^{\varepsilon j}$ 时, $A_iB_j=\emptyset$. 同样, 由 (2.6.48) 我们有: 对充分大的 j, 当 $|k|\geq\theta^{\varepsilon j}$ 时, $A_iB_jC_{ki}=\emptyset$. 令 $T'_{kij}=\inf\{T:T\in A_iB_jC_{ki}\}$

和 $T_{kij}^* = \sup\{T : T \in A_i B_j C_{ki}\}$. 从而

$$
\begin{aligned}
&\limsup_{T\to\infty} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} \\
&\le \limsup_{j\to\infty} \sup_{T\in B_j} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} \\
&\le \limsup_{j\to\infty} \max_{|i|\le\theta^{\varepsilon j}} \sup_{T\in B_jA_i} \sup_{|t|\le b_T} \sup_{0\le s\le a_T} \frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} \\
&\le \limsup_{j\to\infty} \max_{|i|\le\theta^{\varepsilon j}} \sup_{T\in B_jA_i} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{i+1}} \frac{(1+\varepsilon)|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}} \\
&\le \limsup_{j\to\infty} \max_{|i|\le\theta^{\varepsilon j}} \max_{|k|\le\theta^{\varepsilon j}} \sup_{T\in B_jA_iC_{ki}} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{i+1}} (1+\varepsilon)|X(t+s,T) \\
&\quad - X(t,T)|\Big/\Big[H_0(\theta^{i+1},T_{kij}^t)(2\log\theta^j)^{1/2}\Big] \\
&\le \limsup_{j\to\infty} \max_{|i|\le\theta^{\varepsilon j}} \max_{|k|\le\theta^{\varepsilon j}} \sup_{T\in B_jA_iC_{ki}} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{i+1}} (1+\varepsilon)^2|X(t+s,T) \\
&\quad - X(t,T)|\Big/\Big[H_0(\theta^{i+1},T_{kij}^*)(2\log\theta^j)^{1/2}\Big]. \qquad (2.6.54)
\end{aligned}
$$

由命题 2.6.1 和 (2.6.53), 并注意到 $H_1(t,s,T) = H_0(s,T)$, 我们得

$$
\begin{aligned}
&P\Big\{\sup_{T\in B_jA_iC_{ki}} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{i+1}} |X(t+s)-X(t,T)|\Big/ \\
&\qquad \Big[H_0(\theta^{i+1},T_{kij}^*)(2\log\theta^j)^{1/2}\Big] \ge (1+2\varepsilon)^4\Big\} \\
&\quad \le C(\varepsilon) \sup_{T\in B_jA_iC_{ki}} \Big(\frac{b_T}{\theta^{i+1}}+1\Big)\exp\big(-(1+2\varepsilon)^2\log\theta^j\big) \\
&\quad \le C(\varepsilon)\theta^{-4\varepsilon j}. \qquad (2.6.55)
\end{aligned}
$$

从而

$$
\begin{aligned}
&P\Big\{\max_{|i|\le\theta^{\varepsilon j}} \max_{|k|\le\theta^{\varepsilon j}} \sup_{T\in B_jA_iC_{ki}} \sup_{|t|\le b_T} \sup_{0\le s\le\theta^{i+1}} \frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T_{kij}^*)(2\log\theta^j)^{1/2}} \\
&\quad \ge (1+2\varepsilon)^4\Big\} \le C(\varepsilon)\theta^{-\varepsilon j}. \qquad (2.6.56)
\end{aligned}
$$

由 (2.6.54), (2.6.56) 和 Borel-Cantelli 引理得

$$\limsup_{T\to\infty}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\le(1+2\varepsilon)^6\quad \text{a.s.} \tag{2.6.57}$$

由 ε 的任意性, (2.6.53) 得证.

下面证明

$$\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\ge 1\quad \text{a.s.} \tag{2.6.58}$$

注意到

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_Ta_T))^{1/2}}\\
\ge&\liminf_{j\to\infty}\inf_{|i|\le\theta^{\epsilon j}}\inf_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\\
\ge&\liminf_{j\to\infty}\inf_{|i|\le\theta^{\epsilon j}}\inf_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+\theta^{i+1},T)-X(t,T)|}{(1+\varepsilon)H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&-\limsup_{j\to\infty}\sup_{|i|\le\theta^{\epsilon j}}\sup_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+\theta^{i+1},T)-X(t+a_T,T)|}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
\ge&\liminf_{j\to\infty}\inf_{|i|\le\theta^{\epsilon j}}\inf_{T\in B_jA_i}\sup_{0\le l\le\theta^{j-2}}\frac{|X((l+1)\theta^{i+1},T)-X(l\theta^{i+1},T)|}{(1+\varepsilon)H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
&-\limsup_{j\to\infty}\sup_{|i|\le\theta^{\epsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\\
&\qquad\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}.
\end{aligned}\tag{2.6.59}$$

沿着 (2.6.57) 的证明路线, 由 (2.6.46) 我们可以证明

$$\begin{aligned}
&\limsup_{j\to\infty}\sup_{|i|\le\theta^{\epsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\\
\le&\limsup_{j\to\infty}\sup_{|i|\le\theta^{\epsilon j}}\sup_{T\in B_jA_i}\sup_{a_T\le t\le b_T+a_T}\sup_{0\le s\le(\theta-1)\theta^i}\\
&\cdot\frac{\varepsilon|X(t+s,T)-X(t,T)|}{H_0((\theta-1)\theta^i,T)(2\log\theta^j)^{1/2}}\le\varepsilon\quad \text{a.s.}
\end{aligned}\tag{2.6.60}$$

记 $Y(l,T) = X((l+1)\theta^{i+1},T) - X(l\theta^{i+1},T)$. 令 $Z(l,T)$ 为一个两参数 Gauss 过程，满足：对每个固定的 l, $Z(l,T)$ 是独立增量过程且 $Z(l,T) \overset{\mathcal{D}}{=} Y(l,T)$, $EZ(l,T)Z(n,T') = EY(l,T)Y(n,T')$ $(l \neq n)$. 则由 (2.6.45) 有

$$
\begin{aligned}
EY^2(l,T) &= EZ^2(l,T), \\
EY(l,T)Y(n,T) &= EZ(l,T)Z(n,T), \\
EY(l,T)Y(n,T') &= EZ(l,T)Z(n,T'), \quad \forall l \neq n, \\
EY(l,T)Y(l,T') &\geq EY^2(l,T\wedge T') = EZ(l,T)Z(l,T').
\end{aligned}
$$

从而，应用推论 1.2.2 得

$$
\begin{aligned}
&P\Big\{\inf_{T\in B_jA_i}\max_{0\le l\le\theta^{j-2}}\frac{Y(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\} \\
&=1-P\Big\{\bigcap_{T\in B_jA_i}\bigcup_{0\le l\le\theta^{j-2}}\Big\{\frac{Y(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\ge\frac{1}{(1+\varepsilon)^2}\Big\}\Big\} \\
&\le 1-P\Big\{\bigcap_{T\in B_jA_i}\bigcup_{0\le l\le\theta^{j-2}}\Big\{\frac{Z(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\ge\frac{1}{(1+\varepsilon)^2}\Big\}\Big\} \\
&= P\Big\{\inf_{T\in B_jA_i}\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}. \qquad (2.6.61)
\end{aligned}
$$

由 (2.6.48) 易知，对充分大的 j, 当 $|k|\ge\theta^j$ 时， $C_{ki}B_j=\emptyset$. 从而

$$
\begin{aligned}
&P\Big\{\inf_{T\in B_jA_i}\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\} \\
&\le\sum_{|k|\le\theta^j}P\Big\{\inf_{T\in B_jA_iC_{ki}}\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\} \\
&\le\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T^*_{kij})}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\le\frac{\theta}{(1+\varepsilon)^2}\Big\} \\
&\quad+\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}} \\
&\qquad\ge\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}. \qquad (2.6.62)
\end{aligned}
$$

注意到对固定的 l, $Z(l,T)$ 为独立增量过程，我们有

$$\begin{aligned}
E\big(Z(l,T^*_{kij})-Z(l,T'_{kij})\big)^2 &= EZ^2(l,T^*_{kij})-EZ^2(l,T'_{kij})\\
&= EY^2(l,T^*_{kij})-EY^2(l,T'_{kij})\le \theta^{2(k+1)}-\theta^{2k}\\
&\le (\theta^2-1)EY^2(l,T^*_{kij})=(\theta^2-1)H_0^2(\theta^{i+1},T^*_{kij}).
\end{aligned}$$

从而

$$\begin{aligned}
&\sum_{|k|\le\theta^j}P\Big\{\max_{0\le l\le\theta^{j-2}}\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\ge\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}\\
&\le\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}P\Big\{\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\ge\frac{\varepsilon}{2}\Big\}\\
&\le 2\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}P\Big\{\frac{|Z(l,T^*_{kij})-Z(l,T'_{kij})|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\ge\frac{\varepsilon}{2}\Big\}\\
&\le 4\sum_{|k|\le\theta^j}\sum_{l=0}^{\theta^{j-2}}\exp\Big(-\frac{\varepsilon^2\log\theta^j}{4(\theta^2-1)}\Big)\le 4\theta^{-2j}. \qquad (2.6.63)
\end{aligned}$$

这里用到了条件 $1<\theta<1+\varepsilon/32$. 由 (2.6.49) 和 Slepian 引理得

$$\begin{aligned}
&P\Big\{\max_{0\le l\le\theta^{j-2}}\frac{Z(l,T^*_{kij})}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\le\frac{\theta}{(1+\varepsilon)^2}\Big\}\\
&\le\prod_{l=0}^{[\theta^{j-2}]}P\Big\{\frac{Z(l,T^*_{kij})}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\le\frac{\theta}{(1+\varepsilon)^2}\Big\}\\
&\le\prod_{j=0}^{[\theta^{j-2}]}\Big(1-\exp\Big(-\frac{\theta^2}{1+\varepsilon}\log\theta^j\Big)\Big)\\
&\le\exp\big(-\theta^{\varepsilon j/4}\big)\le\theta^{-4j}. \qquad (2.6.64)
\end{aligned}$$

从而，由 (2.6.61)—(2.6.64), 对充分大的 j

$$P\Big\{\inf_{T\in B_jA_i}\max_{0\le l\le\theta^{j-2}}\frac{Y(l,T)}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}\le\frac{1}{(1+\varepsilon)^2}\Big\}\le 5\theta^{-2j}. \qquad (2.6.65)$$

综合 (2.6.59), (2.6.60) 和 (2.6.65), 由 Borel-Cantelli 引理得

$$\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\ge\frac{1}{(1+\varepsilon)^2}-\varepsilon\quad\text{a.s.}\tag{2.6.66}$$

由 ε 的任意性, 得证 (2.6.58), 从而定理 2.6.1 得证.

定理 2.6.2 设条件 (2.6.28) 满足并且对任何 $0\le s\le s_1\le a_T$, $0\le v\le d_T$,$0\le u\le 2c_T$, $|t|\le b_T+a_T$, (2.6.29) 成立. 假设

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{|t|\le b_T+a_T}\sup_{0\le s\le a_T}\sup_{0\le v\le d_T+c_T}\sup_{-\delta c_T\le u\le c_T}\frac{H_2(t+s,a_T,v+u,\delta c_T)+H_2(t+s,\delta a_T,v+u,c_T)}{H_2(t,a_T,v,c_T)}=0,\tag{2.6.67}$$

$$\log\log\Big(a_T+c_T+\frac{1}{d_T}+\frac{1}{b_T}\Big)=o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{d_T}{c_T}\Big)\Big),\quad T\to\infty.\tag{2.6.68}$$

则

$$\limsup_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,u)\big)\big|\Big/H_2(t,a_T,v,c_T)\cdot\Big(2\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(1+\frac{d_T}{c_T}\Big)+\log\log\tilde H_2(t,a_T,v,c_T)\Big)\Big)^{1/2}\le 1\quad\text{a.s.}\tag{2.6.69}$$

这里以及在本节的余下部分中, $\tilde{x}=x+1/x$. 如果进一步还有下面的条件成立: 当 $T\to\infty$ 时, 对 $|t|\le b_T$ 和 $0\le v\le d_T$ 一致成立

$$\log\log\tilde H_2(t,a_T,v,c_T)=o\Big(\log\big(b_T/a_T+1\big)\big(1+d_T/c_T\big)\Big),\tag{2.6.70}$$

且对任何 $s>0$, $u>0$, $j+k\ne m+l$ 成立

$$EX\big(R(js,s,ku,u)\big)X\big(R(ms,s,lu,u)\big)\le 0,\tag{2.6.71}$$

则

$$\lim_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le t\le b_T}\frac{\big|X\big(R(t,a_T,v,c_T)\big)\big|}{H_2(t,a_T,v,c_T)\big(2\log(b_T/a_T+1)(d_T/c_T+1)\big)^{1/2}}=1\quad\text{a.s.},\tag{2.6.72}$$

$$\lim_{T\to\infty}\sup_{\substack{0\le v\le d_T\\ 0\le u\le c_T}}\sup_{\substack{0\le t\le b_T\\ 0\le s\le a_T}}\frac{\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,s,v,u)\big(2\log(b_T/a_T+1)(d_T/c_T+1)\big)^{1/2}}$$

$$= 1 \quad \text{a.s.} \tag{2.6.73}$$

证明 对任意的 $0<\varepsilon<1/2$, 由 (2.6.67), 存在正数 N 使得对 $T\ge N$ 有

$$\sup_{\substack{|t|\le b_T+a_T\\ 0\le s\le a_T}}\sup_{\substack{0\le v\le d_T+c_T\\ -c_T/N\le u\le c_T}}\big[H_2(t+s,a_T,v+u,c_T/N)$$

$$+H_2(t+s,a_T/N,v+u,c_T)\big]\big/H_2(t,a_T,v,c_T)\le\varepsilon. \tag{2.6.74}$$

令 $1<\theta<1+1/N$. 记

$$A_k=\{T:\theta^k<a_T\le\theta^{k+1}\},\quad -\infty<k<\infty,$$

$$B_i=\{T:\theta^i<c_T\le\theta^{i+1}\},\quad -\infty<i<\infty,$$

$$G_j=\left\{T:\theta^j<\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\le\theta^{j+1}\right\},\quad j=0,1,2,\cdots.$$

显然, 由 (2.6.68) 有

$$\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\to\infty,\quad T\to\infty,$$

并且对充分大的 j, 当 $|k|\ge\theta^{\varepsilon j}$ 时有 $A_kG_j=\emptyset$ 和 $B_kG_j=\emptyset$. 从而

$$\limsup_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,u)\big)\big|/$$

$$\cdot H_2(t,a_T,v,c_T)\big(2(\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)+\log\log\tilde H_2(t,a_T,v,c_T))\big)^{1/2}$$

$$\le\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\varepsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,$$

$$v,u)\big)\big|/\cdot H_2(t,a_T,v,c_T)\big(2(\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)$$

$$+\log\log\tilde H_2(t,a_T,v,c_T))\big)^{1/2}$$

$$\le\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\varepsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le d_T}\sup_{0\le u\le\theta^{i+1}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{k+1}}$$

$$\frac{(1+\varepsilon)\big|X\big(R(t,s,v,u)\big)\big|}{H_2(t,\theta^{k+1},v,\theta^{i+1})\big(2(\log\theta^j+\log\log\tilde H_2(t,\theta^{k+1},v,\theta^{i+1}))\big)^{1/2}}.$$

$$\tag{2.6.75}$$

由 (2.6.74) 和命题 2.6.2 得

$$P\Big\{\sup_{|k|,|i|\le\theta^{\varepsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le d_T}\sup_{0\le u\le\theta^{i+1}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{k+1}}\big|X\big(R(t,s,v,u)\big)\big|/$$

$$\big[H_2(t,\theta^{k+1},v,\theta^{i+1})\big(2(\log\theta^j+\log\log\tilde{H}_2(t,\theta^{k+1},v,\theta^{i+1}))\big)^{1/2}\big]\ge(1+\varepsilon)^4\Big\}$$

$$\le C(\varepsilon)\sum_{|k|,|i|\le\theta^{\varepsilon j}}\sup_{T\in A_kB_iG_j}\big(d_T/\theta^{i+1}+1\big)\big(b_T/\theta^{k+1}+1\big)$$

$$\cdot\exp\big(-(1+\varepsilon)^2\log\theta^j\big)$$

$$\le C(\varepsilon)\sum_{|k|,|i|\le\theta^{\varepsilon j}}\theta^j\exp\big(-(1+\varepsilon)^2\log\theta^j\big)\le C(\varepsilon)\theta^{-\varepsilon^2 j}. \qquad (2.6.76)$$

由 (2.6.75), (2.6.76) 和 Borel-Cantelli 引理得

$$\limsup_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\big|X\big(R(t,s,v,u)\big)\big|/$$

$$\cdot H_2(t,a_T,v,c_T)\big(2(\log(\frac{b_T}{a_T}+1)(1+\frac{d_T}{c_T})+\log\log\tilde{H}_2(t,a_T,v,c_T))\big)^{1/2}$$

$$\le(1+\varepsilon)^5\quad\text{a.s.}$$

由此和 ε 的任意性, (2.6.69) 得证.

现在考察 (2.6.72). 注意到

$$\liminf_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0<t\le b_T}\frac{\big|X\big(R(t,a_T,v,c_T)\big)\big|}{H_2(t,a_T,v,c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}}$$

$$\ge\liminf_{j\to\infty}\inf_{|k|,|i|\le\theta^{\varepsilon j}}\inf_{T\in A_kB_iG_j}\sup_{\substack{0\le v\le d_T\\0<t\le b_T}}\big|X\big(R(t,a_T,v,c_T)\big)\big|/$$

$$\big[H_2(t,a_T,v,c_T)\big(2\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}\big]$$

$$\ge\liminf_{j\to\infty}\inf_{|k|,|i|\le\theta^{\varepsilon j}}\inf_{T\in A_kB_iG_j}\sup_{\substack{0\le v\le d_T\\0<t\le b_T}}\big|X\big(R(t,\theta^{k+1},v,\theta^{i+1})\big)\big|/$$

$$\big[(1+\varepsilon)H_2(t,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}\big]$$

$$-\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\varepsilon j}}\sup_{T\in A_kB_iG_j}\sup_{\theta^i\le v\le\theta^i+d_T}\sup_{0\le u\le(\theta-1)\theta^i}\sup_{0<t\le b_T}\sup_{0\le s\le\theta^{k+1}}$$

$$\frac{\left|X\big(R(t,s,v,u)\big)\right|}{H_2(t,\theta^{k+1},v-\theta^i,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\epsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le d_T}\sup_{0\le u\le\theta^{i+1}}\sup_{\theta^k\le t\le\theta^k+b_T}\sup_{0\le s\le(\theta-1)\theta^k}$$

$$\frac{\left|X\big(R(t,s,v,u)\big)\right|}{H_2(t-\theta^k,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$\ge\liminf_{j\to\infty}\inf_{|k|,|i|\le\theta^{\epsilon j}}\min_{m\cdot n\ge\theta^j}\max_{\substack{0\le l\le m\\0\le p\le n}}\left[\left|X\big(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})\big)\right|\right]/$$

$$\left[(1+\varepsilon)H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}\right]$$

$$-\varepsilon\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\epsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le\theta^i+d_T}\sup_{0\le u\le(\theta-1)\theta^i}\sup_{0<t\le b_T}\sup_{0\le s\le\theta^{k+1}}$$

$$\frac{\left|X\big(R(t,s,v,u)\big)\right|}{H_2(t,\theta^{k+1},v,(\theta-1)\theta^i)(2\log\theta^j)^{1/2}}$$

$$-\varepsilon\limsup_{j\to\infty}\sup_{|k|,|i|\le\theta^{\epsilon j}}\sup_{T\in A_kB_iG_j}\sup_{0\le v\le d_T}\sup_{0\le u\le\theta^{i+1}}\sup_{0\le t\le\theta^k+b_T}\sup_{0\le s\le(\theta-1)\theta^k}$$

$$\frac{\left|X\big(R(t,s,v,u)\big)\right|}{H_2(t,(\theta-1)\theta^k,v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$=: I_1-I_2-I_3. \tag{2.6.77}$$

沿着 (2.6.69) 的证明路线，由 (2.6.70) 可以证明

$$I_2+I_3\le 2\varepsilon\quad \text{a.s.} \tag{2.6.78}$$

对 I_1，注意到 (2.6.71)，利用 Slepian 引理可以得到：对充分大的 j 有

$$P\Big\{\min_{mn\ge\theta^j}\max_{0\le l\le m}\max_{0\le p\le n}X\big(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})\big)/$$

$$\left[H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}\right]\le\frac{1}{(1+\varepsilon)^2}\Big\}$$

$$\le\sum_{mn\ge\theta^j}\prod_{0\le l\le m}\prod_{0\le p\le n}P\Big\{X\big(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})\big)/$$

$$\left[H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}\right]\le\frac{1}{(1+\varepsilon)^2}\Big\}$$

$$\le\sum_{m,n:mn\ge\theta^j}\left(1-\exp\Big(-\frac{\log\theta^j}{(1+\varepsilon)^2}\Big)\right)^{(m+1)(n+1)}$$

$$\leq \sum_{m,n:mn\geq\theta^j} \exp\Big(-(m+1)(n+1)\theta^{-j/(1+\varepsilon)^2}\Big)$$
$$\leq \theta^{2j}\exp(-\theta^{j\varepsilon}) \leq \theta^{-j}.$$

由此和 Borel-Cantelli 引理得

$$I_1 \geq \frac{1}{(1+\varepsilon)^3} \quad \text{a.s.} \tag{2.6.79}$$

由上述这些不等式，我们得

$$\liminf_{T\to\infty} \sup_{\substack{0\leq v\leq d_T\\ 0<t\leq b_T}} \frac{\big|X\big(R(t,a_T,v,c_T)\big)\big|}{H_2(t,a_T,v,c_T)\big(2\log(\frac{b_T}{a_T}+1)(1+\frac{d_T}{c_T})\big)^{1/2}}$$
$$\geq \frac{1}{(1+\varepsilon)^3} - 2\varepsilon \quad \text{a.s.} \tag{2.6.80}$$

由 (2.6.80), (2.6.69) 和 (2.6.70) 知 (2.6.72) 和 (2.6.73) 成立．定理 2.6.2 得证．

对于本节一开始给出的例子，我们有下述推论．

推论 2.6.1 令 $\{W(x,y); -\infty<x,y<\infty\}$ 为一标准两参数 Wiener 过程．假设

$$\log\log\Big(Ta_T + \frac{1}{Ta_T} + \frac{1}{b_T}\Big) = o\Big(\log\frac{b_T}{a_T}\Big), \quad T\to\infty.$$

则

$$\lim_{T\to\infty} \sup_{0\leq t\leq b_T} \frac{|W(t+a_T,T)-W(t,T)|}{\big(2Ta_T\log\frac{b_T}{a_T}\big)^{1/2}} = 1 \quad \text{a.s.},$$
$$\lim_{T\to\infty} \sup_{|t|\leq b_T} \sup_{0\leq s\leq a_T} \frac{|W(t+s,T)-W(t,T)|}{\big(2Ta_T\log\frac{b_T}{a_T}\big)^{1/2}} = 1 \quad \text{a.s.}$$

推论 2.6.2 设 $\{W(x,y); -\infty<x,y<\infty\}$ 为一标准两参数 Wiener 过程．假设

$$\log\log\Big(a_T + c_T + a_Tc_T + \frac{1}{a_Tc_T} + \frac{1}{b_T} + \frac{1}{d_T}\Big)$$
$$= o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{d_T}{c_T}+1\Big)\Big), \quad T\to\infty.$$

则

$$\lim_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le t\le b_T}\frac{\left|W\big(R(t,a_T,v,c_T)\big)\right|}{\big(2a_Tc_T\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}}=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le u\le c_T}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{\left|W\big(R(t,s,v,u)\big)\right|}{\big(2a_Tc_T\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}}$$
$$=1\quad \text{a.s.}$$

推论 2.6.3 设 $\{K(x,y);0\le x\le 1,0\le y<\infty\}$ 为一 Kiefer 过程, a_T 和 b_T 为连续函数, 满足 $0\le a_T+b_T\le 1$. 假设

$$\log\log\Big(\frac{1}{b_T}+Ta_T+\frac{1}{Ta_T}\Big)=o\Big(\log\frac{b_T}{a_T}\Big),\quad T\to\infty.$$

则

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\frac{|K(t+a_T,T)-K(t,T)|}{\big(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\big)^{1/2}}=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{|K(t+s,T)-K(t,T)|}{\big(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\big)^{1/2}}=1\quad \text{a.s.}$$

推论 2.6.4 设 $\{K(x,y);0\le x\le 1,0\le y<\infty\}$ 为一 Kiefer 过程, a_T, b_T, c_T, d_T 为连续函数, 满足 $0\le a_T+b_T\le 1$ 和 $0\le a_T\le 1/2$. 假设

$$\log\log\Big(\frac{1}{b_T}+\frac{1}{d_T}+c_T+\frac{1}{a_Tc_T}\Big)$$
$$=o\Big(\log\Big(\frac{b_T}{a_T}+1\Big)\Big(\frac{d_T}{c_T}+1\Big)\Big),\quad T\to\infty.$$

则

$$\lim_{T\to\infty}\sup_{0\le v\le d_T}\sup_{0\le t\le b_T}\frac{\left|K\big(R(t,a_T,v,c_T)\big)\right|}{\big(2a_T(1-a_T)c_T\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}}$$
$$=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{\substack{0\le v\le d_T\\0\le u\le c_T}}\sup_{\substack{0\le t\le b_T\\0\le s\le a_T}}\frac{\left|K\big(R(t,s,v,u)\big)\right|}{\big(2a_T(1-a_T)c_T\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\big)^{1/2}}$$
$$=1\quad \text{a.s.}$$

推论 2.6.1—2.6.4 容易证明，其证明从略.

推论 2.6.5 设 $\{X(t,v); -\infty < t, v < \infty\}$ 为例 2.6.3 所示的 Gauss 过程. 记

$$H^2(a_T,T) := H_1^2(t,a_T,T) = 2\int_0^T \frac{\gamma(x)}{\lambda(x)}\Big(1 - \exp\big(-\lambda(x)a_T\big)\Big)\,dx.$$

假设存在 $c_0 > 0$ 使得对任何 $0 < s \le a_T$ 有

$$\int_{0<x\le T\ \lambda(x)\ge 1/s} \frac{\gamma(x)}{\lambda(x)}dx \le c_0 s \int_{0<x\le T\ \lambda(x)\le 1/s} \gamma(x)\,dx, \quad (2.6.81)$$

且

$$\log\log\Big(a_T + \frac{1}{b_T} + \tilde{H}(a_T,T)\Big) = o\Big(\log\frac{b_T}{a_T}\Big), \quad T \to \infty.$$

则

$$\lim_{T\to\infty} \sup_{0\le t\le T} \frac{|X(t+a_T,T) - X(t,T)|}{H(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \quad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{|X(t+s,T) - X(t,T)|}{H(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}} = 1 \quad \text{a.s.}$$

证明 注意到对任何 $v, u > 0$ 有

$$EX(t,v)X(s,u) = \int_0^{v\wedge u} \exp\big(-\lambda(y)|t-s|\big)\frac{\gamma(y)}{\lambda(y)}\,dy,$$

我们可以验证条件 (2.6.45), (2.6.49), (2.6.28) 满足. 下面证明 $H^2(s,T)/s^\alpha$ 在 $(0,a_T)$ 上关于 s 单调增加，其中 $\alpha = 1/(6(c_0+1))$. 令

$$f(s) = H_1^2(s,T)/s^\alpha = 2\int_0^T \frac{\gamma(x)}{\lambda(x)}\big(1 - \exp(-\lambda(x)s)\big)\,dx/s^\alpha.$$

则由 (2.6.81), 对 $0<\alpha<1/(3(c_0+1))$ 有

$$
\begin{aligned}
f'(s) =& 2s^{-\alpha-1}\Big(-\alpha\int_0^T \frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))dx \\
&+\int_0^T s\gamma(x)\exp(-\lambda(x)s)dx\Big) \\
\geq& 2s^{-\alpha-1}\Big(-\alpha\int_{0\leq x\leq T:\lambda(x)\geq 1/s}\frac{\gamma(x)}{\lambda(x)}\,dx \\
&-\alpha s\int_{0\leq x\leq T:\lambda(x)\leq 1/s}\gamma(x)dx+\frac{s}{3}\int_{0\leq x\leq T:\lambda(x)\leq 1/s}\gamma(x)\,dx\Big) \\
\geq& 2s^{-\alpha-1}\Big(-\alpha(c_0+1)s\int_{0\leq x\leq T:\lambda(x)\leq 1/s}\gamma(x)dx \\
&+\frac{s}{3}\int_{0\leq x\leq T:\lambda(x)\leq 1/s}\gamma(x)dx\Big)>0,
\end{aligned}
$$

这就是要证的. 因此条件 (2.6.46) 满足. 由定理 2.6.1, 推论 2.6.5 得证.

推论 2.6.6 设 $d,b>0$, $\{X(t,v);-\infty<t,v<\infty\}$ 为例 2.6.3 所示的 Gauss 过程. 假设当 $T\to\infty$ 时, $a_T\to 0$, $c_T\to\infty$, 且对任何 $0<x<y\leq 1$ 有

$$\sup_{0<x<d+b}\lambda(x)<\infty, \tag{2.6.82}$$

$$x^{1-\alpha}\gamma(x)\leq c_0y^{1-\alpha}\gamma(y) \text{ 和 } \gamma(y)y^{-1/\alpha}\leq c_0\gamma(x)x^{-1/\alpha}, \tag{2.6.83}$$

其中 $0<\alpha<1$, $c>0$. 则

$$\lim_{T\to\infty}\sup_{0\leq v\leq d}\sup_{0\leq t\leq b}\frac{\big|X\big(R(t,a_T,v,c_T)\big)\big|}{H(a_T,v,c_T)(2\log{(c_Ta_T)}^{-1})^{1/2}}=1\quad \text{a.s.},$$

$$
\begin{aligned}
&\lim_{T\to\infty}\sup_{0\leq v\leq d}\sup_{0\leq u\leq c_T}\sup_{|t|\leq b}\sup_{0\leq s\leq a_T}\frac{\big|X\big(R(t,s,v,u)\big)\big|}{H(a_T,v,c_T)(2\log{(c_Ta_T)}^{-1})^{1/2}} \\
&=1\quad \text{a.s.},
\end{aligned}
$$

其中 $H^2(a_T,v,c_T)=2\int_v^{v+c_T}\frac{\gamma(x)}{\lambda(x)}\big(1-\exp(-\lambda(x)a_T)\big)\,dx$.

证明 记 $M=\sup_{0<x\le d+b}\lambda(x)$. 则由 (2.6.82) 有 $M<\infty$. 由 (2.6.83) 得

$$\int_0^{d+b}\gamma(x)\,dx<\infty. \tag{2.6.84}$$

显然, 对 $0<s\le 1/M,\ 0\le v+c\le d+b$ 有

$$\begin{aligned}\frac{s}{3}\int_v^{v+c}\gamma(x)dx&\le\int_v^{v+c}\frac{\gamma(x)}{\lambda(x)}\Big(1-\exp\big(-\lambda(x)s\big)\Big)dx\\&=\frac{1}{2}H^2(s,v,c)\le s\int_v^{v+c}\gamma(x)dx,\end{aligned}$$

从而条件 (2.6.29) 满足. 下面证明条件 (2.6.67) 也满足. 这只要证

$$\lim_{\delta\to0}\limsup_{T\to\infty}\sup_{0\le v\le d+\frac{b}{2}}\sup_{-\delta c_T\le u\le c_T}\frac{H^2(a_T,v+u,\delta c_T)}{H^2(a_T,v,c_T)}=0. \tag{2.6.85}$$

再次由 (2.6.82), 我们得当 $T\to\infty$ 时, 对 $0<v\le d+b,\ 0<c\le1$ 一致成立

$$\frac{H^2(a_T,v,c)}{2a_T\int_v^{v+c}\gamma(x)dx}\to1.$$

从而等价地, 只要证

$$\lim_{\delta\to0}\limsup_{T\to\infty}\sup_{0\le v\le d+b/2}\sup_{-\delta c_T\le u\le c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)dx}{\int_v^{v+c_T}\gamma(x)dx}=0. \tag{2.6.86}$$

注意到 $\gamma(x)$ 是 $(0,\infty)$ 的正连续函数, 我们有

$$0<\inf_{1/3\le x\le d+b}\gamma(x)\le\sup_{1/3\le x\le d+b}\gamma(x)<\infty.$$

从而

$$\begin{aligned}&\lim_{\delta\to0}\limsup_{T\to\infty}\sup_{1/2\le v\le d+b/2}\sup_{-\delta c_T\le u\le c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)dx}{\int_v^{v+c_T}\gamma(x)dx}\\&\le\lim_{\delta\to0}\delta\frac{\sup_{1/3\le x\le d+b}\gamma(x)}{\inf_{1/3\le x\le d+b}\gamma(x)}=0.\end{aligned} \tag{2.6.87}$$

若 $0<v\le 1/2$, $-\delta c_T\le u\le c_T/2$, 则

$$\int_v^{v+c_T}\gamma(x)dx\ge\int_{v+2c_T/3}^{v+c_T}\gamma(x)dx\ge\frac{c_T}{3}\gamma(x_0),\tag{2.6.88}$$

其中 $v+c_T/3\le x_0\le v+c_T$. 由 (2.6.83) 得

$$\begin{aligned}\int_{v+u}^{v+u+\delta c_T}\gamma(x)dx&\le c_0x_0^{1-\alpha}\gamma(x_0)\int_{v+u}^{v+u+\delta c_T}\frac{1}{x^{1-\alpha}}dx\\&=\frac{c_0}{\alpha}x_0^{1-\alpha}\gamma(x_0)\big((v+u+\delta c_T)^\alpha-(v+u)^\alpha\big)\\&\le\frac{c_0}{\alpha}(v+c_T)^{1-\alpha}\gamma(x_0)\big((v+u+\delta c_T)^\alpha-(v+u)^\alpha\big)\\&\le\frac{6c_0(\delta^\alpha+\delta)c_T\gamma(x_0)}{\alpha}.\end{aligned}\tag{2.6.89}$$

若 $0<v\le\frac{1}{2}$, $\frac{c_T}{2}\le u\le c_T$, 则

$$\int_{v+u}^{v+u+\delta c_T}\gamma(x)dx=\gamma(y_0)\delta c_T,\tag{2.6.90}$$

其中 $v+u\le y_0\le v+u+\delta c_T$. 再次由 (2.6.83), 我们得

$$\begin{aligned}\int_v^{v+c_T}\gamma(x)dx&\ge\int_v^{v+c_T/2}\gamma(x)dx\ge\frac{\gamma(y_0)}{c_0y_0^{1/\alpha}}\int_v^{v+c_T/2}x^{\frac{1}{\alpha}}dx\\&\ge\frac{\alpha\gamma(y_0)}{2c_0y_0^{1/\alpha}}\left(\left(v+\frac{c_T}{2}\right)^{1+1/\alpha}-v^{1+1/\alpha}\right)\\&\ge\frac{\alpha\gamma(y_0)}{2c_0\cdot 2^{1/\alpha}(v+c_T)^{1/\alpha}}\left(\left(v+\frac{c_T}{2}\right)^{1+1/\alpha}-v^{1+1/\alpha}\right)\\&\ge\frac{\alpha\gamma(y_0)c_T}{c_0\cdot(12)^{1+1/\alpha}}.\end{aligned}\tag{2.6.91}$$

综合 (2.6.87)—(2.6.91) 得 (2.6.86). (2.6.67) 得证. 由定理 2.6.2, 推论 2.6.6 得证.

推论 2.6.7 设 $\{X(t,v);-\infty<t<\infty,0\le v<\infty\}$ 为例 2.6.4 所示的 Gauss 过程. 假设对每个 k, $\phi_k(v)$ 是 v 的非降函数,

并且存在 $c_0 > 0$ 使得

$$\sum_{\lambda_k \geq 1/s} \phi_k^2(v)\frac{\gamma_k}{\lambda_k} \leq c_0 s \sum_{\lambda_k \leq 1/s} \phi_k^2(v) \cdot \gamma_k \quad \forall 0 < s \leq 1, v > 0$$

和

$$\log\log\Big(\sum_{k=0}^{\infty} \phi_k^2(T)\frac{\gamma_k}{\lambda_k}\Big) = o\Big(\log\frac{1}{a_T}\Big), \quad T \to \infty$$

成立. 则

$$\lim_{T\to\infty} \sup_{0\leq t\leq 1} \frac{|X(t+a_T,T) - X(t,T)|}{H_1(a_T,T)(2\log\frac{1}{a_T})^{1/2}} = 1 \quad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{|t|\leq 1} \sup_{0\leq s\leq a_T} \frac{|X(t+s,T) - X(t,T)|}{H_1(a_T,T)(2\log\frac{1}{a_T})^{1/2}} = 1 \quad \text{a.s.},$$

其中 $H_1^2(a_T,T) = 2\sum_{k=1}^{\infty} \phi_k^2(T)(1 - e^{-\lambda_k a_T})\frac{\gamma_k}{\lambda_k}$.

推论 2.6.7 易证, 其证明从略.

§2.7 Gauss 过程局部时的连续模

设 $\{X(t); t \geq 0\}$ 为一个实值随机过程. 对实数集上的任意 Borel 集 A, 令

$$H(A,t) = \lambda\{s : 0 \leq s \leq t, X(s) \in A\}, \quad t \geq 0, \tag{2.7.1}$$

称为 X 的占有时, 其中 λ 为 Lebesgue 测度. 如果对每个固定的 t, $H(\cdot,t)$ 相对于 Lebesgue 测度绝对连续, 则它的 Radon-Nikodym 导数称为 $X(\cdot)$ 在 t 处的局部时, 记为 $L(\cdot,t)$, 这时我们称 $X(\cdot)$ 的局部时存在. 由 $L(x,t)$ 的定义知

$$L(x,0) = 0, \quad L(x,s) \leq L(x,t), \quad \forall t \geq s \geq 0, x \in \mathcal{R}. \tag{2.7.2}$$

$$H(A,t) = \int_A L(x,t)\,dx, \tag{2.7.3}$$

$$H(A,t+h) - H(A,t) = \int_A (L(x,t+h) - L(x,t))\,dx, \quad \forall t,h \geq 0. \tag{2.7.4}$$

对于 Wiener 过程的局部时, 已经有许多经典的结果. Hawkes (1971) 得到了关于 t 的连续模: 令 $l(x,t)$ 为 W 的局部时, 则

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\frac{l(0,t+h)-l(0,t)}{\{h\log h^{-1}\}^{1/2}}=1\quad \text{a.s.}\tag{2.7.5}$$

Perkins (1981) 得到了

$$\limsup_{h\to 0}\sup_{0\le t\le 1-h}\sup_{-\infty<x<\infty}\frac{l(x,t+h)-l(x,t)}{\{2h\log\frac{1}{h}\}^{1/2}}=1\quad \text{a.s.}\tag{2.7.6}$$

Csáki, Csörgő, Földes 和 Révész (1983) 证明了在 (2.7.6) 中可用 $\lim_{h\to 0}$ 代替 $\limsup_{h\to 0}$, 并且得到了下述结果:

定理 2.7.1 设 $0<a_T\le T$ 为 $T\ge 0$ 的非降函数, 假设 a_T/T 非增. 则对任何 $x\in R$ 有

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{l(x,t+a_T)-l(x,t)}{\{a_T(\log\frac{T}{a_T}+2\log\log T)\}^{1/2}}=1\quad \text{a.s.}\tag{2.7.7}$$

并且

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{-\infty<x<\infty}\frac{l(x,t+a_T)-l(x,t)}{\{2a_T(\log\frac{T}{a_T}+\log\log T)\}^{1/2}}=1\quad \text{a.s.}\tag{2.7.8}$$

进一步, 如果还假设 $\lim_{T\to\infty}(\log\frac{T}{a_T})/\log\log T=\infty$, 则 (2.7.7) 和 (2.7.8) 中的 $\limsup_{T\to\infty}$ 可用 $\lim_{T\to\infty}$ 代替.

在 (2.7.7) 和 (2.7.8) 中取 $a_T=T$, 我们得到由 Kesten (1965) 证明的重对数律, 即对 Wiener 过程的局部时 $l(\cdot,\cdot)$ 有

$$\limsup_{T\to\infty}\frac{l(x,T)}{(2T\log\log T)^{1/2}}=\limsup_{T\to\infty}\frac{\sup_{-\infty<x<\infty}l(x,T)}{(2T\log\log T)^{1/2}}=1\quad \text{a.s.}\tag{2.7.9}$$

在这一节中, 我们给出关于 Gauss 过程 $\{X(t);t\ge 0\}$ 局部时 $L(x,t)$ 的类似于 (2.7.5), (2.7.6) 和 (2.7.7) 的结果, 这些结果是由

Csörgő, Lin 和 Shao (1995) 得到的.

2.7.1 Gauss 过程局部时的增量的矩估计

设 $\{X(t); t \geq 0\}$ 为一个零均值、具有平稳增量的 Gauss 过程. 记

$$\sigma^2(h) = E(X(t+h) - X(t))^2, \quad t, h \geq 0.$$

我们知道 (参见 Berman 1969 和 Geman 1976): 如果

$$\int_0^t \frac{ds}{\sigma(s)} < \infty \qquad \forall t > 0, \tag{2.7.10}$$

则 X 的局部时 $L(x,t)$ 存在. 此外, 如果 $\sigma^2(h)$ 连续并且是在 $[0,1]$ 上的凹函数, 则 X 的局部时 $L(x,t)$ 存在且是几乎处处联合连续的 (参见 Berman 1972).

下述定理给出了 $L(x,t)$ 关于 t 的增量的矩估计.

命题 2.7.1 设 $\{X(t); t \geq 0\}$ 为一个零均值、具有平稳增量的 Gauss 过程, 且 $X(0) = 0$. 记 $\sigma^2(h) = E(X(t+h) - X(t))^2$. 假设 $\sigma^2(h)$ 非降且在 $(0, h_0)$ 上为凹函数, 并且对某个 $0 < \alpha \leq 1/2$, $c_0 > 0$, $h_0 > 0$ 成立

$$\sigma(ah) \geq c_0 a^\alpha \sigma(h) \quad \forall 0 < a < 1, 0 \leq h \leq h_0. \tag{2.7.11}$$

则对任何 $0 < h \leq h_0$, $x \in \mathcal{R}$ 和任何整数 $m \geq 1$ 有

$$\begin{aligned} &E(L(x,t+h) - L(x,t))^m \\ &\quad \leq \Big(\frac{16h}{c_0\sigma(h)}\Big)^m (m!)^\alpha \exp\Big(-\frac{x^2}{2\sigma^2(t+h)}\Big). \end{aligned} \tag{2.7.12}$$

为了证明这一命题, 我们需要一些引理.

引理 2.7.1 设 $\xi_1, \cdots, \xi_n$ 为二阶矩存在的随机变量, A_n 为它们的协方差矩阵. 则

$$|A_n| \leq |A_{n-1}| Var\xi_n, \tag{2.7.13}$$

其中 $|A_n|$ 表示 A_n 的行列式.

证明 记 $A_n=(a_{ij}:1\le i,j\le n)$, 其中 $a_{ij}=Cov(\xi_j,\xi_j)$. 注意到

$$(a_{1n},\cdots,a_{n-1,n})'(a_{1n},\cdots,a_{n-1,n})$$

为半正定矩阵, 我们得

$$\begin{aligned}|A_n|=&\left|\left(a_{ij}-\frac{a_{ni}a_{nj}}{a_{nn}},1\le i,j\le n-1\right)\right|Var\xi_n\\=&\left|\left(Cov\left(\xi_i-\frac{a_{1i}\xi_n}{a_{nn}},\xi_j-\frac{a_{1j}\xi_n}{a_{nn}}\right),1\le i,j\le n-1\right)\right|Var\xi_n\\=&|(a_{ij},1\le i,j\le n-1)-(a_{1n},\cdots,a_{n-1,n})'(a_{1n},\cdots,a_{n-1,n})|\\&\cdot Var\xi_n\\\le&|(a_{ij},1\le i,j\le n-1)|Var\xi_n\\=&|A_{n-1}|Var\xi_n,\end{aligned}$$

(2.7.13) 得证.

引理 2.7.2 设 $B_n=(b_{ij};1\le i,j\le n)$ 和 $\tilde{B}_{n-1}=(b_{ij};2\le i,j\le n)$ 为实值矩阵. 假设对每个 $1\le i\le n$ 有 $|b_{ii}|\ge\sum_{j\ne i}|b_{ij}|$. 则

$$|B_n|\ge\left(|b_{11}|-\sum_{i=2}^{n}|b_{1i}|\right)|\tilde{B}_{n-1}|\ge|b_{nn}|\prod_{i=1}^{n-1}\left(|b_{ii}|-\sum_{j=i+1}^{n}|b_{ij}|\right).$$

证明见 Price (1951).

引理 2.7.3 设 $\{\xi(t);t\ge0\}$ 为一个零均值、具有平稳增量的 Gauss 过程, 且 $\xi(0)=0$. 记 $\sigma^2(h)=E(\xi(t+h)-\xi(t))^2$. 假设 $\sigma^2(h)$ 非降且在 $(0,h_0)$ 上为凹函数. 对 $t\le t_1<t_2<\cdots<t_n\le t+h_0$ 令 A_n 为 $\xi(t_1),\cdots,\xi(t_n)$ 的协方差矩阵. 则

$$|A_n|\ge(1/2)^n\sigma^2(t_1)\prod_{i=2}^{n}\sigma^2(t_i-t_{i-1}).$$

证明 由 $\sigma^2(h)$ 的凹性得对任何 $0 \le a \le b \le c \le d \le h_0 + t$ 有

$$\sigma^2(d-a) + \sigma^2(c-b) \le \sigma^2(d-b) + \sigma^2(c-a). \qquad (2.7.14)$$

令 $\tilde{A}_n$ 为 $\xi(t_1), \xi(t_2) - \xi(t_1) \cdots, \xi(t_n) - \xi(t_{n-1})$ 的协方差矩阵. 易知 $|A_n| = |\tilde{A}_n|$.

记 $\tilde{A}_n = (a_{ij}, 1 \le i, j \le n)$, 其中

$$a_{11} = E\xi^2(t_1),$$
$$a_{1i} = E\xi(t_1)(\xi(t_i) - \xi(t_{i-1})), \quad 2 \le i \le n,$$
$$a_{ij} = E(\xi(t_i) - \xi(t_{i-1}))(\xi(t_j) - \xi(t_{j-1})), \quad 2 \le i, j \le n.$$

当 $1 < i < j \le n$ 时, 由 (2.7.14) 有

$$\begin{aligned} a_{ij} =& \frac{1}{2}\Big(\sigma^2(t_j - t_{i-1}) + \sigma^2(t_{j-1} - t_i) \\ & - \sigma^2(t_j - t_i) - \sigma^2(t_{j-1} - t_{i-1})\Big) \le 0; \end{aligned}$$

当 $1 < i \le n$ 时, 再次由 (2.7.14) 我们有

$$a_{1i} = \big(\sigma^2(t_{i-1} - t_1) + \sigma^2(t_i) - \sigma^2(t_i - t_1) - \sigma^2(t_{i-1})\big)/2 \le 0.$$

从而对 $1 \le i < n$ 有

$$\begin{aligned} \sum_{j=i+1}^{n} |a_{ij}| =& - \sum_{j=i+1}^{n} a_{ij} \\ =& - E(\xi(t_i) - \xi(t_{i-1}))(\xi(t_n) - \xi(t_i)) \\ =& \frac{1}{2}\Big(\sigma^2(t_n - t_i) + \sigma^2(t_i - t_{i-1}) - \sigma^2(t_n - t_{i-1})\Big) \\ \le& \frac{1}{2}\sigma^2(t_i - t_{i-1}) = \frac{1}{2}a_{ii}, \end{aligned} \qquad (2.7.15)$$

其中 $t_0 = 0$; 对 $2 \le i \le n$ 有

$$\begin{aligned} \sum_{j=1}^{i-1} |a_{ij}| =& - \sum_{j=1}^{i-1} a_{ij} = -E(\xi(t_i) - \xi(t_{i-1}))\xi(t_{i-1}) \\ =& \frac{1}{2}\Big(\sigma^2(t_i - t_{i-1}) - E\xi^2(t_i) + E\xi^2(t_{i-1})\Big) \end{aligned}$$

$$\leq \frac{1}{2}\sigma^2(t_i - t_{i-1}).$$

因此对每个 $1 \leq i \leq n$ 成立

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|.$$

由引理 2.7.2 和 (2.7.15), 结论得证.

如同 Berman (1969, 1974) 一样, 我们利用 Fourier 分析来研究局部时. 令

$$f(u,t) = \int_{-\infty}^{\infty} e^{iux}\, d_x H([0,x],t) = \int_0^t e^{iuX(s)}\, ds, \quad -\infty < u < \infty,$$

为占有时 $H([0,x],t)$ 的 Fourier 变换. 我们可以把 $L(x,t)$ 表示为 $f(u,t)$ 的 Fourier 逆变换, 即

$$\begin{aligned} L(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} f(u,t)\, du \\ &= \frac{1}{2\pi}\int_0^t \int_{-\infty}^{\infty} e^{-iux} e^{iuX(s)}\, du\, ds. \end{aligned} \tag{2.7.16}$$

引理 2.7.4 设 $\{X(t); t \geq 0\}$ 为零均值、具有平稳增量的 Gauss 过程, 其增量的方差函数为 $\sigma^2(h)$. 令 $m \geq 1$ 为一整数, $R(s_1, \cdots, s_m)$ 为 $X(s_1)$, $X(s_2) - X(s_1)$, $\cdots$, $X(s_m) - X(s_{m-1})$ 的协方差矩阵. 则对任何 $x \in \mathcal{R}$, $t \geq 0$ 和 $h > 0$ 有

$$\begin{aligned} &E(L(x,t+h) - L(x,t))^m \\ &\leq \left(\frac{1}{2\pi}\right)^{m/2} m! \int \cdots \int_{t \leq s_1 < s_2 < \cdots < s_m \leq t+h} \exp\left(-\frac{x^2}{2EX^2(s_1)}\right) \\ &\quad \cdot |R(s_1, \cdots, s_m)|^{-1/2} ds_1, \cdots, ds_m. \end{aligned} \tag{2.7.17}$$

证明 记 $v_j = \sum_{i=j}^m u_i$, $1 \leq j \leq m$, $V = (v_1, \cdots, v_m)$. 利用 (2.7.16), 可写

$$
\begin{aligned}
&E(L(x,t+h)-L(x,t))^m\\
&=\left(\frac{1}{2\pi}\right)^m\int_t^{t+h}\cdots\int_t^{t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big(-ix\sum_{j=1}^m u_j\Big)E\\
&\quad\cdot\exp\Big(i\sum_{j=1}^m u_jX(s_j)\Big)\,du_1\cdots du_m ds_1\cdots ds_m\\
&=\left(\frac{1}{2\pi}\right)^m m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-ixv_1)\\
&\quad\cdot E\exp\Big[iv_1X(s_1)+i\sum_{j=2}^m v_j(X(s_j)-X(s_{j-1})\Big]\,dv_1\cdots dv_m ds_1\cdots ds_m\\
&=\left(\frac{1}{2\pi}\right)^m m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big[-ixv_1\\
&\quad-\frac{1}{2}VR(s_1,\cdots,s_m)V'\Big]\,dv_1\cdots dv_m ds_1\cdots ds_m.
\end{aligned}
$$

为证 (2.7.17), 只要证

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big[-ixv_1-\frac{1}{2}VR(s_1,\cdots,s_m)V'\Big]\,dv_1\cdots dv_m\\
&\qquad\le(2\pi)^{m/2}\exp\Big(-\frac{x^2}{2EX^2(s_1)}\Big)|R(s_1,\cdots,s_m)|^{-1/2}.
\end{aligned}\tag{2.7.18}
$$

若 $|R(s_1,\cdots,s_m)|=0$, 显然有 (2.7.18). 因此我们设 $|R(s_1,\cdots,s_m)|>0$, 即 $R(s_1,\cdots,s_m)$ 是正定的. 从而 $R^{-1}(s_1,\cdots,s_m)$ 也是正定的. 令 $(Y_1,\cdots,Y_m)$ 为一个零均值、以 $R^{-1}(s_1,\cdots,s_m)$ 为协方差矩阵的多维正态变量. 则 $(Y_1,\cdots,Y_m)$ 的密度函数为

$$
\left(\frac{1}{2\pi}\right)^{-m/2}|R(s_1,\cdots,s_m)|^{1/2}\exp\Big(-\frac{1}{2}VR(s_1,\cdots,s_m)V'\Big).
$$

从而

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big[-ixv_1-\frac{1}{2}VR(s_1,\cdots,s_m)V'\Big]\,dv_1\cdots dv_m\\
&\quad=(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}Ee^{-ixY_1}\\
&\quad=(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}e^{-(x^2/2)EY_1^2}.
\end{aligned}\tag{2.7.19}
$$

记 $R(s_1,\cdots,s_n)=(\gamma_{ij},1\le i,j\le n)$. 则由引理 2.7.1 得

$$EY_1^2=\frac{|(\gamma_{i,j},2\le i,j\le n)|}{|(\gamma_{ij},1\le i,j\le n)|}\ge\frac{1}{\gamma_{11}}=\frac{1}{EX^2(s_1)}. \tag{2.7.20}$$

由 (2.7.19) 和 (2.7.20), (2.7.18) 得证, 从而 (2.7.17) 得证.

命题 2.7.1 的证明 由引理 2.7.3, 2.7.4 和 (2.7.11) 得

$$\begin{aligned}
&E(L(x,t+h)-L(x,t))^m\\
&\le\Big(\frac{1}{2\pi}\Big)^{m/2}m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\\
&\quad\frac{\exp(-x^2/2EX^2(s_1))2^m}{\sigma(s_1)\prod_{j=2}^m\sigma(s_j-s_{j-1})}ds_1\cdots ds_m\\
&\le\Big(\frac{2}{\pi}\Big)^{m/2}m!\exp\Big(-\frac{x^2}{2\sigma^2(t+h)}\Big)\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\\
&\quad\frac{1}{\sigma(s_1)\prod_{j=2}^m\sigma(s_j-s_{j-1})}ds_1\cdots ds_m\\
&=\Big(\frac{2}{\pi}\Big)^{m/2}m!h^m\exp\Big(-\frac{x^2}{2\sigma^2(t+h)}\Big)\int\cdots\int_{0\le s_1<s_2<\cdots<s_m\le 1}\\
&\quad\frac{1}{\sigma(s_1h+t)\prod_{j=2}^m\sigma((s_j-s_{j-1})h)}ds_1\cdots ds_m\\
&\le\Big(\frac{2}{\pi}\Big)^{m/2}\frac{m!h^m\exp(-x^2/2\sigma^2(t+h))}{\sigma(h+t)\sigma^{m-1}(h)c_0^m}\int\cdots\int_{0\le s_1<s_2<\cdots<s_m\le 1}\\
&\quad\frac{1}{s_1^\alpha\prod_{j=2}^m(s_j-s_{j-1})^\alpha}ds_1\cdots ds_m.
\end{aligned}\tag{2.7.21}$$

通过一些初等的计算 (参见 Ehm 1981), 对 $b_j<1$, $j=1,\cdots,m$ 成立

$$\begin{aligned}
&\int\cdots\int_{0\le s_1<s_2<\cdots<s_m\le 1}\prod_{j=1}^m(s_j-s_{j-1})^{-b_j}ds_1\cdots ds_m\\
&=\Big(\prod_{j=1}^m\Gamma(1-b_j)\Big)\Big/\Gamma\Big(1+m-\sum_{j=1}^m b_j\Big),
\end{aligned}$$

其中 $\Gamma(\cdot)$ 为 gamma 函数. 从而

$$E(L(x,t+h)-L(x,t))^m \le \Big(\frac{2}{\pi c_0^2}\Big)^{m/2} m!\Big(\frac{h}{\sigma(h)}\Big)^m \cdot \exp(-x^2/2\sigma^2(t+h))\frac{\Gamma^m(1-\alpha)}{\Gamma(1+m(1-\alpha))}. \tag{2.7.22}$$

易知, 当 $0<\alpha\le 1/2$ 时,

$$\Gamma(1-\alpha)\le \frac{2}{1-\alpha}\le 4.$$

注意到 $\Gamma(y)$ 在 $(3/2,\infty)$ 上非降, 我们有

$$\Gamma(1+m(1-\alpha))\ge \Gamma(1+[m(1-\alpha)])=[m(1-\alpha)]!.$$

利用 Stirling 公式得

$$\frac{m!}{[m(1-\alpha)]!}\le 2^m\Big(\frac{1}{1-\alpha}\Big)^m (m!)^\alpha \le 4^m(m!)^\alpha.$$

从而

$$E(L(x,t+h)-L(x,t))^m \le \Big(\frac{16h}{c_0\sigma(h)}\Big)^m\cdot(m!)^\alpha \exp\Big(-\frac{x^2}{2\sigma^2(t+h)}\Big).$$

(2.7.12) 得证.

对平稳的 Gauss 过程, 我们有

命题 2.7.2 设 $\{X(t);t\ge 0\}$ 为一个零均值的平稳 Gauss 过程且 $EX^2(0)=1$. 记 $\sigma^2(h)=E(X(t+h)-X(t))^2$. 假设 $\sigma^2(h)$ 是 $(0,h_0)$ 上的非降凹函数, $\sigma^2(h_0)\le 2$ 满足条件 (2.7.11). 则对任何 $0<h\le h_0$, $x\in\mathcal{R}$ 和任何整数 $m\ge 2$ 有

$$E(L(x,t+h)-L(x,t))^m \le \sigma(h)\Big(\frac{16h}{c_0\sigma(h)}\Big)^m (m!)^\alpha \exp(-x^2/2). \tag{2.7.23}$$

证明 只要用下述引理代替引理 2.7.3, 上述命题的证明与命题 2.7.1 的类似, 故从略.

引理 2.7.5 设 $\{\xi(t); t \geq 0\}$ 为零均值平稳 Gauss 过程且 $E\xi^2(0) = 1$. 记 $\sigma^2(h) = E(\xi(t+h) - \xi(t))^2$. 假设 $\sigma^2(h)$ 为 $(0, h_0)$ 上的非降凹函数且 $\sigma^2(h_0) \leq 2$. 对 $t \leq t_1 < t_2 < \cdots < t_n \leq t + h_0$, 令 A_n 为 $\xi(t_1), \cdots, \xi(t_n)$ 的协方差矩阵. 则

$$|A_n| \geq (1/2)^n \prod_{i=2}^{n} \sigma^2(t_i - t_{i-1}). \tag{2.7.24}$$

证明 令 $\tilde{A}_n$ 为 $\xi(t_n), \xi(t_{n-1}) - \xi(t_n), \cdots, \xi(t_1) - \xi(t_2)$ 的协方差矩阵. 则 $|A_n| = |\tilde{A}_n|$. 余下的证明与引理 2.7.3 的完全类似, 故从略.

对于具有平稳增量的 Gauss 过程的局部时增量的最大值, 我们有下述估计.

命题 2.7.3 设 $\{X(t); t \geq 0\}$ 为一个零均值、具有平稳增量的 Gauss 过程且 $X(0) = 0$. 记 $\sigma^2(h) = E(X(t+h) - X(t))^2$. 假设 $\sigma^2(h)$ 为 $(0, 1)$ 上的非降凹函数, 对某 $0 < \alpha \leq 1/2$ 和 $c_0 > 0$ 满足

$$\sigma(ah) \geq c_0 a^\alpha \sigma(h) \quad \forall 0 < a, h \leq 1. \tag{2.7.25}$$

则对任何 $0 < h \leq 1$, $0 \leq t \leq 1$ 和任意整数 $m \geq 4$ 有

$$E\sup_x (L(x, t+h) - L(x, t))^m \leq C_1 h^{-(4/3)\alpha} \left(\frac{524h}{c_0 \sigma(h)}\right)^m (m!)^{1+\alpha}, \tag{2.7.26}$$

其中 $C_1 = 5000(1 + \sigma(2)) c_0^{-8/3} \sigma^{-4/3}(1)$.

为证命题 2.7.3, 我们还需要一些引理.

引理 2.7.6 设 $\{X(t); t \geq 0\}$ 为零均值、具有平稳增量的 Gauss 过程, 其增量的方差函数为 $\sigma^2(h)$. 令 $m \geq 4$ 为一偶数, $R(s_1, \cdots, s_m)$ 为 $X(s_1)$, $X(s_2) - X(s_1)$, $\cdots$, $X(s_m) - X(s_{m-1})$ 的协方差矩阵. 则对任何 $0 < \delta \leq 1$, $xy \geq 0$, $t \geq 0$, $h > 0$ 有

$$E(L(x+y, t+h) - L(x+y, t) - L(x, t+h) + L(x, t))^m$$
$$\leq 3\left(\frac{1}{2\pi}\right)^{m/2} m! |y|^{2\delta} \int \cdots \int_{t \leq s_1 < s_2 < \cdots < s_m \leq t+h} \exp\left(-\frac{x^2}{4EX^2(s_1)}\right)$$

$$\cdot |R(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2})ds_1\cdots ds_m, \tag{2.7.27}$$

其中

$$\begin{aligned}
\rho_m &= \frac{|(\gamma_{ij},1\le i,j\le m-1)|}{|(\gamma_{ij},1\le i,j\le m)|},\\
\rho_1 &= \frac{|(\gamma_{ij},2\le i,j\le m)|}{|(\gamma_{ij},1\le i,j\le m)|},\\
\rho_2 &= \frac{|(\gamma_{ij},1\le i,j\le m; i,j\ne 2)|}{|(\gamma_{ij},1\le i,j\le m)|},\\
R(s_1,\cdots,s_m) &= (\gamma_{ij},1\le i,j\le m).
\end{aligned}$$

证明 由 (2.7.16) 得

$$\begin{aligned}
&L(x+y,t+h)-L(x+y,t)-L(x,t+h)+L(x,t)\\
=&\frac{1}{2\pi}\int_t^{t+h}\int_{-\infty}^{\infty}(e^{-iu(x+y)}-e^{-iux})e^{iuX(s)}\,duds.
\end{aligned}\tag{2.7.28}$$

记 $v_j=\sum_{l=j}^m u_l$, $1\le j\le m$, $V=(v_1,\cdots,v_m)$, $v_{m+1}=0$. 由 (2.7.28), 可写

$$\begin{aligned}
&E(L(x+y,t+h)-L(x+y,t)-L(x,t+h)+L(x,t))^m\\
=&\left(\frac{1}{2\pi}\right)^m\int_t^{t+h}\cdots\int_t^{t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(e^{-i(x+y)u_j}-e^{-ixu_j})\\
&\cdot E\exp\Big(i\sum_{j=1}^m u_jX(s_j)\Big)du_1\cdots du_m ds_1\cdots ds_m\\
=&\left(\frac{1}{2\pi}\right)^m m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(e^{-i(x+y)u_j}\\
&-e^{-ixu_j})E\exp\Big[iu_1X(s_1)+i\sum_{j=2}^m u_j(X(s_j)-X(s_{j-1}))\Big]\\
&\cdot du_1\cdots du_m ds_1\cdots ds_m\\
=&\left(\frac{1}{2\pi}\right)^m m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}
\end{aligned}$$

$$\prod_{j=1}^{m}\left(e^{-i(x+y)(v_j-v_{j+1})}-e^{-ix(v_j-v_{j+1})}\right)$$

$$\cdot\exp\left(-\frac{1}{2}VR(s_1,\cdots,s_m)V'\right)dv_1\cdots dv_m ds_1\cdots ds_m. \quad (2.7.29)$$

为证 (2.7.27), 只要证

$$\begin{aligned} I:=&\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{m}\left(e^{-i(x+y)(v_j-v_{j+1})}-e^{-ix(v_j-v_{j+1})}\right)\right.\\ &\left.\cdot\exp\left(-\frac{1}{2}VR(s_1,\cdots,s_m)V'\right)dv_1\cdots dv_m\right|\\ \le&3(2\pi)^{m/2}\exp\left(-\frac{x^2}{4EX^2(s_1)}\right)|R(s_1,\cdots,s_m)|^{-1/2}\\ &\cdot|y|^{2\delta}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2}). \end{aligned} \quad (2.7.30)$$

若 $|R(s_1,\cdots,s_m)|=0$, 则 (2.7.30) 显然成立. 因此我们可设 $|R(s_1,\cdots,s_m)|>0$. 即 $R(s_1,\cdots,s_m)$ 是正定的, 从而 $R^{-1}(s_1,\cdots,s_m)$ 也正定. 令 $(Y_1,\cdots,Y_m)$ 为零均值、以 $R^{-1}(s_1,\cdots,s_m)$ 为协方差矩阵的多维正态变量, 且 $Y_{m+1}=0$. 则

$$\begin{aligned} I=&(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\\ &\cdot\left|E\prod_{j=1}^{m}\left(e^{-i(x+y)(Y_j-Y_{j+1})}-e^{-ix(Y_j-Y_{j+1})}\right)\right|\\ \le&(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\\ &\cdot\left|E\prod_{j=2}^{m}\left(e^{-i(x+y)(Y_j-Y_{j+1})}-e^{-ix(Y_j-Y_{j+1})}\right)\right.\\ &\left.\cdot E\left(\left(e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)}\right)\middle|Y_2,\cdots,Y_m\right)\right|\\ \le&(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}E\Big\{|e^{-iyY_m}-1|\\ &\cdot\left|E\left(\left(e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)}\right)\middle|Y_2,\cdots,Y_m\right)\right|\Big\}. \end{aligned} \quad (2.7.31)$$

注意到给定 $Y_2,\cdots,Y_m$ 时, Y_1 的条件分布是正态分布, 其均值为条件均值 $E(Y_1|Y_2,\cdots,Y_m)$, 方差为条件方差

$$E(Y_1-E(Y_1|Y_2,\cdots,Y_m))^2=\frac{1}{\gamma_{11}},\quad 其中\gamma_{11}=EX^2(s_1).$$

从而，我们有

$$
\begin{aligned}
&\left|E\left((e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)})\middle|Y_2,\cdots,Y_m\right)\right| \\
=&\left|\exp\left[i(x+y)Y_2-i(x+y)E(Y_1|Y_2,\cdots,Y_m)-\frac{(x+y)^2}{2\gamma_{11}}\right]\right. \\
&\left.-\exp\left[ixY_2-ixE(Y_1|Y_2,\cdots,Y_m)-\frac{x^2}{2\gamma_{11}}\right]\right| \\
\le&\left|\exp\left[iyY_2-iyE(Y_1|Y_2,\cdots,Y_m)-\frac{(x+y)^2}{2\gamma_{11}}\right]-e^{-x^2/2\gamma_{11}}\right| \\
\le&\exp\left[-\frac{(x+y)^2}{2\gamma_{11}}\right]\left|\exp\left[iyY_2-iyE(Y_1|Y_2,\cdots,Y_m)\right]-1\right| \\
&+\left|\exp\left[-\frac{(x+y)^2}{2\gamma_{11}}\right]-\exp\left[-\frac{x^2}{2\gamma_{11}}\right]\right| \\
\le&\exp\left(-\frac{x^2}{2\gamma_{11}}\right)|y|^\delta\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta\right) \\
&+2\exp\left(-\frac{x^2}{4\gamma_{11}}\right)\left|\frac{y}{\sqrt{\gamma_{11}}}\right|^\delta \\
\le&\exp\left(-\frac{x^2}{2\gamma_{11}}\right)|y|^\delta\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right).
\end{aligned}
\tag{2.7.32}
$$

这里我们用到了不等式:

$$
e^{-b^2}-e^{-a^2}\le 2e^{-b^2/2}((a-b)\wedge 1),\quad \forall a\ge b\ge 0.
$$

由 (2.7.32) 和 (2.7.31), 利用引理 2.7.1 我们得

$$
\begin{aligned}
I\le&(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^\delta E\Bigg\{|e^{-iyY_m}-1| \\
&\cdot\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right)\Bigg\} \\
\le&(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^{2\delta}E\Bigg\{|Y_m|^\delta \\
&\cdot\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le (2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\exp\Big(-\frac{x^2}{4\gamma_{11}}\Big)|y|^{2\delta}(EY_m^2)^{\delta/2}\\
&\quad\cdot\Big((EY_2^2)^{\delta/2}+(EY_1^2)^{\delta/2}+\frac{2}{\gamma_{11}^{\delta/2}}\Big)\\
&=(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\exp\Big(-\frac{x^2}{4\gamma_{11}}\Big)|y|^{2\delta}\\
&\quad\cdot\Big(\frac{|(\gamma_{ij},1\le i,j\le m-1)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}\\
&\quad\cdot\Big(\Big(\frac{|(\gamma_{ij},1\le i,j\le m;i,j\ne 2)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}\\
&\quad+\Big(\frac{|(\gamma_{ij},2\le i,j\le m)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}+\frac{2}{\gamma_{11}^{\delta/2}}\Big)\\
&\le 3(2\pi)^{m/2}|R(s_1,\cdots,s_m)|^{-1/2}\exp\Big(-\frac{x^2}{4\gamma_{11}}\Big)|y|^{2\delta}\\
&\quad\cdot\Big(\frac{|(\gamma_{ij},1\le i,j\le m-1)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}\\
&\quad\cdot\Big(\Big(\frac{|(\gamma_{ij},1\le i,j\le m;i,j\ne 2)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}\\
&\quad+\Big(\frac{|(\gamma_{ij},2\le i,j\le m)|}{|(\gamma_{ij},1\le i,j\le m)|}\Big)^{\delta/2}\Big).
\end{aligned}\tag{2.7.33}
$$

(2.7.30) 得证. 从而引理 2.7.6 得证.

引理 2.7.7 沿用引理 2.7.6 的概念和假设，我们有

$$
\begin{aligned}
&E\sup_x(L(x,t+h)-L(x,t))^m\\
&\le 4\Big(1-\Big(\frac{1}{2}\Big)^{(2\delta-1)/m}\Big)^{-m}m!\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\\
&\quad(1+\sqrt{EX^2(s_1)})\cdot|R(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2})\\
&\quad\cdot ds_1\cdots ds_m
\end{aligned}\tag{2.7.34}
$$

对任意的 $1/2<\delta\le 1$, $t\ge 0$, $h>0$ 和偶数 $m\ge 4$ 成立.

证明 显然

$$
\begin{aligned}
&E\sup_x (L(x,t+h)-L(x,t))^m \\
&\le 2^m \sum_{k=-\infty}^{\infty} E(L(k,t+h)-L(k,t))^m \\
&\quad + 2^m \sum_{k=-\infty}^{\infty} E \sup_{0\le y\le 1} (L(k+y,t+h)-L(k+y,t) \\
&\quad - L(k,t+h)+L(k,t))^m. \qquad (2.7.35)
\end{aligned}
$$

由引理 2.7.4, 我们有

$$
\begin{aligned}
&E(L(k,t+h)-L(k,t))^m \\
&\le \left(\frac{1}{2\pi}\right)^{m/2} m! \int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h} \exp\left(-\frac{k^2}{2EX^2(s_1)}\right) \\
&\quad \cdot |R(s_1,\cdots,s_m)|^{-1/2} ds_1\cdots ds_m. \qquad (2.7.36)
\end{aligned}
$$

由引理 2.7.6 和 Móricz (1982) 的定理得

$$
\begin{aligned}
&E\sup_{0\le y\le 1}(L(k+y,t+h)-L(k+y,t)-L(k,t+h)+L(k,t))^m \\
&\le 3\cdot 2^{m+1}\left(1-\left(\frac{1}{2}\right)^{(2\delta-1)/m}\right)^{-m}\left(\frac{1}{2\pi}\right)^{m/2} m! \\
&\quad \cdot \int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h} \exp\left(-\frac{k^2}{4EX^2(s_1)}\right) \\
&\quad \cdot |R(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2}) ds_1\cdots ds_m. \qquad (2.7.37)
\end{aligned}
$$

由 (2.7.35), (2.7.36) 和 (2.7.37), 通过一些初等计算即得 (2.7.34).

命题 2.7.3 的证明 由引理 2.7.1 和 2.7.3 得

$$
\rho_m \le 2\sigma^{-2}(s_m-s_{m-1}), \quad \rho_1 \le 2(EX^2(s_1))^{-1}, \quad \rho_2 \le 2\sigma^{-2}(s_2-s_1),
$$

$$
|R(s,\cdots,s_m)|^{-1/2} \le 2^m(EX^2(s_1)\cdot\sigma^2(s_2-s_1)\cdots\sigma^2(s_m-s_{m-1}))^{-1/2}.
$$

在引理 2.7.7 中取 $\delta=2/3$, 沿着命题 2.7.1 的证明路线可得

$$
\begin{aligned}
&E\sup_x(L(x,t+h)-L(x,t))^m \\
&\le 8\cdot 2^m\Big(1-\Big(\frac{1}{2}\Big)^{1/3m}\Big)^{-m} m!(1+\sigma(2)) \\
&\quad\cdot\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\frac{1}{\sigma(s_1)\sigma(s_2-s_1)\cdots\sigma^{5/3}(s_m-s_{m-1})} \\
&\quad\cdot(\sigma^{-2/3}(s_1)+\sigma^{-2/3}(s_2-s_1))ds_1\cdots ds_m \\
&\le 16\cdot 2^m(6m)^m m!(1+\sigma(2))h^m \\
&\quad\cdot\int\cdots\int_{t\le s_1<s_2<\cdots<s_m\le t+h}\Big\{1/\big[\sigma(s_1h)\sigma((s_2-s_1)h)\cdots\sigma^{5/3} \\
&\quad((s_m-s_{m-1})h)\big]\Big\}\cdot((\sigma^{-2/3}(s_1h)+\sigma^{-2/3}((s_2-s_1)h)ds_1\cdots ds_m \\
&\le 16\cdot 2^m\cdot(6m)^m\cdot m!(1+\sigma(2))\cdot h^m\cdot\sigma(h)^{-m-4/3}c_0^{-m-4/3} \\
&\quad\cdot\int\cdots\int_{0\le s_1<s_2<\cdots<s_m\le 1} s_1^{-\alpha}(s_2-s_1)^{-\alpha}\cdots(s_m-s_{m-1})^{-5\alpha/3} \\
&\quad\cdot(s_1^{-2\alpha/3}+(s_2-s_1)^{-2\alpha/3})ds_1\cdots ds_m \\
&\le 32\cdot 12^m\cdot m^m\cdot m!(h/\sigma(h))^m\cdot c_0^{-m-4/3}\cdot\sigma^{-4/3}(h)\cdot(1+\sigma(2)) \\
&\quad\cdot\frac{\Gamma^{m-2}(1-\alpha)\cdot\Gamma^2(1-\frac{5}{3}\alpha)}{\Gamma(1+m(1-\alpha)-\frac{4}{3}\alpha)} \\
&\le 5000\cdot(1+\sigma(2))c_0^{-8/3}\sigma^{-4/3}(1)h^{-4\alpha/3}\cdot\Big(\frac{524h}{c_0\sigma(h)}\Big)^m\cdot(m!)^{1+\alpha},
\end{aligned}
$$

在第二个不等式中用到了不等式 $1-2^{-1/(3m)}\ge 1/(6m)$ ($m\ge 4$). (2.7.26) 得证.

对于平稳 Gauss 过程的局部时增量的极大值的矩, 我们有下述估计.

命题 2.7.4 设 $\{X(t);t\ge 0\}$ 为零均值平稳 Gauss 过程且 $EX^2(0)=1$. 假设 $\sigma^2(h)$ 为 $(0,1)$ 上的非降凹函数, 满足条件 (2.7.25). 则对任何偶数 $m\ge 4$ 和 $0<h\le 1$, $0\le t\le 1$ 有

$$
E\sup_x(L(x,t+h)-L(x,t))^m\le C_2\cdot h^{-(1/3)\alpha}\Big(\frac{262h}{c_0\sigma(h)}\Big)^m(m!)^{1+\alpha}, \tag{2.7.38}
$$

其中 $C_2 = 10^4 c_0^{-8/3}\sigma^{-1/3}(1)$.

证明与命题 2.7.3 的类似，从略.

2.7.2 Gauss 过程局部时的增量

现在我们对具有平稳增量的 Gauss 过程和平稳 Gauss 过程，给出类似于 (2.7.5) 和 (2.7.7) 的结论.

定理 2.7.2 设 a_T 和 b_T 为 $T \geq 0$ 的非负函数. 记 $a^* = \sup_{T\geq 0} a_T$. 设 $\{X(t); t \geq 0\}$ 为零均值且具有平稳增量的 Gauss 过程. 假设 $X(0) = 0$, $\sigma^2(h)$ 为 $(0, a^*)$ 上的非降连续凹函数. 还假设存在常数 $0 < \alpha \leq 1/2$ 和 $c_0 > 0$ 使得

$$\sigma(ah) \geq c_0 a^\alpha \sigma(h) \quad \forall 0 \leq a \leq 1, 0 < h \leq a^*. \tag{2.7.39}$$

又假设

$$\frac{1 + b_T}{a_T} \to \infty, \quad T \to \infty. \tag{2.7.40}$$

则

$$\limsup_{T\to\infty} \sup_{0\leq t\leq b_T} \frac{L(x, t + a_T) - L(x, t)}{a_T\left(\log \frac{b_T}{a_T} + \log\log(a_T + \frac{1}{a_T})\right)^\alpha / \sigma(a_T)} \leq \frac{160}{c_0} \quad \text{a.s.} \tag{2.7.41}$$

证明定理 2.7.2 之前，我们先给出几个直接的推论.

推论 2.7.1 设 $\{X(t); t \geq 0\}$ 为一个零均值、具有平稳增量的 Gauss 过程. 假设 $X(0) = 0$, $\sigma^2(h)$ 为 $(0, 1)$ 上的非降连续凹函数，对某 $0 < \alpha \leq 1/2$ 和 $c_0 > 0$ 满足

$$\sigma(ah) \geq c_0 a^\alpha \sigma(h) \quad \forall 0 \leq a, h \leq 1. \tag{2.7.42}$$

则

$$\limsup_{h\to 0} \frac{L(0, h)}{h(2\log\log(1/h))^\alpha / \sigma(h)} \leq \frac{160}{c_0} \quad \text{a.s.}, \tag{2.7.43}$$

$$\limsup_{h\to 0} \sup_{0\leq t\leq 1} \frac{L(0, t + h) - L(0, t)}{h(2\log(1/h))^\alpha / \sigma(h)} \leq \frac{160}{c_0} \quad \text{a.s.} \tag{2.7.44}$$

推论 2.7.2 设 $\{Z(t); t \geq 0\}$ 为一个阶为 α 的分数 Wiener 过程, $0 < \alpha \leq 1/2$, 即它是一个零均值、具有平稳增量的 Gauss 过程且 $\sigma^2(h) = h^{2\alpha}$. 则

$$\limsup_{h\to 0} \frac{L(0,h)}{h^{1-\alpha}(2\log\log(1/h))^{\alpha}} \leq 200 \quad \text{a.s.,} \tag{2.7.45}$$

$$\limsup_{h\to 0} \sup_{0\leq t\leq 1} \frac{L(0,t+h)-L(0,t)}{h^{1-\alpha}(2\log(1/h))^{\alpha}} \leq 200 \quad \text{a.s.} \tag{2.7.46}$$

注 2.7.1 考察 $\alpha = 1/2$ 的情形, 即 $Z(\cdot)$ 为标准 Wiener 过程, 具有局部时 $l(x,t)$. 根据 Kesten (1965) 得到的重对数律 (参见 (2.7.9)), 我们有

$$\limsup_{T\to\infty} \frac{l(0,T)}{(2T\log\log T)^{1/2}} = 1 \quad \text{a.s.} \tag{2.7.47}$$

利用 Kesten (1965) 的方法, 我们也可以证明

$$\limsup_{h\to 0} \frac{l(0,h)}{(2h\log\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{2.7.48}$$

这意味着在 (2.7.45) 中我们得到的上界的阶是最佳的.

现在来证明定理 2.7.2. 我们需要一些引理.

引理 2.7.8 在定理 2.7.2 的假设下, 对任何 $0 < h \leq a^*$, $y > 0$, $x \in \mathcal{R}$ 有

$$P\left\{L(x,t+h) - L(x,t) \geq \frac{16h}{c_0\sigma(h)}y\right\} \leq \frac{K_\alpha \exp\{-x^2/(2\sigma^2(t+h))\}}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}}, \tag{2.7.49}$$

其中 K_α 为只依赖于 α 的正常数.

证明 由命题 2.7.1 和下述引理立即可得.

引理 2.7.9 设 ξ 为非负随机变量. 假设对某 $C > 0$, $\alpha > 0$ 和任何 $m \geq 2$ 成立

$$E\xi^m \leq C(m!)^{\alpha}.$$

则对任何 $y>0$ 有

$$P\{\xi>y\}\le \frac{K_\alpha C}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}}, \tag{2.7.50}$$

其中 K_α 为只依赖于 α 的正常数.

证明 当 $0<y\le 2^\alpha$ 时, 由 Chebyshev 不等式得

$$P\{\xi>y\}\le \frac{E\xi^2}{y^2}\le \frac{K_\alpha C}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}}.$$

当 $y>2^\alpha$ 时, 令 $m=[y^{1/\alpha}]$. 由 Stirling 公式和 Chebyshev 不等式得

$$\begin{aligned}P\{\xi>y\}\le&\frac{E\xi^m}{y^m}\le\frac{C(m!)^\alpha}{y^m}\\ \le&\frac{C(3m(m/e)^m)^\alpha}{y^m}\le C(3y^{1/\alpha})^\alpha e^{-m\alpha}\\ \le&C3^\alpha y\exp\{-(y^{1/\alpha}-1)\alpha\}\le K_\alpha C\exp(-y^{1/\alpha}/2)\\ \le&K_\alpha C(\exp(y^{1/\alpha}/4)-1)^{-2\alpha}.\end{aligned}$$

引理得证.

定理 2.7.2 的证明 令 $1<\theta<\frac{5}{4}$. 定义

$$\begin{aligned}&A_k=\{T:\theta^k<a_T\le\theta^{k+1}\},\quad -\infty<k<\infty,\\ &A_{k,j}=\left\{T:\theta^j\le\frac{b_T+a_T}{\theta^k}<\theta^{j+1},T\in A_k\right\},\quad j=0,1,\cdots,\\ &\beta_T=\log\frac{b_T}{a_T}+\log\log\left(a_T+\frac{1}{a_T}\right).\end{aligned}$$

易知

$$\beta_T\ge\beta_{k,j}:=\log\theta^j+\log\log\theta^{|k|},\quad T\in A_{k,j}.$$

注意到对每个固定的 x, $L(x,t)$ 为 t 的非降函数, 我们有

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{0\le t\le b_T}\frac{L(x,t+a_T)-L(x,t)}{a_T\beta_T^\alpha/\sigma(a_T)}\\ \le&\limsup_{|k|+j\to\infty}\sup_{l\ge j}\sup_{T\in A_{k,l}}\sup_{0\le t\le b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^\alpha}\end{aligned}$$

$$\le \limsup_{|k|+j\to\infty} \sup_{l\ge j} \sup_{T\in A_{k,l}} \sup_{0\le t\le(\theta^{l+1}-1)\theta^k} \frac{\sigma(\theta^{k+1})(L(x,t+\theta^{k+1})-L(x,t))}{\theta^k\beta_{k,l}^\alpha}$$

$$\le 2 \limsup_{|k|+j\to\infty} \sup_{l\ge j} \max_{0\le m\le\theta^l} \frac{\sigma(\theta^{k+1})(L(x,(m+1)\theta^{k+1})-L(x,m\theta^{k+1}))}{\theta^k\beta_{k,l}^\alpha}. \tag{2.7.51}$$

利用引理 2.7.8 得

$$P\left\{\sup_{l\ge j}\max_{0\le m\le\theta^l}\left[\sigma(\theta^{k+1})(L(x,(m+1)\theta^{k+1})-L(x,m\theta^{k+1}))\right]/\left[\theta^k\beta_{k,l}^\alpha\right]\right.$$

$$\left.\ge \theta\left(\frac{2\theta}{\alpha}\right)^\alpha\frac{16}{c_0}\right\}\le \sum_{l=j}^\infty\sum_{m=0}^{[\theta^l]}K_\alpha e^{-\theta\beta_{k,l}}\le C\sum_{l=j}^\infty\theta^{l(\theta-1)}\log^{-\theta}\theta^{|k|}$$

$$\le C\theta^{j(\theta-1)}(|k|+1)^{-\theta}, \tag{2.7.52}$$

其中 C 为只依赖于 θ 和 α 的常数. 由 (2.7.51), (2.7.52) 和 Borel-Cantelli 引理得

$$\limsup_{T\to\infty}\sup_{0\le t\le b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^\alpha}$$

$$\le\frac{32}{c_0}\theta(2\theta/\alpha)^\alpha\le\frac{160}{c_0}\theta^2 \quad \text{a.s.}$$

从而

$$\limsup_{T\to\infty}\sup_{0\le t\le b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^\alpha}\le\frac{160}{c_0}\quad\text{a.s.},$$

(2.7.41) 得证.

与引理 2.7.8 类似, 由引理 2.7.9 和命题 2.7.2 可得

引理 2.7.10 在命题 2.7.2 中的假设下, 对任何 $0<h\le h_0$, $t\ge 0$, $y>0$ 成立

$$P\left\{L(x,t+h)-L(x,t)\ge\frac{16h}{c_0\sigma(h)}y\right\}\le\frac{K_\alpha\exp(-x^2/2)\cdot\sigma(h)}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}},$$

其中 K_α 为只依赖于 α 的正常数.

用引理 2.7.10 代替引理 2.7.8, 沿着定理 2.7.2 的证明路线, 我们可得下述关于平稳 Gauss 过程局部时的轨道性质的定理.

定理 2.7.3 设 b_h 是 $(0,1)$ 上的 h 的函数, $\{X(t);\ t \geq 0\}$ 为零均值平稳 Gauss 过程且 $EX^2(0)=1$. 假设 $\sigma^2(h)$ 为 $(0,1)$ 上的非降连续凹函数, 且对某 $0<\alpha\leq 1/2$, $c_0>0$ (2.7.42) 成立. 则对每个 $x\in\mathcal{R}$ 有

$$\limsup_{h\to 0}\sup_{0\leq t\leq b_h}\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))\beta_h^\alpha}\leq\frac{160}{c_0}\quad \text{a.s.}, \tag{2.7.53}$$

其中

$$\beta_h=\log\left(1+\left(\frac{h+b_h}{h}\right)\sigma(h)\log^{3/2}\frac{1}{h}\right). \tag{2.7.54}$$

下面是一些直接的推论:

推论 2.7.3 在定理 2.7.3 的假设下, 对任何 $x\in\mathcal{R}$, $0\leq\theta\leq 1$ 成立

$$\limsup_{h\to 0}\sup_{0\leq t\leq h^\theta}\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))\log^\alpha(1+h^{\theta-1}\sigma(h)\log^{3/2}(1/h))}$$
$$\leq\frac{160}{c_0}\quad \text{a.s.}, \tag{2.7.55}$$

$$\limsup_{h\to 0}\frac{L(x,h)}{h\sigma^{\alpha-1}(h)\log^{2\alpha}(1/h)}=0\quad \text{a.s.} \tag{2.7.56}$$

推论 2.7.4 设 $\{X(t);t\geq 0\}$ 为零均值平稳 Gauss 过程满足 $EX^2(0)=1$. 假设 $\sigma^2(h)$ 为 $(0,1)$ 上的非降连续凹函数, 满足对某 $0<\alpha\leq\frac{1}{2}$, $c_1,c_0>0$ 和任何 $0<h\leq 1$, $c_1h^\alpha\leq\sigma(h)\leq c_0h^\alpha$ 成立. 则对任何 $x\in\mathcal{R}$, $0\leq\theta\leq 1$ 有

$$\limsup_{h\to 0}\sup_{0\leq t\leq h^\theta}\frac{L(x,t+h)-L(x,t)}{h^{1+\alpha(\theta-2+\alpha)}\log^{2\alpha}(1/h)}=0\quad \text{a.s.} \tag{2.7.57}$$

推论 2.7.5 设 $\{X(t); t \geq 0\} = \{\sum_{k=1}^{\infty} X_k(t); t \geq 0\}$, 其中 $\{X_k(t); t \geq 0\}$ 为独立的 O-U 过程, 其系数为 $\gamma_k \geq 0$, $\lambda_k > 0$. 假设 $\Gamma_0 = \sum_{k=1}^{\infty} \gamma_k/\lambda_k = 1$, 并且 (2.7.42) 对某 $0 < \alpha \leq 1/2$, $c_0 > 0$ 成立. 则 (2.7.55) 和 (2.7.56) 成立. 特别地, 对每个 $x \in \mathcal{R}$, $0 \leq \theta \leq 1$ 有

$$\limsup_{h \to 0} \sup_{0 \leq t \leq h^\theta} \frac{L(x, t+h) - L(x,t)}{(h/\sigma(h)) \log^{1/2}(1 + h^{\theta-1}\sigma(h)\log^{3/2}(1/h))} \leq \frac{160}{c_0} \quad \text{a.s.}, \tag{2.7.58}$$

$$\limsup_{h \to 0} \frac{L(x,h)}{h\sigma^{-1/2}(h)\log(1/h)} = 0 \quad \text{a.s.} \tag{2.7.59}$$

注 2.7.2 比较 (2.7.48) 和 (2.7.59), 我们发现具有平稳增量且 $X(0) = 0$ 的 Gauss 过程局部时的极限性质与平稳 Gauss 过程局部时的极限性质完全不同.

2.7.3 Gauss 过程局部时的最大连续模

设 $\{X(t); t \geq 0\}$ 为一个零均值、具有平稳增量的 Gauss 过程. 我们来研究 $L(x,t)$ 关于 $x \in \mathcal{R}$ 的最大值对变量 $t \geq 0$ 的连续模. 下述结果类似于 (2.7.6).

定理 2.7.4 设 $\{X(t); t \geq 0\}$ 为零均值、具有平稳增量的 Gauss 过程且 $X(0) = 0$, $L(x,t)$ 为 $X(\cdot)$ 的局部时. 记 $\sigma^2(h) = E(X(t+h) - X(t))^2$. 假设 $\sigma^2(h)$ 为 $(0,1)$ 上的非降连续凹函数, 并且存在常数 $0 < \alpha \leq 1/2$ 和 $c_0 > 0$ 使得

$$\sigma(ah) \geq c_0 a^\alpha \sigma(h) \quad \forall 0 < a, h \leq 1. \tag{2.7.60}$$

则

$$\limsup_{h \to 0} \sup_{0 \leq t \leq 1} \sup_{-\infty < x < \infty} \frac{L(x, t+h) - L(x,t)}{h(\log(1/h))^{\alpha+1}/\sigma(h)} \leq \frac{524(3+4\alpha)^{\alpha+1}}{c_0} \quad \text{a.s.} \tag{2.7.61}$$

定理 2.7.5 设 $\{X(t); t \geq 0\}$ 为零均值平稳 Gauss 过程且 $EX^2(0) = 1$. 假设 $\sigma^2(h)$ 为 $(0,1)$ 上的非降连续凹函数, 对某 $0 < \alpha \leq 1/2$ 和 $c_0 > 0$, (2.7.60) 成立. 则 (2.7.61) 成立.

定理 2.7.4 和 2.7.5 的证明依赖于下述两个引理, 通过利用命题 2.7.3 (对应地, 命题 2.7.4) 代替命题 2.7.1, 它们的证明与引理 2.7.8 的类似.

引理 2.7.11 在定理 2.7.4 的假设下, 对任何 $y > 1$, $0 \leq t, h \leq 1$ 成立

$$\begin{aligned} P\Big\{ \sup_x (L(x,t+h) - L(x,t)) > y\frac{524h}{c_0\sigma(h)} \Big\} \\ \leq K_\alpha C_1 h^{-4\alpha/3} \exp(-y^{1/(1+\alpha)}/2), \end{aligned} \tag{2.7.62}$$

其中 C_1 如命题 2.7.3 所示, K_α 是只依赖于 α 的正常数.

引理 2.7.12 在定理 2.7.5 的假设下, 对任何 $y > 1$, $0 \leq t, h \leq 1$ 有

$$\begin{aligned} P\Big\{ \sup_x (L(x,t+h) - L(x,t)) > y\frac{524h}{c_0\sigma(h)} \Big\} \\ \leq K_\alpha C_2 h^{-\alpha/3} \exp(-y^{1/(1+\alpha)}/2), \end{aligned} \tag{2.7.63}$$

其中 C_2 如命题 2.7.4 所示, K_α 是只依赖于 α 的正常数.

定理 2.7.4 的证明 令 $1 < \theta < 5/4$. 则

$$\begin{aligned} &\limsup_{h\to 0} \sup_{0\leq t\leq 1} \sup_x \frac{L(x,t+h) - L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha+1}} \\ &\leq \limsup_{k\to\infty} \sup_{\theta^{-k-1}\leq h\leq\theta^k} \sup_{0\leq t\leq 1} \sup_x \frac{L(x,t+h) - L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha+1}} \\ &\leq \limsup_{k\to\infty} \sup_{0\leq t\leq 1} \sup_x \frac{L(x,t+\theta^{-k}) - L(x,t)}{(\theta^{-k-1}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}} \\ &\leq 2\theta \limsup_{k\to\infty} \max_{0\leq j\leq\theta^k} \sup_x \frac{L(x,(j+1)\theta^{-k}) - L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}}. \end{aligned} \tag{2.7.64}$$

由 (2.7.62) 得

$$P\left\{\max_{0\le j\le\theta^k}\sup_x\frac{L(x,(j+1)\theta^{-k})-L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}}\ge\frac{524}{c_0}((3+4\alpha)\theta)^{1+\alpha}\right\}$$
$$\le K_\alpha(\theta^k+1)C_1\cdot\theta^{4\alpha/3}\exp[-(3+4\alpha)\log\theta^k/2]\le K\theta^{-k(\theta-1)}.$$

从而，由 Borel-Cantelli 引理得

$$\limsup_{k\to\infty}\sup_{0\le j\le\theta^k}\sup_x\frac{L(x,(j+1)\theta^{-k})-L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}}$$
$$\le\frac{524}{c_0}((3+4\alpha)\theta)^{1+\alpha}\quad\text{a.s.}\tag{2.7.65}$$

由 (2.7.64) 和 (2.7.65), 并令 θ 充分接近于 1, (2.7.61) 得证.

定理 2.7.5 的证明 证明与定理 2.7.4 的类似，从略.

注 2.7.3 设 $\{W(t);t\ge0\}$ 为标准 Wiener 过程，其局部时为 $l(x,t)$. 由 (2.7.61) 我们得

$$\limsup_{h\to0}\sup_{0\le t\le1}\sup_x\frac{l(x,t+h)-l(x,t)}{h^{1/2}(\log(1/h))^{3/2}}\le262(48)^{3/2}\quad\text{a.s.}$$

比较这一结果和 (2.7.6), 我们发现结论 (2.7.61) 可能不是最好的. 我们相信在定理 2.7.4 和定理 2.7.5 的条件下，(2.7.61) 可由下式代替：对某个常数 c_0^*,

$$\limsup_{h\to0}\sup_{0\le t\le1}\sup_x\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha}}\le c_0^*\quad\text{a.s.}$$

第三章　无穷维 Gauss 过程的连续模和大增量

§3.1　l^p 值 Gauss 过程的连续性

对于无穷维 Gauss 过程的研究，不仅有理论上的兴趣，也有很强的实际背景. 在 §2.1 的第 5 小节中，作为一类随机微分方程无穷组列的解，我们已经引入了无穷维 Ornstein-Uhlenbeck (O-U) 过程. 在这一章中我们将研究这一类过程的连续性、连续模和大增量，并将有关结果推广到较一般的无穷维 Gauss 过程.

设 $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ 是独立的平稳 Gauss 过程序列， $\mathrm{E}X_k(t) = 0$, $\sigma_k^2(h) = \mathrm{E}(X_k(t+h) - X_k(t))^2$, 其中对每一 $k \ge 1$, $\sigma_k(h)$ 是非降连续函数. 首先, 我们假设 $X_k(\cdot)$ 是具有参数 $\gamma_k \ge 0$ 和 $\lambda_k > 0$ 的 O-U 过程, 即 $X_k(\cdot)$ 是一平稳、零均值 Gauss 过程且具有 $\mathrm{E}X_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp\{-\lambda_k|t-s|\}$ 和

$$\sigma_k^2(h) := \mathrm{E}(X_k(t+h) - X_k(t))^2 = \frac{2\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}), \quad h > 0. \tag{3.1.1}$$

(3.1.1) 可推出

$$(2/e)\gamma_k(\lambda_k^{-1} \wedge h) \le \sigma_k^2(h) \le 2\gamma_k(\lambda_k^{-1} \wedge h). \tag{3.1.2}$$

在本节中，我们将研究 $p \ge 1$ 时在空间 l^p 中 $Y(\cdot)$ 的连续性.

3.1.1　l^2 值 O-U 过程

首先我们引述下列不等式 (参见 Iscoe 等 1990), 它可从定理 2.1.6 和推论 2.1.2 得到：存在 $0 < c_p \le C_p < \infty$ 使得

$$
\begin{aligned}
&c_p\Big\{E\|Y(0)\|_{l^p}+\sup_{\|\{y_k\}\|_{l^q}\le 1}\int_0^{1/2}\frac{(\sum_{k=1}^{\infty}y_k^2\sigma_k^2(u))^{1/2}}{u(\log(1/u))^{1/2}}du\Big\}\\
&\quad\le E\sup_{t\in[0,1]}\|Y(t)\|_{l^p}\\
&\quad\le C_p\Big\{E\|Y(0)\|_{l^p}+\sup_{\|\{y_k\}\|_{l^q}\le 1}\int_0^{1/2}\frac{(\sum_{k=1}^{\infty}y_k^2\sigma_k^2(u))^{1/2}}{u(\log(1/u))^{1/2}}du\Big\},
\end{aligned}\tag{3.1.3}
$$

其中 $1/p+1/q=1$.

先来考察 $p=2$ 的情形. 对 $x\ge 0$, 记 $K(x)=\{k\in\mathcal{N};\gamma_k>\lambda_k x\}$, $\lambda(x)=\sup\{\lambda_k;k\in K(x)\}$, 当 $K(x)$ 是空集时, 令 $\lambda(x)=0$. 下述定理是 Fernique(1989) 得到的.

定理 3.1.1 *在 l^2 中 $Y(\cdot)$ 是 a.s. 连续的当且仅当 $\sum_{k=1}^{\infty}\gamma_k/\lambda_k<\infty$ 且*

$$
\int_0^{\infty}\log^+(\lambda(x))dx<\infty.
$$

证明 我们先证充分性. 对任给的 $\delta>0,\rho>0$ 和整数 $N>0$, 写

$$
\begin{aligned}
&P\Big\{\sup_{|t-s|\le\delta,s,t\in[0,1]}\|Y(t)-Y(s)\|_{l^2}\ge 3\rho\Big\}\\
&\quad\le P\Big\{\sup_{|t-s|\le\delta,s,t\in[0,1]}\Big(\sum_{k\le N}|X_k(t)-X_k(s)|^2\Big)^{1/2}\ge\rho\Big\}\\
&\qquad+2P\Big\{\sup_{t\in[0,1]}\sum_{k>N}X_k(t)^2\ge\rho\Big\}.
\end{aligned}\tag{3.1.4}
$$

因为 O-U 过程是连续的, 故当 $\delta\to 0$ 时 (3.1.4) 右边第一项趋于 0. 为估计 (3.1.4) 右边第二个概率, 我们运用 (3.1.3), 但此时我们考察 $k\ge N$ 代替 $k\ge 1$. 对应于 $\mathrm{E}\|Y(0)\|_{l^2}$, 对任给 $\epsilon>0$, 当 $N=N(\epsilon)$ 充分大时有

$$
\sum_{k\ge N}\mathrm{E}X_k(0)^2=\sum_{k\ge N}\gamma_k/\lambda_k<\epsilon.\tag{3.1.5}
$$

在 (3.1.3) 的积分中记 $x=(\log u^{-1})^{1/2}$ 并注意到 (3.1.2), 适当调整常数 c_p 的 C_p, 我们可用如下积分代替该积分

$$M(y):=\int_{\log 2}^{\infty}\Big(\sum_{k\geq N}y_k^2\gamma_k(\lambda_k^{-1}\wedge e^{-x^2})\Big)^{1/2}dx$$

其中 $y=(y_1,y_2,\ldots)$. 显然

$$M(y)=M_1(y)+M_2(y),$$

其中

$$M_1(y)=\int_{\log 2}^{\infty}\Big(\sum_{k\geq N}y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx,$$

$$M_2(y)=\int_{\log 2}^{\infty}\Big(\sum_{k\geq N}y_k^2\gamma_k e^{-x^2}I(\lambda_k\leq e^{x^2})\Big)^{1/2}dx.$$

设 B 是 l^2 中的单位球. 易知

$$\sup_{y\in B}M_2(y)\leq\sum_{k\geq N}\gamma_k/\lambda_k<\varepsilon. \tag{3.1.6}$$

因此只要估计 $M_1(y)$ 即可. 对 $j=\ldots,-1,0,1,\ldots$, 令

$$K_j=\{k:2^{-j-1}<\gamma_k/\lambda_k\leq 2^{-j}\},\ \lambda_j^*=\sup_{k\in K_j}\lambda_k,$$

$$J:=J(N)=\sup\Big\{j:\sup_{k\geq N}\gamma_k/\lambda_k\leq 2^{-j}\Big\}.$$

那么由条件 $\int_{-\infty}^{\infty}\log^+(\lambda(x))dx<\infty$, 当 N 充分大 (因此 J 也充分大) 时

$$\begin{aligned}\sup_{y\in B}M_1(y)&\leq\sup_{y\in B}\int_{\log 2}^{\infty}\Big(\sum_{j\geq J}2^{-j}\sum_{k\in K_j}y_k^2I(\lambda_j^*>e^{x^2})\Big)^{1/2}dx\\&\leq\sup_{z\in B}\int_{\log 2}^{\infty}\sum_{j\geq J}2^{-j/2}|z_j|I(\lambda_j^*>e^{x^2})dx\\&\leq\sup_{z\in B}\sum_{j\geq J}2^{-j/2}(\log^+\lambda_j^*)^{1/2}|z_j|\\&\leq\Big(\sum_{j\geq J}2^{-j}\log^+\lambda_j^*\Big)^{1/2}<\varepsilon.\end{aligned}\tag{3.1.7}$$

结合 (3.1.5)—(3.1.7) 选取 N 充分大, (3.1.4) 右边可任意小. 因此我们推得在 l^2 中 $Y(\cdot)$ a.s. 连续.

其次, 我们来证必要性. 假设在 l^2 中 $Y(\cdot)$ 是 a.s 连续的. 则它是 a.s. 局部有界的. 由定理 2.1.1, 我们有 $E\sup_{t\in[0,1]}\|Y(t)\|_{l^2}<\infty$. 因此由 (3.1.3) 可得

$$E\|Y(0)\|_{l^2}=\sum_{k=1}^{\infty}\gamma_k/\lambda_k<\infty,\qquad L:=\sup_{y\in B}L(y)<\infty,\tag{3.1.8}$$

其中

$$L(y)=\int_0^{\infty}\Big(\sum_{k=1}^{\infty}y_k^2\gamma_k(\lambda_k^{-1}\wedge e^{-x^2})\Big)^{1/2}dx.\tag{3.1.9}$$

令 $H_n=\{k:\exp(n^2)\le\lambda_k<\exp((n+1)^2)\}$, $s_n=\sup_{k\in H_n}\gamma_k/\lambda_k$, $z_n^2=\sum_{k\in H_n}y_k^2$. 那么

$$\begin{aligned}L_1(y):&=\int_0^{\infty}\Big(\sum_{k=1}^{\infty}y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx\\&=\sum_{n=1}^{\infty}\int_{n-1}^{n}\Big(\sum_{k=1}^{\infty}y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx\\&\ge\sum_{n=1}^{\infty}\Big(\sum_{m\ge n}\sum_{k\in H_m}y_k^2\gamma_k\lambda_k^{-1}\Big)^{1/2}=:L_1'(y).\end{aligned}$$

$$L_1'(y)\le\sum_{n=1}^{\infty}\Big(\sum_{m\ge n}z_m^2s_m\Big)^{1/2}.$$

另一方面, 对任一 $z'\in B$, 若定义 $k_m=\sup\{k:\gamma_k/\lambda_k=s_m,k\in H_m\}$, 且对 $k\in H_m$, 设 $y'_{k_m}=z'_m$, $y'_k=0$, $k\ne k_m$, 则 $y'=(y'_1,y'_2,\dots)\in B$. 因此我们有

$$\sup_{y\in B}L_1(y)\ge\sup_{y\in B}L_1'(y)=\sup_{z\in B}\sum_{n=1}^{\infty}\Big(\sum_{m\ge n}z_m^2s_m\Big)^{1/2}=:L_2(s).$$

我们可设 $\{s_m\}$ 是递减的, 不然的话存在一递减的序列 $\{s'_m\}$, 使

得 $s_m \le s'_m$ 且 $L_2(s)=L_2(s')$. 设 $z_m^2 = ms_m/\sum_{k=1}^{\infty} ks_k$. 则

$$\begin{aligned}
L_2(s)^2 &\ge \Big(\sum_{n=1}^{\infty}\Big(\sum_{m\ge n} ms_m^2\Big)^{1/2}\Big)\Big(\sum_{k=1}^{\infty} ks_k\Big)^{-1/2} \\
&\ge \Big(\sum_{n=1}^{\infty}\Big(n\sum_{m=n}^{2n-1} s_m^2\Big)^{1/2}\Big)\Big(\sum_{k=1}^{\infty} ks_k\Big)^{-1/2} \\
&\ge \Big(\sum_{n=1}^{\infty} ns_{2n-1}\Big)\Big(\sum_{k=1}^{\infty} ks_k\Big)^{-1/2} \ge \frac{1}{2}\Big(\sum_{n=1}^{\infty} ns_n\Big)^{1/2}. \quad (3.1.10)
\end{aligned}$$

注意到由 (3.1.8) 有 $L_2(s)<\infty$, 我们得

$$\sum_{n=1}^{\infty} ns_n < \infty. \tag{3.1.11}$$

现在我们来证

$$\int_0^{\infty} \log^+(\lambda(x))dx < \infty. \tag{3.1.12}$$

显然由 (3.1.8) 可得 $\sup_{k\ge 1}\gamma_k/\lambda_k<\infty$. 注意到 $\lambda(x)$ 是非增的函数, 因此当 $x>\sup_{k\ge1}\gamma_k/\lambda_k$ 时 $\lambda(x)=0$. 记 $x_0=\sup\{x:\lambda(x)>e\}$, 则 $x_0<\infty$. 故为证 (3.1.12) 只需证明

$$\int_0^{x_0} \log(\lambda(x))dx < \infty. \tag{3.1.13}$$

若 $\sup_{k\ge1}\lambda_k<\infty$, 此时 $\lambda(x)$ 是有界的, 易见 (3.1.13) 正确. 若 $\sup_{k\ge1}\lambda_k=\infty$, 则当 $x\to 0$ 时 $\lambda(x)\to\infty$. 令 $y=\log\lambda(x)$, 则 $x=\lambda^{-1}(e^y)$. 我们有

$$\begin{aligned}
\int_0^{x_0} \log(\lambda(x))dx &= x\log(\lambda(x))\Big|_0^{x_0} - \int_0^{x_0} x d\log\lambda(x) \\
&= \lambda^{-1}(e^y)y\Big|_{\infty}^{1} + \int_1^{\infty}\lambda^{-1}(e^y)dy.
\end{aligned}$$

此时我们有

$$y\lambda^{-1}(e^y)\to 0 \quad 当 \quad y\to\infty \text{ 时},$$

且由 (3.1.11)

$$\int_1^\infty \lambda^{-1}(e^y)dy = \sum_{n=1}^\infty \int_{n^2}^{(n+1)^2} \lambda^{-1}(e^y)dy$$
$$\le \sum_{n=1}^\infty ((n+1)^2 - n^2) \max_{k\in H_n} \gamma_k/\lambda_k$$
$$\le 2\sum_{n=1}^\infty (n+1)s_n < \infty.$$

综合这些结果我们得 (3.1.13). 必要性得证. 定理 3.1.1 证毕.

利用这一定理, 我们可给出在 l^2 中 $Y(\cdot)$ 连续的若干充分条件. 其中之一如下:

推论 3.1.1 假若

$$\sum_{k=1}^\infty (\gamma_k/\lambda_k)(1+\log^+\lambda_k) < \infty,$$

那么 $Y(\cdot)$ 在 l^2 中连续.

注 3.1.1 Fernique (1990a) 将定理 3.1.1 推广至 $p \ge 2$ 的情形: 在 l^p $(p \ge 2)$ 中, $Y(\cdot)$ 是 a.s. 连续的当且仅当

$$\sum_{k=1}^\infty (\gamma_k/\lambda_k)^{p/2} < \infty,$$

$$\int_0^\infty x^{p/2-1}(\log^+\lambda(x))^{p/2}dx < \infty.$$

3.1.2 l^p 值 O-U 过程

在本小节中我们将利用 Dirichlet 型理论来得出在 l^p $(p \ge 1)$ 中 $Y(\cdot)$ 具有连续样本轨道的条件. 过程 $Y(\cdot)$ 有状态空间 $\mathcal{R}^\infty$ 且有不变测度

$$m = \prod_{k=1}^\infty N(0, \gamma_k/\lambda_k).$$

$\mathcal{R}^\infty$ 中的点记作 $x=(x_1,x_2,\ldots)$. 与过程 $Y(\cdot)$ 相关的 Dirichlet 型 $\mathcal{E}$ 由定义在 $L^2(Y,P)$ 上的下列双线性型的闭包给出:

$$\mathcal{E}(u,v)=\frac{1}{2}\int\sum_{k=1}^{\infty}\gamma_k\Big(\frac{\partial u}{\partial x_k}\Big)\Big(\frac{\partial v}{\partial x_k}\Big)dm,$$

其中

$$u,v\in\mathcal{D}(\mathcal{E}):=\{u\in L^2;\, u=\varphi(x_1,\ldots,x_k),\, \varphi\in C_0^\infty(\mathcal{R}^k),\, k\geq 1\},$$

$C_0^\infty(\mathcal{R}^k)$ 是 $\mathcal{R}^k$ 上具有紧支撑的、且有任意阶连续导数的连续函数空间. 为证 $Y(\cdot)$ 的连续性我们需要下述 Fukushima (1980) 的引理. 在 $\mathcal{D}(\mathcal{E})$ 上由 $\mathcal{E}_1(u,v)=\mathcal{E}(u,v)+(u,v)_{L^2}$ 定义型 $\mathcal{E}_1$.

引理 3.1.1 每个 $u\in\mathcal{D}(\mathcal{E})$ 有一拟连续 (quasi-continuous) 修正 $\tilde{u}$ 使得

$$P\{\text{ 映射}t\to\tilde{u}(Y(t))\text{ 是连续的}\}=1.$$

进一步, 若在 $\mathcal{E}_1$ 型中 $u_n\to u$, 则对每一 $T>0$ 存在一个子列 $\{u_{n_k}\}$ 使得

$$P\{\text{在}[0,T]\text{上},\tilde{u}_{n_k}(Y(t))\text{ 一致收敛于}\tilde{u}(Y(t))\}=1.$$

下列结果是 Schmuland (1990) 得到的. 定义

$$\delta_k=4^{p/2}\Big(\gamma_k/\lambda_k\Big)^{p/2}\log^{p/2}\Big(\gamma_k\Big(\gamma_k/\lambda_k\Big)^{p/2-1}\vee e\Big).$$

定理 3.1.2 若 $\sum_{k=1}^{\infty}\delta_k<\infty$, 则在 l^p 中 $Y(\cdot)$ a.s. 连续.

证明 由假设知 $\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2}<\infty$, 这就推出 $Y(\cdot)\in l^p$ a.s. 定义

$$u_n=\sum_{k=1}^{n}|x_k(t)|^p\vee\delta_k.$$

函数 u_n 属于闭的型 $\mathcal{E}$ 的定义域，且

$$\frac{\partial u_n(t)}{\partial x_k(t)} = p|x_k(t)|^{p-1}I\{|x_k(t)|^p > \delta_k\}I\{k \le n\}.$$

所以，对 $n > m$ 有

$$\begin{aligned}
&\mathcal{E}(u_n - u_m, u_n - u_m) \\
&= \frac{p^2}{2}\int \sum_{k=m+1}^{n} \gamma_k |x_k|^{2(p-1)} I\{|x_k|^p > \delta_k\} dm(x) \\
&\le c \sum_{k=m+1}^{n} \gamma_k \left(\frac{\gamma_k}{\lambda_k}\right)^{p-1} \exp\left\{-\frac{1}{4}\delta_k^{2/p} \Big/ \left(\frac{\gamma_k}{\lambda_k}\right)\right\} \\
&= c \sum_{k=m+1}^{n} \left(\frac{\gamma_k}{\lambda_k}\right)^{p/2}.
\end{aligned} \tag{3.1.14}$$

其中的不等式利用了下列事实：当 $x \to \infty$ 时

$$\int_x^\infty y^p e^{-y^2/2} dy \le e^{-x^2/2} P(x) = O(e^{-x^2/4}),$$

这里 $P(\cdot)$ 是某一多项式. 现在 $u_n \uparrow u = \sum_{k=1}^\infty |x_k|^p \vee \delta_k$ 且 $u \le \|x\|_{l^p}^p + \sum_{k=1}^\infty \delta_k$, 它属于 L^2, 故在 L_2 中 $u_n \to u$. 由 (3.1.14) 可得 $\{u_n\}$ 是 $\mathcal{E}$-Cauchy 序列，故在 $\mathcal{E}_1$ 范数意义下， $u - u_n \to 0$.

应用引理 3.1.1, 对任给的 $T > 0$, 我们有

$$\begin{aligned}
&P\Big\{\sup_{0\le t\le T} \sum_{k=n+1}^{\infty} |X_k(t)|^p \overset{n\to\infty}{\longrightarrow} 0\Big\} \\
&\qquad \ge P\Big\{\sup_{0\le t\le T} (u - u_n)Y(t) \overset{n\to\infty}{\longrightarrow} 0\Big\} = 1,
\end{aligned}$$

这就给出了所要证的结论.

推论 3.1.2 若对 $1 \le p < 2$, $\sum_{k=1}^\infty (\gamma_k/\lambda_k)^{p/2} \log(\lambda_k \vee e)^{p/2} < \infty$, 或者对 $2 \le p < \infty$, $\sum_{k=1}^\infty (\gamma_k/\lambda_k)^{p/2} \log(\gamma_k \vee e)^{p/2} < \infty$, 那么 $Y(\cdot)$ 在 l^p 中是 a.s. 连续的.

注 3.1.2 由上面的证明可见 $Y(\cdot)$ 的分量间的独立性对于定理 3.1.2 的结论不是必要的.

3.1.3 l^p 值 Gauss 过程

设 $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ 为本节开始定义的 Gauss 过程序列. 令

$$\sigma(p,h)=\Big(\sum_{k=1}^{\infty}\sigma_k^p(h)\Big)^{1/p},\quad p\geq 1,$$
$$\sigma^*(h)=\max_{k\geq 1}\sigma_k(h),$$
$$\tilde{\sigma}(p,h)=\begin{cases}\sigma(\frac{2p}{2-p},h), & \text{当}1\leq p<2\text{时},\\ \sigma^*(h), & \text{当}p\geq 2\text{时},\end{cases}$$
$$\delta_p^p=\mathrm{E}|N(0,1)|^p,\qquad p\geq 1.$$

由于 $\mathrm{E}\|Y(t+h)-Y(t)\|_{l^p}^p=\delta_p^p\sum_{k=1}^{\infty}\sigma_k^p(h)$, 故对固定的 t 和 h, $Y(t+h)-Y(t)\in l^p$ $(p\geq 1)$ a.s. 成立当且仅当

$$\sigma(p,h)<\infty, \tag{3.1.15}$$

且对每一个 t, $Y(t)\in l^p$ a.s. 当且仅当 (3.1.15) 和

$$\sum_{k=1}^{\infty}E|X_k(0)|^p<\infty. \tag{3.1.16}$$

引理 3.1.2 对 $p\geq 1$ 和任何 $t,x,h\geq 0$ 有

$$P\{\|Y(t+h)-Y(t)\|_{l^p}\geq\delta_p\sigma(p,h)+x\tilde{\sigma}(p,h)\}\leq 2\exp(-x^2/2).$$

证明 设 $\{\xi_n;n\geq 1\}$ 是独立正态随机变量列, $E\xi_n=0$ 且 $\sum_{k=1}^{\infty}(E\xi_k^2)^{p/2}<\infty$. 众所周知

$$\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}=\sup_{\|a\|_{l^q}\leq 1}\sum_{k=1}^{\infty}\xi_k a_k, \tag{3.1.17}$$

其中 $q=p/(p-1), a=(a_1,a_2,\ldots)\in l^q$. 应用 Borell 不等式 (定理 1.1.1), 我们有

$$\begin{aligned}
&P\Big\{\Big|\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}-E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}\Big|\ge x\Big\}\\
&=P\Big\{\Big|\sup_{\|a\|_{l^q}\le 1}\sum_{k=1}^{\infty}a_k\xi_k-E\sup_{\|a\|_{l^q}\le 1}\sum_{k=1}^{\infty}a_k\xi_k\Big|\ge x\Big\}\\
&\le 2\exp\Big\{-\frac{x^2}{2\sup_{\|a\|_{l^q}\le 1}E(\sum_{k=1}^{\infty}a_k\xi_k)^2}\Big\}\\
&=2\exp\Big\{-\frac{x^2}{2\sup_{\|a\|_{l^q}\le 1}\sum_{k=1}^{\infty}a_k^2E\xi_k^2}\Big\}.
\end{aligned}$$

注意到

$$\begin{aligned}
&\sup_{\|a\|_{l^q}\le 1}\sum_{k=1}^{\infty}a_k^2E\xi_k^2\\
&\le\begin{cases}\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{\frac{q}{q-2}}\Big)^{\frac{q-2}{q}}\sup\limits_{\|a\|_{l^q}\le 1}\Big(\sum_{k=1}^{\infty}|a_k|^q\Big)^{\frac{2}{q}}, & \text{当}1\le p<2\text{时},\\ \max\limits_{k\ge 1}E\xi_k^2\sup\limits_{\|a\|_{l^q}\le 1}\sum_{k=1}^{\infty}a_k^2, & \text{当}p\ge 2\text{时}\end{cases}\\
&=\begin{cases}\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/(2-p)}\Big)^{(2-p)/p}, & \text{当}1\le p<2\text{时},\\ \max\limits_{k\ge 1}E\xi_k^2, & \text{当}p\ge 2\text{时},\end{cases}
\end{aligned}$$

我们得

$$\begin{aligned}
&P\Big\{\Big|\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}-E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}\Big|\ge x\Big\}\\
&=\begin{cases}2\exp\Big\{-\frac{x^2}{2(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/(2-p)})^{(2-p)/p}}\Big\}, & \text{当}1\le p<2\text{时},\\ 2\exp\Big\{-\frac{x^2}{2\max_{k\ge 1}E\xi_k^2}\Big\}, & \text{当}p\ge 2\text{时}.\end{cases}
\end{aligned}\tag{3.1.18}$$

因为由 Hőlder 不等式有

$$E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p} \le \Big(\sum_{k=1}^{\infty}E|\xi_k|^p\Big)^{1/p} = \delta_p\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/2}\Big)^{1/p},$$

由 (3.1.18), 引理得证.

下面我们利用引理 3.1.2 来建立 Fernique 型不等式. 我们考察较一般的过程, 它不必是 Gauss 的. 下述引理是 Csáki, Csörgő 和 Shao (1992) 得到的.

引理 3.1.3 设 $\mathcal{B}$ 为一可分的 Banach 空间, 其范数为 $\|\cdot\|$; 设 $\{\Gamma(t); -\infty < t < \infty\}$ 是取值于 $\mathcal{B}$ 中的随机过程, P 是由 $\Gamma(\cdot)$ 生成的概率测度. 假设 $\Gamma(\cdot)$ 关于 $\|\cdot\|$ 是 P a.s. 可分的, 且对 $|t| \le t_0$, $0 < x^* \le x$ 和 $0 < h \le h_0$ 存在非负单调不减函数 $\sigma_1(h)$ 和 $\sigma_2(h)$, 使得对某 $K, \gamma, \beta > 0$

$$P\Big\{\|\Gamma(t+h)-\Gamma(t)\| \ge x\sigma_1(h)+\sigma_2(h)\Big\} \le K\exp(-\gamma x^{\beta}). \quad (3.1.19)$$

则对任何 $0 \le T \le t_0$, $0 < a \le h_0$, $x \ge x^*$ 和 $k \ge 3$ 有

$$\begin{aligned} P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a}\|\Gamma(t+s)-\Gamma(t)\| &\ge x(\sigma_1(a)+\sigma_1(a,k)) \\ &\quad + \sigma_1^*(a,k)+\sigma_2(a)+\sigma_2(a,k)\Big\} \\ &\le 4\Big(\frac{T}{a}+1\Big)K\cdot 2^{2^{k+1}}\exp(-\gamma x^{\beta}), \quad (3.1.20) \end{aligned}$$

其中

$$\sigma_1(a,k) = 2^{3+1/\beta}\int_{2^{k-3}}^{\infty}\frac{\sigma_1(ae^{-z})}{z}dz,$$

$$\sigma_2(a,k) = 6\int_{2^{k-3}}^{\infty}\frac{\sigma_2(ae^{-z})}{z}dz,$$

$$\sigma_1^*(a,k) = 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}\beta\int_{2^{\frac{k-2}{\beta}}}^{\infty}\sigma_1(ae^{-z^{\beta}})dz.$$

证明 对任何正实数 t 和 $k \ge 3$ 令 $t_j = a\big[t\cdot\frac{2^{2^j}}{a}\big]/2^{2^j}$, $R = 2^{2^k}$. 记 $\mathcal{T} = \{t_j : t \ge 0, j \ge 1\}$. 对每一 $t, s \in \mathcal{T}$ 我们有

$$\begin{aligned} &\|\Gamma(t+s)-\Gamma(t)\| \\ &\quad \le \|\Gamma((t+s)_k)-\Gamma(t_k)\| + \|\Gamma(t+s)-\Gamma((t+s)_k)\| \\ &\qquad + \|\Gamma(t)-\Gamma(t_k)\| \end{aligned}$$

$$\leq \|\Gamma((t+s)_k) - \Gamma(t_k)\| + \sum_{j=0}^{\infty} \|\Gamma((t+s)_{k+j+1}) - \Gamma((t+s)_{k+j})\|$$

$$+ \sum_{j=0}^{\infty} \|\Gamma(t_{k+j+1}) - \Gamma(t_{k+j})\|,$$

事实上，上述两个级数都是有限和. 由于 $\Gamma(\cdot)$ 关于 $\|\cdot\|$ 的 a.s. 可分性，我们有

$$\begin{aligned}
&\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma(t+s)-\Gamma(t)\| \\
&\quad = \sup_{\substack{t\in\mathcal{T} \\ 0\leq t\leq T}}\sup_{\substack{s\in\mathcal{T} \\ 0\leq s\leq a}}\|\Gamma(t+s)-\Gamma(t)\| \\
&\quad \leq \sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\qquad + \sum_{j=0}^{\infty}\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| \\
&\qquad + \sum_{j=0}^{\infty}\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|.
\end{aligned}$$

由于

$$\sup_{0\leq t\leq T}\sup_{0\leq s\leq a(1-\frac{1}{R})}|(t+s)_k - t_k| \leq a,$$

$$\sup_{0\leq t\leq T}\sup_{a(1-1/R)\leq s\leq a}\left|(t+s)_k - \Big(t+a\Big(1-1/R\Big)\Big)_k\right| \leq 2a\cdot 2^{-2^k},$$

$$\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}|(t+s)_{k+j+1}-(t+s)_{k+j}| \leq a\cdot 2^{-2^{k+j}},$$

$$\begin{aligned}
&\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\quad \leq \sup_{0\leq t\leq T}\sup_{0\leq s\leq a(1-\frac{1}{R})}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\qquad + \sup_{0\leq t\leq T}\sup_{a(1-\frac{1}{R})\leq s\leq a}\|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)\|,
\end{aligned}$$

由 (3.1.19) 得，对每一 $x \ge x^*$ 和 $x_j \ge x^*$ 有

$$P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a(1-\frac{1}{R})}\|\Gamma((t+s)_k)-\Gamma(t_k)\|\ge x\sigma_1(a)+\sigma_2(a)\Big\}$$

$$\le 2KR^2(T/a+1)\exp(-\gamma x^\beta),$$

$$P\Big\{\sup_{0\le t\le T}\sup_{a(1-1/R)\le s\le a}\|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)\|$$

$$\ge x\sigma_1(2a/R)+\sigma_2(2a/R)\Big\}$$

$$\le 2KR(T/a+1)\exp(-\gamma x^\beta),$$

$$P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le a}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\|$$

$$\ge x_j\sigma_1(a/2^{2^{k+j}})+\sigma_2(a/2^{2^{k+j}})\Big\}$$

$$\le 2K2^{2^{k+j+1}}(T/a+1)\exp(-\gamma x_j^\beta),$$

以及

$$P\Big\{\sup_{0\le t\le T}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|\ge x_j\sigma_1(a/2^{2^{k+j}})+\sigma_2(a/2^{2^{k+j}})\Big\}$$

$$\le K2^{2^{k+j+1}}(T/a+1)\exp(-\gamma x_j^\beta).$$

现在令 $\gamma x_j^\beta = \gamma x^\beta + 2^{k+j+1}$. 则

$$\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(\gamma x_j^\beta)=\sum_{j=0}^{\infty}2^{2^{k+j+1}}e^{-2^{k+j+1}}e^{-\gamma x^\beta}\le\exp(-\gamma x^\beta).$$

由 x_j 的定义，我们知

$$x_j\le 2^{\frac{1}{\beta}}x+(2/\gamma)^{\frac{1}{\beta}}2^{\frac{k+j+1}{\beta}},$$

$$x\sigma_1\Big(\frac{2a}{R}\Big)+\sigma_2\Big(\frac{2a}{R}\Big)+2\sum_{j=0}^{\infty}x_j\sigma_1\Big(\frac{a}{2^{2^{k+j}}}\Big)+2\sum_{j=0}^{\infty}\sigma_2\Big(\frac{a}{2^{2^{k+j}}}\Big)$$

$$\le x\Big(\sigma_1\Big(\frac{2a}{2^{2^k}}\Big)+2^{1+\frac{1}{\beta}}\sum_{j=0}^{\infty}\sigma_1\Big(\frac{a}{2^{2^{k+j}}}\Big)\Big)$$

$$+2\Big(\frac{2}{\gamma}\Big)^{\frac{1}{\beta}}\sum_{j=0}^{\infty}2^{\frac{k+j+1}{\beta}}\sigma_1\Big(\frac{a}{2^{2^{k+j}}}\Big)+\sigma_2\Big(\frac{2a}{R}\Big)+2\sum_{j=0}^{\infty}\sigma_2\Big(\frac{a}{2^{2^{k+j}}}\Big),$$

$$\sigma_1\left(\frac{2a}{2^{2^k}}\right)+2^{1+\frac{1}{\beta}}\sum_{j=0}^{\infty}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)\le\left(1+2^{\frac{1}{\beta}+1}\right)\sum_{j=0}^{\infty}\sigma_1\left(\frac{2a}{2^{2^{k+j}}}\right)$$

$$\le\left(1+2^{\frac{1}{\beta}+1}\right)\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}}\frac{\sigma_1(\frac{2a}{2^z})}{z}dz/\ln 2$$

$$\le 2^{3+\frac{1}{\beta}}\int_{2^{k-1}}^{\infty}\frac{\sigma_1(\frac{2a}{2^z})}{z}dz$$

$$\le 2^{3+\frac{1}{\beta}}\int_{2^{k-3}}^{\infty}\frac{\sigma_1(ae^{-z})}{z}dz$$

$$=\sigma_1(a,k),$$

$$2\left(\frac{2}{\gamma}\right)^{\frac{1}{\beta}}\sum_{j=0}^{\infty}2^{\frac{k+j+1}{\beta}}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)$$

$$\le\frac{2^{\frac{2}{\beta}+1}(\frac{2}{\gamma})^{\frac{1}{\beta}}}{\beta(2^{\frac{1}{\beta}}-1)}\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}}z^{\frac{1}{\beta}-1}\sigma_1(a2^{-z})dz$$

$$\le\frac{2^{\frac{1}{\beta}+1}(\frac{4}{\gamma})^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}}-1}\int_{2^{\frac{k-1}{\beta}}}^{\infty}\sigma_1\left(a2^{-z^{\beta}}\right)dz$$

$$\le 4\left(\frac{14}{\gamma}\right)^{\frac{1}{\beta}}\cdot\beta\int_{2^{\frac{k-2}{\beta}}}^{\infty}\sigma_1\left(ae^{-z^{\beta}}\right)dz$$

$$=\sigma_1^*(a,k),$$

以及

$$2\sum_{j=0}^{\infty}\sigma_2\left(\frac{a}{2^{2^{k+j}}}\right)+\sigma_2\left(\frac{2a}{2^{2^k}}\right)\le 6\int_{2^{k-3}}^{\infty}\frac{\sigma_2(ae^{-z})}{z}dz$$

$$=\sigma_2(a,k).$$

综合上述不等式得证 (3.1.20). 引理 3.1.3 证毕.

利用引理 3.1.2 和 3.1.3, 我们来给出 $Y(\cdot)\in l^p$ $(p\ge 1)$ 连续的下述充分条件.

定理 3.1.3 假设条件 (3.1.16) 满足, 且

$$\int_1^{\infty}\frac{\sigma(p,e^{-z})}{z}dz<\infty, \tag{3.1.21}$$

$$\int_1^\infty \tilde{\sigma}(p, e^{-z^2})dz < \infty. \tag{3.1.22}$$

则 $Y(\cdot) \in l^p$ $(p \geq 1)$ 具有 a.s. 连续的样本轨道.

证明 只需证明对任给的 $\varepsilon > 0, T > 0$

$$\lim_{h\to 0} P\Big\{\sup_{|t|\leq T}\sup_{0\leq s\leq h}\sum_{k=1}^{\infty}|X_k(t+s)-X_k(t)|^p \geq \varepsilon^p\Big\} = 0,$$

即

$$\lim_{h\to 0}\lim_{N\to\infty} P\Big\{\sup_{|t|\leq T}\sup_{0\leq s\leq h}\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p \geq \varepsilon^p\Big\} = 0. \tag{3.1.23}$$

显然, 由 (3.1.22) 可得对每一 $k \geq 1$, $\int_1^\infty \sigma_k(e^{-z^2})dz < \infty$. 故由定理 2.1.6 知对每一 $k \geq 1$, $X_k(\cdot)$ 是 a.s. 连续的 Gauss 过程, 故对每一 N, 在 l^p 中, $Y_N(\cdot) := \{X_k(\cdot)\}_{k=1}^N$ 也是 a.s. 连续的. 应用引理 3.1.2 和 3.1.3, 对任一 $x > 0$ 和 $N \geq 1$ 有

$$\begin{aligned}P\Big\{\sup_{|t|\leq T}\sup_{0\leq s\leq h}\Big(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\Big)^{1/p} &\geq x(\tilde{\sigma}(p,h)+\tilde{\sigma}_1(p,h))\\ &+\tilde{\sigma}_2(p,h)+\delta_p(\sigma(p,h)+\sigma_1(p,h))\Big\}\\ &\leq 8(T+1)h^{-32}\exp(-x^2/2),\end{aligned}$$

其中

$$\tilde{\sigma}_1(p,h) = 16\int_{\log h^{-1}}^\infty \frac{\tilde{\sigma}(p, he^{-z})}{z}dz,$$

$$\tilde{\sigma}_2(p,h) = 11\int_{(\log h^{-1})^2}^\infty \tilde{\sigma}(p, he^{-z^2})dz,$$

$$\sigma_1(p,h) = 6\int_{\log h^{-1}}^\infty \frac{\sigma(p, he^{-z})}{z}dz.$$

因 $\tilde{\sigma}(p,h)$ 关于 h 是非减的, 由 (3.1.22) 即得

$$\tilde{\sigma}(p,h) = o((\log h^{-1})^{-1/2}), \quad h \to 0.$$

由此进一步得

$$\tilde{\sigma}_1(p,h) = o((\log h^{-1})^{-1/2}), \quad h \to 0.$$

另外，由 (3.1.22) 和 (3.1.21) 我们有

$$\tilde{\sigma}_2(p,h) + \delta_p(\sigma(p,h) + \sigma_1(p,h)) \to 0, \quad h \to 0.$$

从而，我们得

$$\begin{aligned}
&\lim_{h\to 0}\lim_{N\to\infty} P\left\{\sup_{|t|\le T}\sup_{0\le s\le h}\left(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\right)^{1/p} \ge \varepsilon\right\} \\
&\quad \le \lim_{h\to 0}\lim_{N\to\infty} P\left\{\sup_{|t|\le T}\sup_{0\le s\le h}\left(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\right)^{1/p} \ge \varepsilon/2\right. \\
&\qquad \left. + \tilde{\sigma}_2(p,h) + \delta_p(\sigma(p,h)+\sigma_1(p,h))\right\} \\
&\quad \le 8(T+1)\lim_{h\to 0} h^{-32}\exp\left\{-\frac{\varepsilon^2}{8(\tilde{\sigma}(p,h)+\tilde{\sigma}_1(p,h))^2}\right\} \\
&\quad \le 8(T+1)\lim_{h\to 0} h^{-32}\exp(-34\log h^{-1}) \\
&\quad = 0.
\end{aligned}$$

这就证明了 (3.1.23), 定理 3.1.3 得证.

§3.2　$\mathcal{B}$ 值随机过程的增量

在本节中我们将介绍由 Csörgő 和 Shao (1994) 建立的关于随机过程大增量和小增量的一般定理. 设 $\mathcal{B}$ 是可分 Banach 空间, 具有范数 $\|\cdot\|$, 令 $\{\Gamma(t); -\infty < t < \infty\}$ 为取值于 Banach 空间 $\mathcal{B}$ 的随机过程. 设 P 是由 $\Gamma(\cdot)$ 生成的概率测度.

定理 3.2.1　*设 $a_T, b_T, C_T, \sigma_1(T), \sigma_2(T)$ 是非负连续函数. 假设 a_T 和 b_T 或同为 T 的非降函数, 或同为 T 的非增函数, 且*

$$C_T + \sigma_1(T) + \sigma_1^{-1}(T) \to \infty \quad 当 \quad T \to \infty, \tag{3.2.1}$$

又对任何满足下式的 x 有

$$\Big(\frac{1}{\gamma}\Big(\log C_T+\log\log\Big(\sigma_1(T)+\frac{1}{\sigma_1(T)}\Big)\Big)\Big)^{\frac{1}{\beta}}\le x$$
$$\le(1+\delta)\Big(\frac{1}{\gamma}\Big(\log C_T+\log\log\Big(\sigma_1(T)+\frac{1}{\sigma_1(T)}\Big)\Big)\Big)^{\frac{1}{\beta}}+\delta\frac{\sigma_2(T)}{\sigma_1(T)},\tag{3.2.2}$$

其中 $\gamma,\beta,\delta>0$, 成立

$$P\Big\{\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\|\Gamma(t+s)-\Gamma(t)\|\ge x\sigma_1(T)+\sigma_2(T)\Big\}$$
$$\le C_T\exp(-\gamma x^{\beta}).\tag{3.2.3}$$

则有

$$\limsup_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|\le 1\quad\text{a.s.},\tag{3.2.4}$$

其中 $\alpha_T^{-1}=\sigma_1(T)(\frac{1}{\gamma}(\log C_T+\log\log(\sigma_1(T)+\frac{1}{\sigma_1(T)})))^{\frac{1}{\beta}}+\sigma_2(T)$.

证明 不失一般性可设 $0<\delta<1/2$ 且 a_T 和 b_T 都是非降的. 令 $1<\theta<1+\delta/2$. 定义

$$A_k=\{T:\theta^k<\sigma_1(T)\le\theta^{k+1}\},\quad -\infty<k<\infty,$$
$$A_{k,j}=\{T:2^j\le C_T<2^{j+1},T\in A_k\},\quad j\ge 0,$$
$$A_{k,j,i}=\{T:\theta^i<\sigma_2(T)\le\theta^{i+1},T\in A_{k,j}\},\quad -\infty<i<\infty,$$
$$T_{k,j,i}=\sup\{T:T\in A_{k,j,i}\}.$$

记 $a(T)=a_T$, $b(T)=b_T$. 注意到 (3.2.1) 满足, 并利用 a_T, b_T, C_T, $\sigma_1(T)$ 和 $\sigma_2(T)$ 的连续性, 我们有

$$\limsup_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|$$
$$\le\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{T\in A_{k,j}}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|$$
$$\le\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{T\in A_{k,j}}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\|\Gamma(t+s)-\Gamma(t)\|/$$
$$[\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\sigma_2(T)]$$

$$\leq \max\Big\{\limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i\leq k}\sup_{T\in A_{k,j,i}}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\|\Gamma(t+s)-\Gamma(t)\|/$$

$$\big[\theta^k(\frac{1}{\gamma}(\log(2^j\log\theta^{|k|})))^{1/\beta}\big],$$

$$\limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i>k}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T_{k,j,i})}\|\Gamma(t+s)-\Gamma(t)\|/$$

$$\big[\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\theta^i\big]\Big\}. \tag{3.2.5}$$

首先我们来证

$$\limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i>k}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T_{k,j,i})}\|\Gamma(t+s)-\Gamma(t)\|/$$

$$\big[\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\theta^i\big]$$

$$\leq \theta\limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i>k}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T_{k,j,i})}\alpha_{T_{k,j,i}}\|\Gamma(t+s)-\Gamma(t)\|$$

$$\leq \theta^2 \quad \text{a.s.} \tag{3.2.6}$$

由 (3.2.2) 得

$$P\Big\{\sup_{j\geq l}\sup_{i>k}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T_{k,j,i})}\alpha_{T_{k,j,i}}\|\Gamma(t+s)-\Gamma(t)\|\geq\theta\Big\}$$

$$\leq \sum_{j\geq l}\sum_{i>k}C_{T_{k,j,i}}\exp\Big(-\gamma\Big(((\theta-1)\frac{\sigma_2(T_{k,j,i})}{\sigma_1(T_{k,j,i})}$$

$$+\theta\Big(\frac{1}{\gamma}(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\Big)^{\frac{1}{\beta}}\Big)^{\beta}\Big)$$

$$\leq \sum_{j\geq l}\sum_{i>k}C_{T_{k,j,i}}\exp\Big(-\gamma\Big(((\theta-1)\theta^{i-k-1}$$

$$+\theta\Big(\frac{1}{\gamma}(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\Big)^{\frac{1}{\beta}}\Big)^{\beta}\Big)$$

$$\leq \sum_{j\geq l}\sum_{\substack{i>k\\ \theta^{i-k}\leq j^{2/\beta}+k^2}}C_{T_{k,j,i}}\exp\Big(-\theta^{\beta}(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\Big)$$

$$+\sum_{j\geq l}\sum_{\theta^{i-k}>j^{2/\beta}+k^2}C_{T_{k,j,i}}\exp(-\gamma(\theta-1)^{\beta}\theta^{(i-k-1)\beta})$$

$$
\begin{aligned}
&\le c\Big(\sum_{j\ge l}(j^{2/\beta}+\log|k|)2^{-j(\theta^\beta-1)}(|k|+1)^{-\theta^\beta}\\
&\qquad+\sum_{j\ge l}2^{-j}\exp\Big(-\gamma\Big(1-\frac{1}{\theta}\Big)^\beta(j^{2/\beta}+k^2)^\beta\Big)\Big)\\
&\le c2^{-\frac{l(\theta^\beta-1)}{2}}(1+|k|)^{-\frac{\theta^\beta+1}{2}},
\end{aligned}\tag{3.2.7}
$$

其中常数 c 仅依赖于 θ,β 和 γ. 所以

$$
\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{|k|=0}^{\infty}P\Big\{\sup_{j\ge l}\sup_{i>k}\sup_{0\le t\le b(T_{k,j,i})}\sup_{0\le s\le a(T_{k,j,i})}\alpha_{T_{k,j,i}}\\
&\qquad\cdot\|\Gamma(t+s)-\Gamma(t)\|>\theta\Big\}<\infty.
\end{aligned}\tag{3.2.8}
$$

从 (3.2.8) 和 Borel-Cantelli 引理即得 (3.2.6).

其次，我们来证

$$
\begin{aligned}
&\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{T\in A_{k,j},\sigma_2(T)\le\theta^{k+1}}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log(2^j\log\theta^{|k|})))^{1/\beta}}\\
&\qquad\le\theta^2\quad\text{a.s.}
\end{aligned}\tag{3.2.9}
$$

令

$$T_{k,j}=\sup\{T:T\in A_{k,j},\sigma_2(T)\le\theta^{k+1}\}.$$

则

$$\sigma_2(T_{k,j})\le\theta^{k+1}$$

且

$$
\begin{aligned}
&\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{T\in A_{k,j},\sigma_2(T)\le\theta^{k+1}}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log(2^j\log\theta^{|k|})))^{1/\beta}}\\
&\le\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{0\le t\le b(T_{k,j})}\sup_{0\le s\le a(T_{k,j})}\|\Gamma(t+s)-\Gamma(t)\|/\\
&\qquad[\theta^k(\frac{1}{\gamma}(\log(2^j\log\theta^{|k|})))^{1/\beta}+\sigma_2(T_{k,j})]\\
&\le\theta\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{0\le t\le b(T_{k,j})}\sup_{0\le s\le a(T_{k,j})}\alpha_{T_{k,j}}\|\Gamma(t+s)-\Gamma(t)\|.
\end{aligned}
$$

(3.2.9) 的余下部分证明可沿着 (3.2.6) 的同一思路而得. 现在 (3.2.4) 可由 (3.2.5), (3.2.6), (3.2.9) 和 δ 的任意性得到. 定理 3.2.1 证毕.

下一定理指出 a_T 和 b_T 同为 T 的非降或同为 T 的非增函数的假设在某些条件下可以去掉.

定理 3.2.2 设 $a_T, b_T, \sigma_1(T)$ 和 $\sigma_2(T)$ 是非负连续函数. 假设

$$\frac{b_T}{a_T}+\sigma_1(T)+\frac{1}{\sigma_1(T)}\to\infty \quad 当 \quad T\to\infty, \tag{3.2.10}$$

且对任何 $b\ge b_T$ 及 $\gamma,\beta,\delta,\ A>0$ 和满足下式的 x

$$\begin{aligned}\Big(\frac{1}{\gamma}\Big(\log\Big(\frac{b}{a_T}+1\Big)+\log\log\Big(\sigma_1(T)+\frac{1}{\sigma_1(T)}\Big)\Big)\Big)^{\frac{1}{\beta}}&\le x\\ \le(1+\delta)\Big(\frac{1}{\gamma}\Big(\log\Big(\frac{b}{a_T}+1\Big)&+\log\log\Big(\sigma_1(T)+\frac{1}{\sigma_1(T)}\Big)\Big)\Big)^{\frac{1}{\beta}}\\ &+\delta\frac{\sigma_2(T)}{\sigma_1(T)}\end{aligned}$$

成立

$$\begin{aligned}P\Big\{\sup_{0\le t\le b}\sup_{0\le s\le a_T}\|\Gamma(t+s)-\Gamma(t)\|\ge x\sigma_1(T)+\sigma_2(T)\Big\}\\ \le A(1+\frac{b}{a_T})\exp(-\gamma x^{\beta}).\end{aligned} \tag{3.2.11}$$

那么

$$\limsup_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\alpha_T^*\|\Gamma(t+s)-\Gamma(t)\|\le 1 \quad \text{a.s.}, \tag{3.2.12}$$

其中

$$\begin{aligned}1/\alpha_T^*=\sigma_1(T)(\tfrac{1}{\gamma}(&\log(b_T/a_T+1)\\ &+\log\log(\sigma_1(T)+\sigma_1^{-1}(T))))^{1/\beta}+\sigma_2(T).\end{aligned}$$

证明 设 $C_T=1+b_T/a_T$. 不失一般性假设 $0<\delta<1/2$. 设 $\theta, A_k, A_{k,j}, A_{k,j,i}$ 如定理 3.2.1 的证明中一样. 令

$$b(T_{k,j,i})=\sup\{b_T: T\in A_{k,j,i}\}$$

和

$$a(T_{k,j,i}^*)=\sup\{a_T: T\in A_{k,j,i}\}.$$

易知 $b(T_{k,j,i}) \geq b(T^*_{k,j,i})$ 且

$$2^j \leq \frac{b(T^*_{k,j,i})}{a(T^*_{k,j,i})}+1 \leq \frac{b(T_{k,j,i})}{a(T^*_{k,j,i})}+1 \leq \frac{b(T_{k,j,i})}{a(T_{k,j,i})}+1 \leq 2^{j+1}.$$

因此，沿着定理 3.2.1 的证明路线可得

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\alpha^*_T\|\Gamma(t+s)-\Gamma(t)\| \\
&\leq \limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{T\in A_{k,j}}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\alpha^*_T\|\Gamma(t+s)-\Gamma(t)\| \\
&\leq \limsup_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i}\sup_{T\in A_{k,j,i}}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T^*_{k,j,i})}\alpha^*_T \\
&\quad\cdot\|\Gamma(t+s)-\Gamma(t)\| \\
&\leq \theta^2 \quad \text{a.s.}
\end{aligned}$$

由 δ 的任意性，这就证明了 (3.2.12).

利用引理 3.1.3, 我们可给出上述定理的若干推论，为此先证明下述引理.

引理 3.2.1 设 $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ 和 $\sigma_2(h)$ 如引理 3.1.3 中定义，且假设对某 $\alpha>0$, $\sigma_1(x)/x^\alpha$ 和 $\sigma_2(x)/x^\alpha$ 在 $(0,h_0)$ 中是拟增的. 那么对任何 $0<\varepsilon<1$ 存在 $C=C(\varepsilon,\beta,\gamma,\alpha)$ 使得对每一 $x\geq\max(1,\frac{x^*}{1-\varepsilon})$, $0\leq T\leq t_0$ 和 $0<h\leq h_0$ 有

$$\begin{aligned}
&P\Big\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}\|\Gamma(t+s)-\Gamma(t)\|\geq x\sigma_1(h)+(1+\varepsilon)\sigma_2(h)\Big\} \\
&\qquad\qquad\leq CK\Big(\frac{T}{h}+1\Big)\exp\Big(-\frac{\gamma x^\beta}{1+\varepsilon}\Big).
\end{aligned} \tag{3.2.13}$$

证明 因 $\sigma_1(x)/x^\alpha$ 和 $\sigma_2(x)/x^\alpha$ 在 $(0,h_0)$ 中是拟增的，即存在正数 c_0 使得对所有的 $0<t\leq 1$ 有

$$\sigma_i(ht)\leq c_0t^\alpha\sigma_i(h), \quad i=1,2. \tag{3.2.14}$$

由 (3.2.14) 易知

$$\begin{aligned}
&\sigma_i(h,k)\leq 2^{3+\frac{1}{\beta}}c_0e^{-\alpha(k-3)}\alpha^{-1}\sigma_i(h), \\
&\sigma^*_i(h,k)\leq 4(\frac{14}{\gamma})^{\frac{1}{\beta}}\beta c_0e^{-\alpha(k-2)}\alpha^{-1}\sigma_1(h), \quad i=1,2.
\end{aligned}$$

因此对 $\delta=\min(\varepsilon, 1-(1+\varepsilon)^{-1/\beta})$, 可取 k 使得

$$\sigma_2(h,k)+\sigma_1(h,k)+\sigma_1^*(h,k)\le\frac{\delta}{2}(\sigma_1(h)+\sigma_2(h)).$$

由引理 3.1.3, 我们有

$$\begin{aligned}&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma(t+s)-\Gamma(t)\|\ge x\sigma_1(h)+(1+\varepsilon)\sigma_2(h)\Big\}\\&\le P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma(t+s)-\Gamma(t)\|\\&\ge x(1-\delta)(\sigma_1(h)+\sigma_1(h,k))+\sigma_1^*(h,k)+\sigma_2(h)+\sigma_2(h,k)\Big\}\\&\le 4K\Big(\frac{T}{h}+1\Big)2^{2^{k+1}}\exp(-\gamma(x(1-\delta))^{\beta})\\&\le 4K\Big(\frac{T}{h}+1\Big)2^{2^{k+1}}\exp\Big(-\frac{\gamma x^{\beta}}{1+\varepsilon}\Big).\end{aligned}$$

现在令 $C=C(\varepsilon,\beta,\gamma,\alpha)=4\cdot 2^{2^{k+1}}$, 就得证引理成立.

注 3.2.1 严格地讲 (3.2.13) 中的常数 C 不仅依赖于 $\varepsilon,\alpha,\gamma,\beta$, 也与 (3.2.14) 中的 c_0 有关. 但为方便计在以后仍写 $C=C(\varepsilon,\alpha,\gamma,\beta)$.

引理 3.2.2 设 $\{\Gamma(t); -\infty<t<\infty\}$, $\sigma_1(h)$ 和 $\sigma_2(h)$ 如引理 3.1.3 中定义, 并设对某 $\alpha>1/\beta$, $\sigma_1(x)(\log x^{-1})^{\alpha}$ 和 $\sigma_2(x)(\log x^{-1})^{\alpha}$ 在 $(0,1/2)$ 上是拟增的, 即存在 $c_0>0$ 使得对每一 $0<t\le 1$, $0<h\le\frac{1}{2}$ 有

$$\sigma_i(ht)\le c_0\sigma_i(h)\Big(\log\frac{1}{h}\Big)^{\alpha}\Big/\Big(\log\frac{1}{h}+\log\frac{1}{t}\Big)^{\alpha},\quad i=1,2. \tag{3.2.15}$$

则对任给的 $\varepsilon>0$, 对每一 $x\ge\max(x^*,1)$, $0\le T\le t_0$ 和 $0<h\le\min(e^{-8/\varepsilon},h_0,1/2)$, 有

$$\begin{aligned}P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le h}\|\Gamma(t+s)-\Gamma(t)\|&\ge x\sigma_1(h)(1+c_1c_0)\\&+\sigma_2(h)(1+c_1c_0)+c_2c_0\sigma_1(h)(\log h^{-1})^{1/\beta}\Big\}\\&\le 8K\Big(\frac{T}{h}+1\Big)\frac{1}{h^{2\varepsilon}}\exp(-\gamma x^{\beta}),\end{aligned}\tag{3.2.16}$$

其中

$$c_1 = 2^{3+1/\beta}\Big(1+\frac{\varepsilon}{8}\Big)^{-\alpha}\Big(\frac{1}{\alpha}+\log\Big(1+\frac{8}{\varepsilon}\Big)\Big),$$
$$c_2 = 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}\beta\frac{\beta\alpha}{\beta\alpha-1}\Big(1+\frac{\varepsilon}{4}\Big)^{-(\beta\alpha-1)/\beta}.$$

证明 在引理 3.1.3 中令 $2^k=\varepsilon\log h^{-1}$. 由 (3.2.15), 对 $i=1,2$ 我们有

$$\begin{aligned}
\sigma_i(h,k) &\le c_0 2^{3+1/\beta}\int_{2^{k-3}}^{\infty}\frac{\sigma_i(h)(\log 1/h)^{\alpha}}{z(\log(h+z)^{-1})^{\alpha}}dz\\
&= c_0 2^{3+1/\beta}\sigma_i(h)\int_{\frac{\varepsilon}{8}\log\frac{1}{h}}^{\infty}\frac{(\log h^{-1})^{\alpha}}{z(\log(h^{-1}+z)^{-1})^{\alpha}}dz\\
&= c_0 2^{3+1/\beta}\sigma_i(h)\int_{\varepsilon/8}^{\infty}\frac{1}{z(z+1)^{\alpha}}dz\\
&\le c_0 2^{3+1/\beta}\sigma_i(h)\Big(\int_{\varepsilon/8}^{1+\varepsilon/8}\frac{1}{z(1+\varepsilon/8)^{\alpha}}dz+\int_{1+\varepsilon/8}^{\infty}\frac{dz}{z^{1+\alpha}}\Big)\\
&\le c_0 2^{3+1/\beta}\sigma_i(h)\Big(1+\varepsilon/8\Big)^{-\alpha}\Big(\frac{1}{\alpha}+\log\Big(1+\frac{8}{\varepsilon}\Big)\Big)\\
&= c_0c_1\sigma_i(h)
\end{aligned}$$

和

$$\begin{aligned}
\sigma_1^*(h,k) &\le 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}\beta c_0\int_{2^{\frac{k-2}{\beta}}}^{\infty}\frac{\sigma_1(h)\log^{\alpha}h^{-1}}{(z^{\beta}+\log h^{-1})^{\alpha}}dz\\
&\le 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}\beta c_0\sigma_1(h)\Big(\log\frac{1}{h}\Big)^{1/\beta}\int_{(\varepsilon/4)^{1/\beta}}^{\infty}\frac{1}{(1+z^{\beta})^{\alpha}}dz\\
&\le 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}\beta c_0\sigma_1(h)\Big(\log\frac{1}{h}\Big)^{\frac{1}{\beta}}\\
&\quad\cdot\Big(\int_{(\varepsilon/4)^{1/\beta}}^{(1+\varepsilon/4)^{1/\beta}}\frac{dz}{(1+\varepsilon/4)^{\alpha}}+\int_{(1+\varepsilon/4)^{1/\beta}}^{\infty}\frac{dz}{z^{\alpha\beta}}\Big)\\
&\le 4\Big(\frac{14}{\gamma}\Big)^{1/\beta}c_0\sigma_1(h)(\log h^{-1})^{\frac{1}{\beta}}\frac{\beta^2\alpha}{\beta\alpha-1}\Big(1+\frac{\varepsilon}{4}\Big)^{-\alpha-1/\beta}\\
&= c_0\sigma_1(h)(\log h^{-1})^{1/\beta}c_2.
\end{aligned}$$

由引理 3.1.3 即得 (3.2.16).

不等式 (3.2.13) 和 (3.2.16) 使我们能够去研究当 h 很小时 $\Gamma(\cdot)$ 的增量，我们将看到某些结果是出人意料的. (3.2.13) 和 (3.2.16) 的下述修正是为研究 $\Gamma(\cdot)$ 的大增量作准备的.

引理 3.2.3 设 $\{\Gamma(t); -\infty < t < \infty\}$, $\sigma_1(h)$ 和 $\sigma_2(h)$ 如引理 3.1.3 中定义，其中 $t_0 = h_0 = \infty$. 假设对某 $\alpha > 0$, $\sigma_1(x)/x^\alpha$ 和 $\sigma_2(x)/x^\alpha$ 在 $(0, \infty)$ 上是拟增的. 则对任何 $0 < \varepsilon < 1$, 存在 $C = C(\varepsilon, \gamma, \alpha, \beta)$ 使得对每一 $x \geq \max(1, x^*/(1-\varepsilon))$ 和 $T, a > 0$, 有

$$P\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma(t+s)-\Gamma(t)\| \geq x\sigma_1(a) + (1+\varepsilon)\sigma_2(a)\right\}$$
$$\leq CK\left(\frac{T}{a}+1\right)\exp\left(-\frac{\gamma x^\beta}{1+\varepsilon}\right). \tag{3.2.17}$$

引理 3.2.4 设 $\{\Gamma(t); -\infty < t < \infty\}$, $\sigma_1(x)$ 和 $\sigma_2(x)$ 如引理 3.1.3 中定义, 其中 $t_0 = h_0 = \infty$. 假设对某 $\alpha > 1/\beta$, $\sigma_1(x)/(\log x)^\alpha$ 和 $\sigma_2(x)/(\log x)^\alpha$ 在 $(2, \infty)$ 上是拟增的, 且 $\int_1^\infty \sigma_1(e^{-z^\beta})dz < \infty$, $\int_1^\infty \sigma_2(e^{-z})dz < \infty$. 则存在正常数 c_1, c_2 和 a_0 使得对每一 $x \geq \max(1, x^*)$ 和 $T \geq a \geq a_0$ 有

$$P\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq a}\|\Gamma(t+s)-\Gamma(t)\| \geq xc_1\sigma_1(a) + c_1\sigma_2(a)\right\}$$
$$\leq c_2KT\exp(-\gamma x^\beta). \tag{3.2.18}$$

(3.2.17) 和 (3.2.18) 的证明类似于引理 3.2.1 和引理 3.2.2 的证明，从略.

综合定理 3.2.2 和上述引理，我们可得

推论 3.2.1 假设 $\Gamma(\cdot)$ 关于范数 $\|\cdot\|$ 是 P-a.s. 连续的，且存在非负连续函数 $\sigma_1(h)$ 和 $\sigma_2(h)$ 使得对每一 $t \geq 0, h > 0$ 和 $x \geq x^* > 0$ 及某 $K, \gamma, \beta > 0$ 有

$$P\{\|\Gamma(t+s)-\Gamma(t)\| \geq x\sigma_1(h) + \sigma_2(h)\} \leq K\exp(-\gamma x^\beta).$$

此外，假设对某 $\alpha > 0$, $\sigma_1(x)/x^\alpha$ 和 $\sigma_2(x)/x^\alpha$ 在 $(0,\infty)$ 上是拟增的，且有连续函数 a_T 和 b_T，使得

$$\frac{b_T}{a_T} + \sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \to \infty, \quad T \to \infty.$$

则

$$\limsup_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \beta_T \|\Gamma(t+s) - \Gamma(t)\| \le 1 \quad \text{a.s.,} \qquad (3.2.19)$$

其中 $$\beta_T^{-1} = \sigma_1(a_T)(\gamma^{-1}(\log(1+b_T/a_T) + \log\log(\sigma_1(a_T) \\ + \sigma_1(a_T)^{-1})))^{1/\beta} + \sigma_2(a_T).$$

证明 由引理 3.2.3, 对每一 $0 < \varepsilon < 1$, $b > 0$ 和 $x \ge \max(1, x^*/(1-\varepsilon))$ 我们有

$$P\Big\{\sup_{0\le t\le b} \sup_{0\le s\le a_T} \|\Gamma(t+s) - \Gamma(t)\| \ge x\sigma_1(a_T) + (1+\varepsilon)\sigma_2(a_T)\Big\} \\ \le K\Big(1 + \frac{b}{a_T}\Big)\exp\Big(-\frac{\gamma x^\beta}{1+\varepsilon}\Big).$$

因而，由定理 3.2.2 得

$$\limsup_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \beta_T(\varepsilon)\|\Gamma(t+s) - \Gamma(t)\| \le 1 \quad \text{a.s.,} \qquad (3.2.20)$$

其中 $\beta_T^{-1}(\varepsilon) = \sigma_1(a_T)(\frac{1+\varepsilon}{\gamma}(\log(1+b_T/a_T) + \log\log(\sigma_1(a_T) + \sigma_1(a_T)^{-1})))^{1/\beta} + (1+\varepsilon)\sigma_2(a_T)$. 由 ε 的任意性，从 (3.2.20) 即得 (3.2.19).

推论 3.2.2 设 $\{\Gamma(t); -\infty < t < \infty\}$, $\sigma_1(h)$ 和 $\sigma_2(h)$ 如引理 3.2.1 中所示，其中 $t_0 = 1$. 那么我们有

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \theta_h \|\Gamma(t+s) - \Gamma(t)\| \le 1 \quad \text{a.s.,} \qquad (3.2.21)$$

其中 $\theta_h^{-1} = \sigma_1(h)\big(\gamma^{-1}\big(\log h^{-1} + \log\log\sigma_1(h)^{-1}\big)\big)^{1/\beta} + \sigma_2(h)$.

推论 3.2.3 设 $\{\Gamma(t); -\infty < t < \infty\}$, $\sigma_1(h)$ 和 $\sigma_2(h)$ 如引理 3.2.2 中所示. 则存在正常数 C 使得

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \theta_h \|\Gamma(t+s) - \Gamma(t)\| \le C \quad \text{a.s.} \tag{3.2.22}$$

推论 3.2.2 和 3.2.3 的证明类似于推论 3.2.1 的证明, 从略.

§3.3 l^p 值 Gauss 过程的增量

设 $\{Y(t); t \ge 0\} = \{X_k(t); t \ge 0\}_{k=1}^{\infty}$ 是独立 Gauss 过程序列, $EX_k(t) = 0$ 且有平稳增量 $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, 其中对每一 $k \ge 1$, 假设 $\sigma_k(h)$ 是非降连续函数. 我们将继续使用 3.1.3 小节中的记号.

3.3.1 当 $\sigma(p,h)/h^\alpha$ 和 / 或 $\tilde{\sigma}(p,h)/h^\alpha$ 是拟增时的增量

首先考察当 $\sigma(p,h)$ 和 / 或 $\tilde{\sigma}(p,h)$ 无界时的情形. 下面两命题在主要定理的证明中将被用到.

命题 3.3.1 设 a_T $(T>0)$ 是正连续函数且 b_T $(T>0)$ 是非负连续函数. 令 $a^* = \sup_{T>0} a_T$. 假设对某 $\alpha > 0$, 在 $(0, a^*)$ 上 $\sigma(p,h)/h^\alpha$ 和 $\tilde{\sigma}(p,h)/h^\alpha$ 是拟增的且

$$\frac{1+b_T}{a_T} + a_T \to \infty \quad 当 \quad T \to \infty. \tag{3.3.1}$$

那么我们有

$$\limsup_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \beta(p,T) \|Y(t+s) - Y(t)\|_{l^p} \le 1 \quad \text{a.s.}, \tag{3.3.2}$$

其中 $\beta(p,T)^{-1} = \delta_p \sigma(p, a_T) + \tilde{\sigma}(p, a_T)(2(\log(b_T/a_T) + \log\log(a_T + 1/a_T)))^{1/2}$.

证明 回顾到

$$
\begin{aligned}
&\tilde{\sigma}(p,h)\\
&=\begin{cases}\delta^{-1}_{2p/(2-p)}\Big(E\Big(\sum\limits_{k=1}^{\infty}|X_k(t+h)-X_k(t)|^{2p/(2-p)}\Big)\Big)^{(2-p)/2p}, & \text{若}1\le p<2,\\ \max\limits_{k\ge1}(E(X_k(t+h)-X_k(t))^2)^{1/2}, & \text{若}p\ge 2,\end{cases}
\end{aligned}
$$

利用 Minkowski 不等式, 对每一 h 和 $p\ge1$, 我们得

$$
\tilde{\sigma}(p,2h)\le 2\tilde{\sigma}(p,h). \tag{3.3.3}
$$

由 (3.3.3) 对每一 $h>0$ 得

$$
\tilde{\sigma}(p,h)+\frac{1}{\tilde{\sigma}(p,h)}\le 4\Big(h+\frac{1}{h}\Big)\Big(\tilde{\sigma}(p,1)+\frac{1}{\tilde{\sigma}(p,1)}\Big). \tag{3.3.4}
$$

由 (3.3.4), 引理 3.1.3 和定理 3.1.3 即得 (3.3.2).

注 3.3.1 设 $\sigma_*(p,h)$ 和 $\tilde{\sigma}_*(p,h)$ 是非降函数使得对每一 $h>0$, $\sigma(p,h)\le\sigma_*(p,h)$ 且 $\tilde{\sigma}(p,h)\le\tilde{\sigma}_*(p,h)$. 假设对某 $\alpha>0$, $\sigma_*(p,h)/h^{\alpha}$ 和 $\tilde{\sigma}_*(p,h)/h^{\alpha}$ 在 $(0,a^*)$ 上是拟增的. 显然, 若 $\sigma(p,h)$ 和 $\tilde{\sigma}(p,h)$ 分别被 $\sigma_*(p,h)$ 和 $\tilde{\sigma}_*(p,h)$ 代替, (3.1.17) 仍正确. 因此, 当 $\sigma_*(p,a_T)$ 和 $\tilde{\sigma}_*(p,a_T)$ 分别被 $\sigma(p,a_T)$ 和 $\tilde{\sigma}(p,a_T)$ 代替时, (3.3.2) 仍正确.

命题 3.3.2 设 a_T 和 b_T 是正连续函数. 假设 (3.1.15) 及下式成立

$$
\frac{\log(b_T/a_T)}{\log\log(a_T+(1/a_T))}\to\infty,\quad T\to\infty, \tag{3.3.5}
$$

且对每一 $\varepsilon>0$ 还满足

$$
\begin{aligned}
&\limsup_{T\to\infty}\max_{(b_T/a_T)^{\varepsilon}\le j\le b_T/a_T}\max_{k\ge1}\{\sigma_k^{-2}(a_T)\\
&\cdot E[(X_k(a_T)-X_k(0))(X_k(ja_T)-X_k((j-1)a_T))]\}\le 0. \quad (3.3.6)
\end{aligned}
$$

则有

$$
\liminf_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}}\ge 1\quad \text{a.s.} \tag{3.3.7}
$$

证明 令 $1 < \theta < 65/64$. 定义

$$A_k = \left\{T; 2^k \le \frac{b_T}{a_T} \le 2^{k+1}\right\}, \quad k \ge 0,$$
$$A_{k,j} = \{T; \theta^{j-1} \le \tilde{\sigma}(p, a_T) \le \theta^j, T \in A_k\}, \quad -\infty < j < \infty,$$
$$b(T_{k,j}) = \inf\{b_T; T \in A_{k,j}\}, \quad a_{k,j} = a(T^*_{k,j}) = \inf\{a_T; T \in A_{k,j}\}.$$

由 (3.3.4) 和 (3.3.5) 可知, 对充分大的 k 有

$$\text{当} |j| \ge e^k \text{ 时}, \quad A_{k,j} = \phi. \tag{3.3.8}$$

另外, 易知

$$2^k \le \frac{b(T_{k,j})}{a(T_{k,j})} \le \frac{b(T_{k,j})}{a(T^*_{k,j})} \le \frac{b(T^*_{k,j})}{a(T^*_{k,j})} \le 2^{k+1}. \tag{3.3.9}$$

从而

$$\begin{aligned}
&\liminf_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}} \\
&\ge \liminf_{k\to\infty} \inf_j \inf_{T\in A_{k,j}} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}} \\
&\ge \liminf_{k\to\infty} \min_{|j|\le e^k} \sup_{0\le t\le b(T_{k,j})} \sup_{0\le s\le a_{k,j}} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\theta^j(2\log 2^{k+1})^{1/2}} \\
&\ge \liminf_{k\to\infty} \min_{|j|\le e^k} \max_{0\le i\le 2^{k(2-\theta)}} \|Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}/ \\
&\quad \theta\tilde{\sigma}(p,a_{k,j})(2\log 2^k)^{1/2}.
\end{aligned} \tag{3.3.10}$$

下面分别对 $1 \le p < 2$ 和 $2 \le p < \infty$ 两种情形进行证明.

情形 I. $1 \le p < 2$. 此时由 (3.1.17) 我们有

$$\begin{aligned}
&\|Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j}) - Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p} \\
&\quad \ge \left[\sum_{v=1}^{\infty} \sigma_v(a_{k,j})^{2p/(2-p)}\right]^{-(p-1)/p} \sum_{v=1}^{\infty} \sigma_v(a_{k,j})^{2(p-1)/(2-p)} \\
&\qquad \cdot (X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j}) - X_v(i2^{k(\theta-1)}a_{k,j})).
\end{aligned} \tag{3.3.11}$$

对 $k=1,2,\ldots,|j|\le e^k,\ 0\le i\le 2^{k(2-\theta)}$, 考察

$$
\begin{aligned}
&\xi(k,j;i)\\
&=\Big[\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2(p-1)/(2-p)}(X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j})\\
&\quad -X_v(i2^{k(\theta-1)}a_{k,j}))\Big]\Big/\Big[\tilde{\sigma}(p,a_{k,j})(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)})^{(p-1)/p}\Big]\\
&=\Big[\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2(p-1)/(2-p)}(X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j})\\
&\quad -X_v(i2^{k(\theta-1)}a_{k,j}))\Big]\Big/\Big(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)}\Big)^{1/2}.
\end{aligned}
$$

对固定的 j,k 和 $0\le i<m\le 2^{k(2-\theta)}$, 由于 $\{X_v(t);t\ge 0\}_{v=1}^{\infty}$ 是具有平稳增量的独立 Gauss 过程序列, 我们有

$$
\begin{aligned}
&E\{\xi(k,j;i)\xi(k,j;m)\}\\
&=\Big(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)}\Big)^{-1}\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{4(p-1)/(2-p)}\\
&\quad\cdot E\{(X_v(a_{k,j})-X_v(0))(X_v((m-i)2^{k(\theta-1)}a_{k,j}+a_{k,j})\\
&\quad -X_v((m-i)2^{k(\theta-1)}a_{k,j}))\}.
\end{aligned}
\tag{3.3.12}
$$

注意到

$$
\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{4(p-1)/(2-p)}\sigma_v^2(a_{k,j})=\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)},
$$

利用假设 (3.3.6), 由 (3.3.12), 对每一 $|j|\le e^k,0\le i<m\le 2^{k(2-\theta)}$, 当 k 充分大时有

$$
E\xi(k,j;i)\xi(k,j;m)\le\theta-1. \tag{3.3.13}
$$

又显然有

$$
E\xi^2(k,j;i)=1. \tag{3.3.14}
$$

设 $\{\eta_i; 0\le i\le 2^{k(2-\theta)}\}$ 和 τ 为独立正态随时机变量序列，均值为 0 且 $E\eta_i^2 = 2-\theta, 0\le i\le 2^{k(2-\theta)}$, $E\tau^2=\theta-1$. 定义 $\tau_i=\tau+\eta_i$, $0\le i\le 2^{k(2-\theta)}$. 注意到对充分大的 k

$$E\xi^2(k,j;i) = E\tau_i^2 = 1, \quad 0\le i\le 2^{k(2-\theta)},$$

$$E\{\xi(k,j;i)\xi(k,j;m)\} \le E\{\tau_i\tau_m\}, \quad 0\le i\ne m\le 2^{k(2-\theta)},$$

从而，由 Slepian 不等式对充分大的 k 有

$$\begin{aligned}
&P\Big\{\max_{0\le i\le 2^{k(2-\theta)}} \xi(k,j;i) \le ((2-\theta)^2 - 2(\theta-1)^{1/2})(2\log 2^k)^{1/2}\Big\}\\
&\quad\le P\Big\{\max_{0\le i\le 2^{k(2-\theta)}} \tau_i \le ((2-\theta)^2 - 2(\theta-1)^{1/2})(2\log 2^k)^{1/2}\Big\}\\
&\quad\le P\Big\{\max_{0\le i\le 2^{k(2-\theta)}} \eta_i \le (2-\theta)^2(2\log 2^k)^{1/2}\Big\}\\
&\qquad + P\{\tau\ge 2(\theta-1)^{1/2}(2\log 2^k)^{1/2}\}\\
&\quad\le \Big(\Phi((2-\theta)^{3/2}(2\log 2^k)^{1/2})\Big)^{2^{k(2-\theta)}} + \exp(-4\log 2^k)\\
&\quad\le 2^{-4k} + \left(1 - \frac{\exp(-(2-\theta)^3\log 2^k)}{3(1+(2-\theta)^{3/2}(2\log 2^k)^{1/2})}\right)^{2^{k(2-\theta)}}\\
&\quad\le 2^{-4k} + \exp\Big(-\frac{2^{k(2-\theta)}\cdot 2^{-k(2-\theta)^3}}{k}\Big)\\
&\quad\le 2^{-4k} + \exp\Big(-\frac{2^{k(2-\theta)(\theta-1)}}{k}\Big).
\end{aligned} \tag{3.3.15}$$

综合上述不等式，并应用 Borel-Cantelli 引理得

$$\begin{aligned}
&\liminf_{k\to\infty}\min_{|j|\le e^k}\max_{0\le i\le 2^{k(2-\theta)}} \frac{\|Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j}) - Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\theta\tilde\sigma(p,a_{k,j})(2\log 2^k)^{1/2}}\\
&\quad\ge \frac{(2-\theta)^2 - 2(\theta-1)^{1/2}}{\theta} \quad \text{a.s.}
\end{aligned} \tag{3.3.16}$$

由 (3.3.10) 和 $1<\theta<65/64$ 的任意性就得 (3.3.7) 成立.

情形 II. $p\ge 2$. 取 $N_{k,j}$ 使得 $\sigma_{N_{k,j}}(a_{k,j}) = \sigma^*(a_{k,j})$. 显然

$$\frac{\|Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\tilde{\sigma}(p,a_{k,j})}$$
$$\geq \frac{X_{N_{k,j}}(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-X_{N_{k,j}}(i2^{k(\theta-1)}a_{k,j})}{\sigma_{N_{k,j}}(a_{k,j})}.$$

沿着情形 I 的证明，我们可得此时 (3.3.7) 仍成立.

注 3.3.2 从命题 3.3.2 和证明我们也可推得：若 (3.3.5) 被下述条件代替:

$$\log\log\Big(a_T+\frac{1}{a_T}\Big)=O\Big(\log\frac{b_T}{a_T}\Big) \quad \text{且} \quad \frac{b_T}{a_T}\to\infty, \quad T\to\infty,$$

则 (3.3.7) 仍成立. 此外，若条件 (3.3.5) 和 (3.3.6) 被下述条件代替:

$$\frac{\log(b_T/a_T)}{\log\log\log(a_T+1/a_T)}\to\infty, \quad T\to\infty$$

且对每一 $k\geq 1$, $0\leq a<b\leq c$

$$E\{(X_k(a)-X_k(0))(X_k(c)-X_k(b))\}\leq 0,$$

那么 (3.3.7) 仍成立.

现在我们给出本节中的主要结果，它们是由 Csörgő 和 Shao (1993) 得到的.

定理 3.3.1 设 a_T $(T>0)$ 是正连续函数. 令 $a^*=\sup_{T>0}a_T$. 假设对某 $\alpha>0$, $\tilde{\sigma}(p,h)/h^\alpha$ 和 $\sigma(p,h)/h^\alpha$ 在 $(0,a^*)$ 上是拟增的，并且对每一 $\varepsilon>0$ 有

$$\lim_{T\to\infty}\frac{\log(T/a_T)}{\log\log T}=\infty, \tag{3.3.17}$$

$$\sigma(p,a_T)=o\Big(\tilde{\sigma}(p,a_T)\Big(\log\frac{T}{a_T}\Big)^{1/2}\Big), \quad T\to\infty, \tag{3.3.18}$$

$$\limsup_{T\to\infty}\max_{(T/a_T)^\varepsilon\leq j\leq T/a_T}\max_{k\geq 1}\{\sigma_k^{-2}(a_T)\cdot E[(X_k(a_T)-X_k(0))(X_k(ja_T)$$
$$-X_k((j+1)a_T))]\}\leq 0. \tag{3.3.19}$$

那么我们有

$$\lim_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde\sigma(p,a_T)(2\log(T/a_T))^{1/2}}=1\quad \text{a.s.}\tag{3.3.20}$$

定理 3.3.2 设 a_T $(T>0)$ 是正连续函数, 满足 (3.3.17). 假设对某 $\alpha>0$, $\sigma(p,h)/h^\alpha$ 和 $\tilde\sigma(p,h)/h^\alpha$ 在 $(0,a^*)$ 上是拟增的, 且

$$\tilde\sigma(p,a_T)\Big(\log\frac{T}{a_T}\Big)^{1/2}=o(\sigma(p,a_T)),\quad T\to\infty.\tag{3.3.21}$$

那么我们有

$$\lim_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,a_T)}=1\quad \text{a.s.}\tag{3.3.22}$$

定理 3.3.3 假设对某 $\alpha>0$, $\tilde\sigma(p,h)/h^\alpha$ 在 $(0,1)$ 上是拟增的. 还假设对每一 $\varepsilon>0$

$$\sigma(p,h)=o\Big(\tilde\sigma(p,h)\Big(\log\frac{1}{h}\Big)^{1/2}\Big),\quad h\to 0,\tag{3.3.23}$$

$$\limsup_{h\to0}\max_{h^{-\varepsilon}\le j\le h^{-1}}\max_{k\ge1}\frac{E[(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))]}{\sigma_k^2(h)}\le 0.\tag{3.3.24}$$

那么我们有

$$\lim_{h\to0}\sup_{0\le t\le1}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde\sigma(p,h)(2\log(1/h))^{1/2}}=1\quad \text{a.s.}\tag{3.3.25}$$

定理 3.3.4 假设对某 $\alpha>0$, $\sigma(p,h)/h^\alpha$ 在 $(0,1)$ 上是拟增的, 且

$$\tilde\sigma(p,h)(\log\frac{1}{h})^{1/2}=o(\sigma(p,h)),\quad h\to0.\tag{3.3.26}$$

那么我们有

$$\lim_{h\to0}\sup_{0\le t\le1}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}=1\quad \text{a.s.}\tag{3.3.27}$$

注 3.3.3 若对任意的 $0 \le a \le b \le c \le d < \infty$ 和每一 $k \ge 1$ 有

$$E\{(X_k(b) - X_k(a))(X_k(d) - X_k(c))\} \le 0, \tag{3.3.28}$$

显然，(3.3.19) 和 (3.3.24) 满足. 特别地，若对每一 $k \ge 1$, $\sigma_k^2(h)$ 在 $(0, \infty)$ 上是凹函数，则 (3.3.28) 成立，从而 (3.3.19) 和 (3.3.24) 被满足. 事实上，条件 (3.3.19) 或 (3.3.24) 的确比条件 (3.3.28) 弱.

注 3.3.4 值得指出的是：在定理 3.3.1 和 3.3.3 中的正则化常数是完全不同于定理 3.3.2 和 3.3.4 的. 后两定理的结论看起来是令人惊奇的. 然而，在定理 3.3.2 和定理 3.3.4 的条件下，分别有

$$\|Y(a_T) - Y(0)\|_{l^p} \sim \delta_p \sigma(p, a_T), \quad T \to \infty$$

和

$$\|Y(h) - Y(0)\|_{l^p} \sim \delta_p \sigma(p, h), \quad h \to 0.$$

由此可得，它们的结论很像大数律. 另一方面，定理 3.3.1 和 3.3.3 的结论分别可与标准 Wiener 过程的大增量和连续模相比较. (参见定理 0.1,0.2 或 Csörgő 和 Révész 1981, 第一章).

定理 3.3.1 的证明 这是命题 3.3.1 和 3.3.2 的直接推论.

定理 3.3.2 的证明 由命题 3.3.1 我们有

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p, a_T)} \le 1 \quad \text{a.s.}$$

故只需证明

$$\liminf_{T\to\infty} \sup_{0\le s\le a_T} \frac{\|Y(s) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_T)} \ge 1 \quad \text{a.s.} \tag{3.3.29}$$

设 $1 < \theta < 65/64$. 定义

$$B_k = \left\{T;\ 2^k \le \frac{T}{a_T} \le 2^{k+1}\right\}, \quad k \ge 0,$$

$$B_{k,j} = \{T;\ \theta^j \le \sigma(p, a_T) \le \theta^{j+1}, T \in B_k\}, \quad -\infty < j < \infty,$$

$$a_{k,j} = a(T_{k,j}) = \inf\{a_T;\ T \in B_{k,j}\}.$$

与 (3.3.4) 类似，我们有

$$\sigma(p,a_T)\le 2(1+a_T)\sigma(p,1),$$

因此并注意到 (3.3.17), 当 k 充分大时有

$$B_{k,j}=\phi,\quad 若\ |j|>e^k.$$

所以

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le s\le a_T}\frac{\|Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\\
&\ge\liminf_{k\to\infty}\min_{|j|\le e^k}\inf_{T\in B_{k,j}}\sup_{0\le s\le a_T}\frac{\|Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\\
&\ge\liminf_{k\to\infty}\min_{|j|\le e^k}\frac{\|Y(a_{k,j})-Y(0)\|_{l^p}}{\theta\delta_p\sigma(p,a_{k,j})}.
\end{aligned}\tag{3.3.30}$$

应用 Hölder 不等式得

$$\begin{aligned}
E\|Y(a_T)-Y(0)\|_{l^p}&\ge\frac{(E\|Y(a_T)-Y(0)\|_{l^p}^p)^{(2p-1)/p}}{(E\|Y(a_T)-Y(0)\|_{l^p}^{2p})^{(p-1)/p}}\\
&\ge\frac{(\delta_p^p\sum_{k=1}^\infty\sigma_k^p(a_T))^{(2p-1)/p}}{\left((\delta_p^p\sum_{k=1}^\infty\sigma_k^p(a_T))^2+\delta_{2p}^{2p}\sum_{k=1}^\infty\sigma_k^{2p}(a_T)\right)^{(p-1)/p}}\\
&\ge\frac{(\delta_p^p\sum_{k=1}^\infty\sigma_k^p(a_T))^{(2p-1)/p}}{\left((\delta_p^p\sum_{k=1}^\infty\sigma_k^p(a_T))^2+\delta_{2p}^{2p}\sigma^{*p}(a_T)\sum_{k=1}^\infty\sigma_k^p(a_T)\right)^{(p-1)/p}}\\
&=\frac{\delta_p\sigma(p,a_T)}{\left(1+\delta_{2p}^{2p}\delta_p^{-p}(\sigma^*(a_T)/\sigma(p,a_T))^p\right)^{(p-1)/p}}.
\end{aligned}\tag{3.3.31}$$

从而，由 (3.3.21)

$$\liminf_{T\to\infty}\frac{E\|Y(a_T)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\ge 1.\tag{3.3.32}$$

由 (3.1.18) 和 (3.3.32) 即得, 对每一充分大的 k 及 $|j| \le e^k$,

$$
\begin{aligned}
P&\left\{\frac{\|Y(a_{k,j})-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_{k,j})}\le 2-\theta\right\}\\
&\le P\Big\{\|Y(a_{k,j})-Y(0)\|_{l^p}-E\|Y(a_{k,j})-Y(0)\|_{l^p}\\
&\qquad\le -\frac{\theta-1}{2}\delta_p\sigma(p,a_{k,j})\Big\}\\
&\le 2\exp\Big(-\frac{(\theta-1)^2\delta_p^2\sigma^2(p,a_{k,j})}{\delta\tilde{\sigma}^2(p,a_{k,j})}\Big)\\
&\le 2\exp\Big(-4\log\frac{T_{k,j}}{a(T_{k,j})}\Big)\\
&\le 2\cdot 2^{-4k},
\end{aligned}
\tag{3.3.33}
$$

结合 Borel-Cantelli 引理得

$$
\liminf_{k\to\infty}\min_{|j|\le e^k}\frac{\|Y(a_{k,j})-Y(0)\|_{l^p}}{\theta\delta_p\sigma(p,a_{k,j})}\ge\frac{2-\theta}{\theta}\quad \text{a.s.} \tag{3.3.34}
$$

由 (3.3.30), (3.3.34) 和 $\theta>1$ 的任意性得证 (3.3.29) 成立.

定理 3.3.3 的证明 由命题 3.3.2, 我们有

$$
\liminf_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}}\ge 1\quad \text{a.s.}
$$

故只需证明

$$
\limsup_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}}\le 1\quad \text{a.s.} \tag{3.3.35}
$$

对任一固定的 $\varepsilon>0$, 令 $\sigma_*(p,h)=\varepsilon\sup_{0\le s\le h}\tilde{\sigma}(p,h)(\log(1/s))^{1/2}$, $0<h\le 1$. 注意到 $\tilde{\sigma}(p,h)/h^{\alpha}$ 是拟增的, 存在与 ε 无关的常数 c_0, 使得对 $0<h<1$ 有

$$
\begin{aligned}
\varepsilon\sigma(p,h)\Big(\log\frac{1}{h}\Big)^{1/2}\le\sigma_*(p,h)&=\varepsilon\sup_{0\le s\le h}\frac{\tilde{\sigma}(p,s)}{s^{\alpha}}s^{\alpha}\Big(\log\frac{1}{s}\Big)^{1/2}\\
&\le\varepsilon c_0\tilde{\sigma}(p,h)\Big(\log\frac{1}{h}\Big)^{1/2}.
\end{aligned}
\tag{3.3.36}
$$

其次，$\sigma_*(p,h)$ 是非降的，$\sigma_*(p,h)/h^{\alpha/2}$ 是拟增的，且由 (3.3.23)，对充分小的 h 成立 $\sigma(p,h) \le \sigma_*(p,h)$. 从而由注 3.3.1，我们得

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}+\sigma_*(p,h)} \le 1 \quad \text{a.s.}$$

这样，由 (3.3.36) 有

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} \le 1+\varepsilon c_0.$$

由 ε 的任意性得证 (3.3.35) 成立.

定理 3.3.4 的证明 类似于 (3.3.35) 的证明，由命题 3.3.1 我们有

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)} \le 1 \quad \text{a.s.} \tag{3.3.37}$$

另一方面，沿着 (3.3.29) 的证明思路，我们也可得

$$\liminf_{h\to 0} \sup_{0\le s\le h} \frac{\|Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,h)} \ge 1 \quad \text{a.s.} \tag{3.3.38}$$

综合 (3.3.37) 和 (3.3.38) 就得证 (3.3.27). 定理证毕.

作为上述定理的应用，我们来给出 l^p 值分数 Wiener 过程的连续模及 l^p 值分数 O-U 过程的连续模.

设 $\{\xi(t); t\ge 0\}$ 为具有平稳增量、零均值的 Gauss 过程. 称 $\xi(t)$ 是阶为 γ 的分数 Wiene 过程 (或称为自相似 Gauss 过程)，若 $E\xi^2(t)=t^{2\gamma}$，其中 $0<\gamma<1$. 当 $\gamma=1/2$ 时，$\xi(t)$ 就是熟知的 Wiener 过程.

设 $p\ge 1$, $\{c_n; n\ge 1\}$ 为非负数列. 令

$$c(p)=\Big(\sum_{k=1}^{\infty} c_k^p\Big)^{1/p}, \tag{3.3.39}$$

$$\tilde{c}(p) = \begin{cases} c\left(\dfrac{2p}{2-p}\right), & 当 1 \le p < 2, \\ \max\limits_{k\ge 1} c_k, & 当 p \ge 2. \end{cases} \tag{3.3.40}$$

设 $\{Y(t);t \ge 0\} = \{c_k\xi_k(t);t \ge 0\}_{k=1}^{\infty}$, 其中 $\xi_k(t)$ 是独立的 γ 阶 $(0<\gamma<1)$ 分数 Wiener 过程. 记 $\sigma_k^2(h) = c_k^2 E\xi_k^2(h) = c_k^2 h^{2\gamma}$. 定义 $\sigma(p,h)$ 和 $\tilde{\sigma}(p,h)$ 如前. 显然, 我们有

$$\sigma(p,h) = h^{\gamma}c(p), \quad \tilde{\sigma}(p,h) = h^{\gamma}\tilde{c}(p). \tag{3.3.41}$$

假设

$$0 < \sum_{k=1}^{\infty} c_k^p < \infty. \tag{3.3.42}$$

则由定理 3.1.3, $Y(\cdot) \in l^p$ 具有 a.s 连续的样本轨道. 注意到对每一 $k \ge 1$, $a > 0$, $j > 2$ 有

$$\begin{aligned} &\frac{Ec_k\xi_k(a)(c_k\xi_k(ja) - c_k\xi_k((j-1)a))}{\sigma_k^2(a)} \\ &= \frac{E\xi_1(a)(\xi_1(ja) - \xi_1((j-1)a))}{E\xi_1^2(a)} \\ &= \frac{1}{2}((j+1)^{2\gamma} + (j-1)^{2\gamma} - 2j^{2\gamma}) \end{aligned}$$

且

$$\lim_{j\to\infty}((j+1)^{2\gamma} + (j-1)^{2\gamma} - 2j^{2\gamma}) = 0,$$

条件 (3.3.19) 和 (3.3.24) 被满足. 从而由定理 3.3.1, 3.3.3, 我们得到下述推论.

推论 3.3.1 设 $p \ge 1$, $\{\xi_k(t);t\ge 0\}$ 为独立的 γ $(0<\gamma<1)$ 阶分数 Wiener 过程. 设 $\{Y(t);t \ge 0\} = \{c_k\xi_k(t);t \ge 0\}_{k=1}^{\infty}$. 假设 (3.3.42) 被满足. 则对任何满足 $\lim_{T\to\infty}\log(T/a_T)/\log\log T = \infty$ 的正连续函数 a_T, 我们有

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{h^{\gamma}\tilde{c}(p)(2\log(1/h))^{1/2}} = 1 \quad \text{a.s.}, \tag{3.3.43}$$

$$\lim_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{a_T^\gamma\tilde{c}(p)(2\log(T/a_T))^{1/2}}=1\quad \text{a.s.} \tag{3.3.44}$$

设 $\{Y(t);t\ge 0\}=\{X_k(t);t\ge 0\}_{k=1}^\infty$ 为独立 O-U 过程，具有系数 γ_k 和 λ_k. 易知

$$\{X_k(t);t\ge 0\}_{k=1}^\infty \quad 和 \quad \left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}};t\ge 0\right\}_{k=1}^\infty$$

有相同的分布，其中 $\{W_k(t)\}_{k=1}^\infty$ 是独立标准 Wiener 过程. 因此，不失一般性，可写

$$X_k(t)=\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}},\quad t\ge 0,k=1,2,\ldots$$

且保持 $Y(\cdot)$ 的轨道性质不变. 这一关系式和分数 Wiener 过程的概念自然导致引入分数 O-U 过程并研究它们的轨道性质.

设 $\{\xi(t);t\ge 0\}$ 为 γ $(0<\gamma<1)$ 阶分数 Wiener 过程. 平稳 Gauss 过程 $\{X(t);t\ge 0\}$ 称为 γ 阶的具有系数 a 和 b 的 O-U 过程，若

$$\{X(t);t\ge 0\} \quad 和 \quad \left\{\left(\frac{a}{b}\right)^{1/2}\frac{\xi(e^{2bt})}{e^{2\gamma bt}};t\ge 0\right\}$$

具有相同的分布，即 $EX(t)=0$, 且对所有 $t,s\ge 0$ 有

$$E\{X(t)X(s)\}=\frac{a}{2b}(e^{2\gamma b(t-s)}+e^{2\gamma b(s-t)}-|e^{b(t-s)}-e^{b(s-t)}|^{2\gamma}), \tag{3.3.45}$$

其中 $a\ge 0,b>0$.

显然，当 $\gamma=1/2$ 时，$\{X(t);t\ge 0\}$ 就是通常的 O-U 过程.

下面设 $\{Y(t);t\ge 0\}=\{X_k(t);t\ge 0\}_{k=1}^\infty$ 为 γ 阶独立分数 O-U 过程序列，具有系数 γ_k 和 λ_k, 其中 $0<\gamma<1$, $\gamma_k\ge 0$, $\lambda_k>0$. 对 $h\ge 0$, $k=1,2,\ldots$, 令

$$\begin{aligned}\sigma_k^2(h)&=E(X_k(t+h)-X_k(t))^2\\&=\frac{\gamma_k}{\lambda_k}(2+(e^{\lambda_k h}-e^{-\lambda_k h})^{2\gamma}-e^{2\gamma\lambda_k h}-e^{-2\gamma\lambda_k h}).\end{aligned}$$

设 $p \geq 1$, 定义 $\sigma(p,h), \tilde{\sigma}(p,h)$ 的 δ_p 如前. 由定理 3.3.1 我们得

推论 3.3.2 假设对某 $\alpha > 0$, $\tilde{\sigma}(p,h)/h^{\alpha}$ 在 $(0,1)$ 上是拟增的. 若

$$\sigma(p,h) = o\Big(\tilde{\sigma}(p,h)\Big(\log\frac{1}{h}\Big)^{1/2}\Big), \quad h \to 0, \tag{3.3.46}$$

那么

$$\lim_{h\to 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{3.3.47}$$

假设对某 $\alpha > 0$, $\sigma(p,h)/h^{\alpha}$ 在 $(0,1)$ 上是拟增的. 若

$$\tilde{\sigma}(p,h)\Big(\log\frac{1}{h}\Big)^{1/2} = o(\sigma(p,h)), \quad h \to 0, \tag{3.3.48}$$

那么

$$\lim_{h\to 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p \sigma(p,h)} = 1 \quad \text{a.s.} \tag{3.3.49}$$

证明 由定理 3.3.3 和 3.3.4, 只需验证

$$\limsup_{h\to 0} \max_{\log(1/h)\leq j\leq 1/h} \max_{k\geq 1} \frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))\}}{\sigma_k^2(h)} \leq 0. \tag{3.3.50}$$

我们可以验证, 若 $0 < \gamma \leq 1/2$, $\sigma_k^2(h)$ 在 $(0,\infty)$ 上是凹函数, 那么这时 (3.3.50) 被满足.

下面考察 $1/2 < \gamma < 1$ 的情形. 对每一 $h > 0$, $j \geq 6$, $k \geq 1$ 和对某 $(j-2)\lambda_k h \leq \xi \leq j\lambda_k h$, 我们有

$$\begin{aligned}
&\frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))\}}{\sigma_k^2(h)} \\
&= \frac{f(j\lambda_k h) + f((j-2)\lambda_k h) - 2f((j-1)\lambda_k h)}{2(2+f(\lambda_k h))} \\
&= \frac{f''(\xi)(\lambda_k h)^2}{2(2+f(\lambda_k h))}.
\end{aligned}$$

其中

$$f(x) = (e^x - e^{-x})^{2\gamma} - e^{2\gamma x} - e^{-2\gamma x}.$$

通过一些初等计算可以证明 (3.3.50) 成立.

下述结论通过系数 γ_k, λ_k 和阶 γ 应满足的条件, 给出了推论 3.3.2 的某些特殊情形.

推论 3.3.3 假设

$$0 < \sum_{k=1}^{\infty} (\gamma_k/\lambda_k)^{p/2} \lambda_k^{\gamma p} < \infty.$$

那么我们有

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\Gamma(p,\gamma) h^{\gamma} (2\log(1/h))^{1/2}} = 1 \quad \text{a.s.},$$

其中

$$\Gamma(p,\gamma) = \begin{cases} \left(\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k}\right)^{p/(2-p)} (2\lambda_k)^{2\gamma p/(2-p)}\right)^{(2-p)/2p}, & \text{当} 1 \le p < 2 \text{ 时}, \\ \max_{k\ge 1} (\gamma_k/\lambda_k)^{1/2} \cdot (2\lambda_k)^{\gamma}, & \text{当} p \ge 2 \text{ 时}. \end{cases}$$

特别地, 取 $\gamma = 1/2, p = 2$, 我们得到 O-U 过程生成的 l^2 模平方过程:

$$\chi^2(t) = \|Y(t)\|_{l^2}^2 = \sum_{k=1}^{\infty} X_k^2(t).$$

令

$$\Gamma_0 = E\chi^2(t) = \sum_{k=1}^{\infty} (\gamma_k/\lambda_k),\ \Gamma_1 = \sum_{k=1}^{\infty} \gamma_k,\ \Gamma_2 = \sum_{k=1}^{\infty} \gamma_k^2/\lambda_k,$$

$$B = \max_{k\ge 1} \gamma_k,\ M = \max_{k\ge 1} \gamma_k^2/\lambda_k.$$

那么从推论 3.3.3 可得

推论 3.3.4 假设 $\Gamma_0 < \infty, \Gamma_1 < \infty$, 则

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{2(Bh\log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

Csörgő 和 Lin (1990) 也研究过当 $h \to 0$ 而 t 的值域趋向无穷时的另一形式的连续模. 此时正则化因子相当不一样.

定理 3.3.5 假设 $\Gamma_0 < \infty, \Gamma_2 < \infty$ 且当 $h \to 0$ 时 $T_h \uparrow$ 连续地递增趋向无穷. 则

$$\limsup_{h\to 0} \sup_{|t|\le T_h} \sup_{0\le s\le h} \frac{|\chi^2(t+s)-\chi^2(t)|}{(8Mh)^{1/2}2\log(T_h/h)} \le 1.$$

若 T_h 还满足

$$(\log T_h)/\log(1/h) \to \infty, \quad h \to 0,$$

则有

$$\lim_{h\to 0} \sup_{|t|\le T_h} \frac{|\chi^2(t+h)-\chi^2(t)|}{(8Mh)^{1/2}2\log T_h} = 1 \quad \text{a.s.},$$

$$\lim_{h\to 0} \sup_{|t|\le T_h} \sup_{0\le s\le h} \frac{|\chi^2(t+s)-\chi^2(t)|}{(8Mh)^{1/2}2\log T_h} = 1 \quad \text{a.s.}$$

注 3.3.5 此时, 例如我们可取

$$T_h = \exp\{\log(1/h)\log\log\cdots\log(1/h)\},$$

其中对充分小的 $h > 0$, $\log\log\cdots\log(1/h)$ 是指可取对数任意有限次, 这样, 过程 $\chi^2(\cdot)$ 的模为 $(8Mh)^{1/2}2\log(1/h)\log\log\cdots\log(1/h)$.

定理 3.3.5 的证明的关键是下述引理.

引理 3.3.1 假设 $\Gamma_0 < \infty$ 且 $\Gamma_2 < \infty$. 对任意的 $\varepsilon > 0$ 存在 $h(\varepsilon) > 0$ 和 $C = C(\varepsilon) > 0$ 使得对任何 $T_h > h(\varepsilon)$, $h < h(\varepsilon)$, 对任一固定的 t 和任何 $x \ge (8/\varepsilon^2)(\Gamma_2/M)^{1/2}$ 有

$$P\{|\chi^2(t+h)-\chi^2(t)| \ge x(8Mh)^{1/2}\} \ge \frac{1}{7x}\exp\Big(-\frac{x}{1+\varepsilon}\Big),$$

$$P\{\sup_{|t|\le T_h} \sup_{0\le s\le h} |\chi^2(t+s)-\chi^2(t)| \ge x(8Mh)^{1/2}\}$$
$$\le (CT_h/h)\exp\Big(-\frac{x}{1+\varepsilon}\Big).$$

定理 3.3.5 和引理 3.3.1 和证明不在此陈述了.

3.3.2 当 $\sigma(p,h)$ 有界时的大增量

对某些 Gauss 过程, 上面定理的条件不被满足. 例如, 对 O-U 过程, 当 $h\to\infty$ 时 $\sigma(p,h)$ 是有界的. 现在我们对这类过程 $Y(\cdot)$ 来建立有关大增量的结果. 为此目的, 我们先给出引理 3.1.4 的一个精细的修正.

引理 3.3.2 设 B 是范数为 $\|\cdot\|$ 的 Banach 空间, $\{\Gamma(t);t\geq 0\}$ 是取值于 B 中的随机过程, P 是由 $\Gamma(\cdot)$ 生成的概率测度. 假设 $\Gamma(\cdot)$ 关于范数是 P-a.s. 连续的, 且对任何 $t\geq 0$, $h\geq 0$, $0<x^*\leq x$ 存在非负单调非降函数 $\sigma_1(h)$ 和 $\sigma_2(h)$ 使得对某 $K,\gamma,\beta>0$

$$P\{\|\Gamma(t+h)-\Gamma(t)\|\geq x\sigma_1(h)+\sigma_2(h)\}\leq K\exp(-\gamma x^\beta). \quad (3.3.51)$$

则对任给的 $2\leq\alpha<e$, $0<\tau<1$, 对任何 $0\leq h\leq T$, $x\geq x^*$, 整数 $m\geq 3$ 和充分大的 $k=k(\tau)$ 有

$$\begin{aligned}P\Big\{&\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}\|\Gamma(t+s)-\Gamma(t)\|\geq x(\sigma_1(h+d(m,k)^{-1}h)\\&+\sigma_1(h,m,k))+\sigma_1^*(h,m,k)+\sigma_2(h+d(m,k)^{-1}h)+\sigma_2(h,m,k)\Big\}\\\leq{}& 4K\Big(\frac{T}{h}+1\Big)d(m,k)^2\exp(-\gamma x^\beta),\end{aligned} \quad (3.3.52)$$

其中

$$d(m,k)=\underbrace{\alpha^{\cdot^{\cdot^{\alpha^k}}}}_{m\text{ 次}},$$

$$\sigma_1(h,m,k)=2^{2+\frac{1}{\beta}}d(m-3,k)^{-1}\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_1(h\alpha^{-\alpha^y})}{y}dy,$$

$$\begin{aligned}\sigma_1^*(h,m,k)={}&2\Big(\frac{2}{\gamma}\Big)^{1/\beta}\Big(1-\Big(\frac{d(m-1,k+1-\tau)}{d(m-1,k+1)}\Big)^{1/\beta}\Big)^{-1}\\&\cdot\int_{d(m-1,k+1-\tau)^{1/\beta}}^{\infty}\sigma_1(h\alpha^{-y^\beta})dy,\end{aligned}$$

$$\sigma_2(h,m,k)=4d(m-3,k)^{-1}\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_2(h\alpha^{-\alpha^y})}{y}dy.$$

证明 固定整数 $m \geq 3$. 依照引理 3.1.3 的证明, 对任意的实数 t, 令

$$t_j = [td(m,j)/h]/(d(m,j)/h).$$

我们有

$$\|\Gamma(t+s) - \Gamma(t)\| \leq \|\Gamma((t+s)_k) - \Gamma(t_k)\| + \sum_{j=0}^{\infty} \|\Gamma((t+s)_{k+j+1}) - \Gamma((t+s)_{k+j})\| + \sum_{j=0}^{\infty} \|\Gamma(t_{k+j+1}) - \Gamma(t_{k+j})\|.$$

由于

$$\sup_{0\leq t\leq T} \sup_{0\leq s\leq h} |(t+s)_k - t_k| \leq h + d(m,k)^{-1}h,$$

$$\sup_{0\leq t\leq T} \sup_{0\leq s\leq h} |(t+s)_{k+j+1} - (t+s)_{k+j}| \leq hd(m,k+j+1)^{-1},$$

对任给的 $x \geq x^*$ 和 $x_j \geq x^*$, 由 (3.3.51) 得

$$P\Big\{\sup_{0\leq t\leq T} \sup_{0\leq s\leq h} \|\Gamma((t+s)_k) - \Gamma(t_k)\| \geq x\sigma_1(h + d(m,k)^{-1}h) + \sigma_2(h + d(m,k)^{-1}h)\Big\}$$
$$\leq 2Kd(m,k)^2\Big(\frac{T}{h} + 1\Big)\exp(-\gamma x^{\beta}),$$

$$P\Big\{\sup_{0\leq t\leq T} \sup_{0\leq s\leq h} \|\Gamma((t+s)_{k+j+1}) - \Gamma((t+s)_{k+j})\| \geq x_j\sigma_1(hd(m,k+j+1)^{-1}) + \sigma_2(hd(m,k+j+1)^{-1})\Big\}$$
$$\leq 2Kd(m,k+j+1)\Big(\frac{T}{h} + 1\Big)\exp(-\gamma x_j^{\beta})$$

和

$$P\Big\{\sup_{0\leq t\leq T} \sup_{0\leq s\leq h} \|\Gamma(t_{k+j+1}) - \Gamma(t_{k+j})\| \geq x_j\sigma_1(hd(m,k+j+1)^{-1}) + \sigma_2(hd(m,k+j+1)^{-1})\Big\}$$
$$\leq 2Kd(m,k+j+1)\Big(\frac{T}{h} + 1\Big)\exp(-\gamma x_j^{\beta}).$$

现在令 $\gamma x_j^\beta = \gamma x^\beta + d(m-1,k+j+1)$. 则

$$
\begin{aligned}
&\sum_{j=0}^{\infty} d(m,k+j+1)\exp(-\gamma x_j^\beta) \\
&=\sum_{j=0}^{\infty} d(m,k+j+1)\exp(-d(m-1,k+j+1))\exp(-\gamma x^\beta) \\
&\le \exp(-\gamma x^\beta)
\end{aligned}
$$

且

$$
\begin{aligned}
&2\sum_{j=0}^{\infty} x_j\sigma_1(hd(m,k+j+1)^{-1}) \\
\le& 2^{1+1/\beta}x\sum_{j=0}^{\infty}\sigma_1(hd(m,k+j+1)^{-1}) \\
&+2\Big(\frac{2}{\gamma}\Big)^{1/\beta}\sum_{j=0}^{\infty} d(m-1,k+j+1)^{1/\beta}\sigma_1(hd(m,k+j+1)^{-1}) \\
\le& 2^{1+1/\beta}x\sum_{j=0}^{\infty}(d(m-3,k+j+1)-d(m-3,k+j+1-\tau))^{-1} \\
&\cdot\int_{d(m-2,k+j+1-\tau)}^{d(m-2,k+j+1)}\frac{\sigma_1(h\alpha^{-\alpha^y})}{y}dy/\log\alpha \\
&+2\Big(2/\gamma\Big)^{1/\beta}\sum_{j=0}^{\infty}\Big(1-\Big(\frac{d(m-1,k+j+1-\tau)}{d(m-1,k+j+1)}\Big)^{1/\beta}\Big)^{-1} \\
&\cdot\int_{d(m-2,k+j+1-\tau)}^{d(m-2,k+j+1)}\sigma_1(h\alpha^{-\alpha^y})d\alpha^{\frac{1}{\beta}y} \\
\le& 2^{2+\frac{1}{\beta}}d(m-3,k)^{-1}x\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_1(h\alpha^{-\alpha^y})}{y}dy \\
&+2\Big(2/\gamma\Big)^{\frac{1}{\beta}}\Big(1-\Big(\frac{d(m-1,k+1-\tau)}{d(m-1,k+1)}\Big)^{1/\beta}\Big)^{-1} \\
&\cdot\int_{d(m-1,k+1-\tau)}^{\infty}\sigma_1(h\alpha^{-y^\beta})dy,
\end{aligned}
$$

此外还有

$$2\sum_{j=0}^{\infty}\sigma_2(hd(m,k+j+1)^{-1})$$
$$\leq 4d(m-3,k)^{-1}\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_2(h\alpha^{-\alpha^y})}{y}dy.$$

综合上述所有不等式得证 (3.3.52).

注 3.3.6 从引理 3.3.2 的证明容易看出, 由 (3.3.51) 可推出对任何 $T\geq 0$, $h\geq 0$, $x\geq x^*$, $k>0$ 和整数 $m\geq 3$ 有

$$P\Big\{\sup_{0\leq s\leq h}\|\Gamma(T+s)-\Gamma(T)\|\geq x(\sigma_1(h+d(m,k)^{-1}h)+\sigma_1(h,m,k))$$
$$+\sigma_1^*(h,m,k)+\sigma_2(h+d(m+k)^{-1}h)+\sigma_2(h,m,k)\Big\}$$
$$\leq 4Kd(m,k)\exp(-\gamma x^{\beta}).$$

利用这一引理, 我们可证明下列定理, 它是由林正炎 (1997a) 给出的. 令

$$L_m x=\underbrace{\log_\alpha\cdots\log_\alpha}_{m\text{ 次}}x.$$

定理 3.3.6 设 $Y(\cdot)$ 是本节开始所定义的过程. 设 a_T 是 T 的正连续拟增函数, $a_T\to\infty$ $(T\to\infty)$. 假设

$$\tilde{\sigma}(p,T)\to\tilde{\sigma}<\infty,\quad \sigma(p,T)=o\Big(\Big(\log\frac{T}{a_T}\Big)^{1/2}\Big),\quad T\to\infty, \tag{3.3.53}$$

$$\int_1^{\infty}\tilde{\sigma}(p,\alpha^{-x^2})dx<\infty,\quad \int_1^{\infty}\sigma(p,\alpha^{-\alpha^x})/x\,dx<\infty. \tag{3.3.54}$$

又假设存在 $0<\delta<1/\alpha$ 和整数 $m\geq 1$ 使得

$$a_T\leq Td(m,(L_mT)^{\delta+1/\alpha})^{-1}, \tag{3.3.55}$$

且存在 $a_0>0$ 使对任何 $a\geq a_0$, 每一 $i\geq 1$ 有

$$\max_{k\geq 1}E(X_k(ia)-X_k((i-1)a))(X_k(ja)-X_k((j-1)a))\leq 0. \tag{3.3.56}$$

那么我们有

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p\big(2\log(T/a_T)\big)^{1/2}}=1\quad\text{a.s.},\tag{3.3.57}$$

$$\lim_{T\to\infty}\sup_{0\le t\le T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p\big(2\log(T/a_T)\big)^{1/2}}=1\quad\text{a.s.},\tag{3.3.58}$$

$$\limsup_{T\to\infty}\frac{\|Y(T+a_T)-Y(T)\|_{l^p}}{\tilde{\sigma}_p\big(2\log(T/a_T)\big)^{1/2}}=1\quad\text{a.s.}\tag{3.3.59}$$

若条件 (3.3.56) 和 (3.3.55) 被下述条件代替:

$$\limsup_{a\to\infty}m(a)\le 0,\tag{3.3.56$'$}$$

其中 $m(a)=\max_{k\ge1}\max_{j>i\ge1}E(X_k(ia)-X_k((i-1)a))(X_k(ja)$ $-X_k((j-1)a))$, 且对某 $0<\delta<1/\alpha$,

$$a_T\le T(T^{-\delta_T}\wedge d(m,(L_mT)^{\delta+1/\alpha})^{-1}),\tag{3.3.55$'$}$$

其中 $\delta_T\to0$ $(T\to\infty)$ 且 $(0\vee m(T))/\delta_T\to0$ $(T\to\infty)$, 那么 (3.3.57)—(3.3.59) 仍成立.

注 3.3.7 易知

$$d(m,(L_mT)^{\delta+1/\alpha})=T^{(L_1T)^{-1+(L_2T)^{\cdot^{\cdot^{\cdot^{-1+(L_mT)^{-1+\delta+1/\alpha}}}}}}}.\tag{3.3.60}$$

记 $d(m,(L_mT)^{\delta+1/\alpha})=T^{(L_1T)^{-1+o(T,m)}}$. 则 $o(T,m)\to0$ 且 $\frac{o(T,m+1)}{o(T,m)}\to0$ $(T\to\infty)$. 因此当 $T\to\infty$ 时对任给 $\varepsilon>0$ 有

$$d(m,(L_mT)^{\delta+1/\alpha})=\alpha^{(L_1T)^{o(T,m)}}\le\alpha^{(L_1T)^{\varepsilon}},\tag{3.3.61}$$

$$d(m+1,(L_{m+1}T)^{\delta+1/\alpha})/d(m,(L_mT)^{\delta+1/\alpha})\to0.$$

定理 3.3.6 的证明 我们指出由条件 (3.3.55) 可推出

$$\frac{\log(T/a_T)}{\log\log T}\to\infty,\quad T\to\infty.\tag{3.3.62}$$

首先，我们来证

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p\big(2\log(T/a_T)\big)^{1/2}}\le 1 \quad \text{a.s.} \qquad (3.3.63)$$

由引理 3.1.2 可知引理 3.3.2 中的 (3.3.51) 对 $K=2,\gamma=\frac{1}{2},\beta=2$, $\sigma_1(h)=\tilde{\sigma}(p,h)$ 和 $\sigma_2(h)=\delta_p\sigma(p,h)$ 成立. 对 (3.3.55) 中给定的 $\delta>0$ 和 (3.3.52) 中的 $0<\tau<1$, 令 δ_1 满足 $1-1/\alpha-\delta<\delta_1<1-1/\alpha$ 且使得 $\tau_1:=\tau-\log(1-\delta_1)<1$. 记 $\varepsilon_T=d(m-1,(L_m a_T)^{1-\delta_1})/L_1a_T$. 在 (3.3.52) 中取 $k=L_{m+2}a_T^{\varepsilon_T}$ 和 $k_1=L_{m+2}a_T$. 那么 $d(m+2,k)=a_T^{\varepsilon_T}$ 且

$$\begin{aligned}
k-\tau&=L_m\Big[(L_2a_T)\Big(1+\frac{L_1\varepsilon_T}{L_2a_T}\Big)\Big]-\tau\\
&=L_{m+2}a_T+L_1(1-\delta_1)-\tau\\
&=k_1-\tau_1.
\end{aligned}$$

因此利用条件 (3.3.54), 对任给的 $\varepsilon>0$, 假定 T(等价地 k_1) 充分大，我们有

$$\begin{aligned}
&\sigma_1(a_T,m+2,k)\\
&\quad=2^{5/2}d(m-1,k)^{-1}\int_{d(m,k+1-\tau)}^{\infty}\frac{\sigma_1(a_T\alpha^{-\alpha^x})}{x}dx\\
&\quad\le 2^{5/2}d(m-1,k)^{-1}\int_{1}^{\infty}\frac{\sigma_1(\alpha^{\alpha^{d(m,k_1)}-\alpha^{xd(m,k_1+1-\tau_1)}})}{x}dx\\
&\quad\le\frac{\varepsilon}{2}\tilde{\sigma}_p.
\end{aligned}$$

类似地对充分大的 T

$$\begin{aligned}
&\sigma_1^*(a_T,m+2,k)\\
&\quad=4\Big(1-\Big(\frac{d(m+1,k+1-\tau)}{d(m+1,k+1)}\Big)^{1/2}\Big)^{-1}\\
&\qquad\cdot\int_{d(m+1,k+1-\tau)}^{\infty}\sigma_1(a_T\alpha^{-x^2})dx\le\varepsilon,
\end{aligned}$$

$$\sigma_2(a_T,m+2,k)=4d(m-1,k)^{-1}\int_{d(m,k+1-\tau)}^{\infty}\frac{\sigma_2(a_T\alpha^{-\alpha^x})}{x}dx\le\varepsilon.$$

因为当 (3.3.55) 成立时，对充分大的 T,

$$a_T^{\varepsilon+\varepsilon_T} \le T^{\varepsilon+\varepsilon_T} d(m,(L_m T)^{\delta+1/\alpha})^{-\varepsilon} \le T^{\varepsilon}.$$

所以，由 (3.3.62), 条件 (3.3.53) 和引理 3.3.2, 即得对任给的 $c>0$ 和充分大的 T 有

$$\begin{aligned}
&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le ca_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/ca_T))^{1/2}}\ge 1+2\varepsilon\Big\}\\
\le&P\Big\{\sup_{0\le t\le T}\sup_{0\le s\le ca_T}\|Y(t+s)-Y(t)\|_{l^p}\ge(1+\varepsilon)\\
&\quad\cdot\Big(2\Big[\log\frac{T}{ca_T}+\log\log T\Big]\Big)^{1/2}(\sigma_1(ca_T(1+d(m+2,k)^{-1}))\\
&\quad+\sigma_1(ca_T,m+2,k))+\sigma_1^*(ca_T,m+2,k)\\
&\quad+\sigma_2(ca_T(1+d(m+2,k)^{-1}))+\sigma_2(ca_T,m+2,k)\Big\}\\
\le&\frac{9T}{ca_T}d(m+2,k)^2\exp\Big\{-(1+\varepsilon)^2\Big(\log\frac{T}{ca_T}+\log\log T\Big)\Big\}\\
\le&9c^{2\varepsilon}T^{-2\varepsilon}a_T^{2(\varepsilon+\varepsilon_T)}(\log T)^{-(1+2\varepsilon)}\\
\le&9c^{2\varepsilon}(\log T)^{-(1+2\varepsilon)}.
\end{aligned}$$

对某 $\theta>1$, 记 $T_j=\theta^j$. 那么由 Borel-Cantelli 引理我们有

$$\limsup_{j\to\infty}\sup_{0\le t\le T_j}\sup_{0\le s\le ca_{T_j}}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T_j/ca_{T_j}))^{1/2}}\le 1+2\varepsilon\quad\text{a.s.}\tag{3.3.64}$$

注意到 a_T 是拟增的，由 (3.3.64) 我们得 (3.3.63). 由 (3.3.62) 和命题 3.3.2 知，相反的等式也成立．因而得证 (3.3.57) 成立.

为证 (3.3.58), 只需证明

$$\liminf_{T\to\infty}\sup_{0\le t\le T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}\ge 1\quad\text{a.s.}\tag{3.3.65}$$

假设条件 (3.3.55) 和 (3.3.56) 被满足．对某 $h>0$, 记 $B_{nk}=\{T;kh\le a_T<(k+1)h,n-1\le T<n\}$, $a_n'=\inf\{a_T;n-1\le T<n\}$,

$a_n^* = \sup\{a_T; n-1 \le T < n\}$. 则

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le t\le T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}\\
&\ge \liminf_{n\to\infty}\min_{a_n'/h-1\le k\le a_n^*/h}\inf_{T\in B_{nk}}\sup_{0\le t\le T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}\\
&\ge \liminf_{n\to\infty}\min_{a_n'/h-1\le k\le a_n^*/h}\sup_{0\le t\le n-1}\frac{\|Y(t+kh)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(n/kh))^{1/2}}\\
&\quad-\limsup_{n\to\infty}\sup_{0\le t\le n}\sup_{0\le s\le h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log((n-1)/a_n^*))^{1/2}}\\
&=: L_1 - L_2.
\end{aligned}$$

注意到当 $h\to 0$ 时, $\tilde{\sigma}(p,h)/\tilde{\sigma}_p \to 0$ 且由 a_T 是拟增的, $a_n^* \le ca_n$. 那么类似于 (3.3.63) 的证明, 当 h 充分小时我们有

$$L_2 \le \epsilon \quad \text{a.s.} \tag{3.3.66}$$

考察 L_1. 假设 $1\le p<2$. 我们有 (参见 (3.1.17))

$$\begin{aligned}
&\|Y((j+1)kh)-Y(jkh)\|_{l^p}\\
&\qquad\ge \frac{\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\left(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{\frac{p-1}{p}}}.
\end{aligned}$$

记

$$\begin{aligned}
\xi(j,k) &= \frac{\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\tilde{\sigma}(p,kh)\left(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{\frac{p-1}{p}}}\\
&= \frac{\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\left(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{1/2}}.
\end{aligned}$$

那么由条件 (3.3.56), 对 $j>i\ge 1$, 当 k 充分大时有

$$\begin{aligned}
&E\xi(i,k)\xi(j,k)\\
=&\Big(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{-1}\Big(\sum_{v=1}^{\infty}\sigma_v(kh)^{\frac{4(p-1)}{2-p}}\Big)E(X_v((i+1)kh)-X_v(ikh))\\
&\cdot(X_v((j+1)kh)-X_v(jkh))\le 0,
\end{aligned}\tag{3.3.67}$$

故由 Slepian 不等式，回顾 B_{nk} 的定义并注意到条件 (3.3.55) 及 a_T 的拟增性，存在 $C>0$ 使得对充分大的 n 有

$$
\begin{aligned}
&P\Big\{\min_{a_n'/h-1\le k\le a_n^*/h}\max_{0\le j\le n/2kh}\xi(j,k)\le(1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\\
&\le\sum_{k=[a_n'/h]-1}^{[a_n^*/h]}P\Big\{\max_{0\le j\le n/2kh}\xi(j,k)\le(1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\\
&\le\sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\Big(1-\exp\Big\{-(1-\varepsilon)\log\frac{n}{kh}\Big\}\Big)^{n/2kh}\\
&\le Ca_n\exp\{-C(n/a_n)^{\varepsilon}\}\le Cn^{-2}.
\end{aligned}\tag{3.3.68}
$$

由此即得

$$
L_1\ge 1-\varepsilon\quad\text{a.s.}\tag{3.3.69}
$$

假设 $p\ge 2$. 取 N_k 使得 $\sigma_{N_k}(kh)=\sigma^*(kh)$. 显然对充分大的 k 有

$$
\frac{\|Y((j+1)kh)-Y(jkh)\|_{l^p}}{\tilde\sigma_p}\ge(1-\varepsilon)\frac{X_{N_k}((j+1)kh)-X_{N_k}(jkh)}{\sigma_{N_k}(kh)}.
$$

按 $1\le p<2$ 情形时的证法，我们也有 (3.3.69). 至此 (3.3.65) 已被证明了，因此在条件 (3.3.55) 和 (3.3.56) 下，我们完成了 (3.3.57) 和 (3.3.58) 的证明.

现在考察 (3.3.59). 这只需证明

$$
\limsup_{T\to\infty}\frac{\|Y(T+a_T)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}\ge 1\quad\text{a.s.}\tag{3.3.70}
$$

令 $a_T'=a_0[a_T/a_0]$, 其中 a_0 由条件 (3.3.56) 确定. 那么

$$
\begin{aligned}
\limsup_{T\to\infty}\frac{\|Y(T+a_T)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}&\ge\limsup_{T\to\infty}\frac{\|Y(T+a_T')-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T'))^{1/2}}\\
&\quad-\limsup_{T\to\infty}\sup_{0\le s\le a_0}\frac{\|Y(T+s)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}=:I_1-I_2.
\end{aligned}\tag{3.3.71}
$$

注意到注 3.3.5, 按 (3.3.63) 的证明方法, 我们有

$$I_2 = 0 \quad \text{a.s.} \tag{3.3.72}$$

设 $t_0 = 1$. 由 $t_k = t_{k-1} + a'_{t_{k-1}}$, $k = 1, 2, \ldots$, 定义 t_k. 则

$$\begin{aligned}&\frac{\|Y(t_k + a'_{t_k}) - Y(t_k)\|_{l^p}}{\tilde{\sigma}(p, a'_{t_k})} \\ &\quad\geq \frac{\sum_{v=1}^{\infty} \sigma_v(a'_{t_k})^{\frac{2(p-1)}{2-p}} (X_v(t_k + a'_{t_k}) - X_v(t_k))}{\left(\sum_{v=1}^{\infty} \sigma_v(a'_{t_k})^{\frac{2p}{2-p}}\right)^{1/2}} =: \zeta_k.\end{aligned}$$

再次运用条件 (3.3.56), 只要 i 充分大, 对 $j > i$ 我们有

$$E\zeta_i\zeta_j \leq 0.$$

令 $D_n = \{k; \frac{1}{2}n \leq t_k \leq n - 1\}$. 显然, 由条件 (3.3.55), 对 $k \in D_n$, 当 $n \to \infty$ 时 $a_{t_k} = o(n)$. 因此对充分大的 n 有

$$\sum_{k \in D_n} a_{t_k} \geq \sum_{k \in D_n} (t_k - t_{k-1}) - \max_{k \in D_n} a_{t_k} \geq \frac{1}{3} n.$$

其次, 由条件 (3.3.55) 可知, 对任何 $A > 0$ 和充分大的 T 有

$$a_T \leq T(\log T)^{-A}. \tag{3.3.73}$$

由 Slepian 不等式, 对 n, $n - 1 < T \leq n$, 当 $n \to \infty$ 时

$$\begin{aligned}&P\left\{\sup_{T/2 \leq t \leq T} \frac{\|Y(t + a'_t) - Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(t/a_t))^{1/2}} \leq 1 - \varepsilon\right\} \\ &\quad\leq P\left\{\max_{k \in D_n} \zeta_k/(2\log(t_k/a_{t_k}))^{1/2} \leq 1 - \frac{\varepsilon}{2}\right\} \\ &\quad\leq \prod_{k \in D_n} P\left\{\zeta_k \leq \left(1 - \frac{\varepsilon}{2}\right)(2\log(t_k/a_{t_k}))^{1/2}\right\} \\ &\quad\leq \prod_{k \in D_n} \left(1 - \exp\left\{-\left(1 - \frac{\varepsilon}{2}\right)\log(t_k/a_{t_k})\right\}\right) \\ &\quad\leq \exp\left\{-\sum_{k \in D_n} (a_{t_k}/t_k)^{1-\varepsilon/2}\right\} \leq \exp\left\{-\frac{1}{3}\log n\right\} \to 0.\end{aligned}$$

由此即得

$$I_1 \geq 1-\varepsilon \quad \text{a.s.} \tag{3.3.74}$$

把 (3.3.72) 和 (3.3.74) 代入 (3.3.71) 中就得 (3.3.70), 因此 (3.3.59) 得证.

当条件 (3.3.55) 和 (3.3.56) 分别被 (3.3.55′) 和 (3.3.56′) 代替时, (3.3.69) 的证明是类似的. 我们仅考察 $1 \leq p < 2$ 的情形. 注意到已证的事实, 不失一般性我们假设 $m(a) \geq 0$ 对所有充分大的 a 成立. 令

$$m_1(a) = m(a)\Big(\sum_{v=1}^{\infty}\sigma_v(a)^{\frac{2p}{2-p}}\Big)^{-1}\sum_{v=1}^{\infty}\sigma_v(a)^{\frac{4(p-1)}{2-p}}.$$

设 $\eta_j = \eta_j^{(k)}, j = 0,1,\ldots,[n/2kh]$, 和 $\tau = \tau^{(k)}$ 是独立正态随机变量, 均值为零, 且 $E\eta_j^2 = 1 - m_1(kh)$, $E\tau^2 = m_1(kh)$. 定义 $\xi_i = \eta_i + \tau$. 则 $E\xi_i^2 = 1$ 且

$$E\xi(i,k)\xi(j,k) = E\xi_i\xi_j = m_1(kh), \quad j-i \geq 1.$$

从而对充分大的 n 有

$$\begin{aligned}
&P\Big\{\min_{a_n'/h-1\leq k\leq a_n^*/h}\ \max_{0\leq j\leq n/2kh}\xi(j,k) \leq (1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\\
&\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} P\Big\{\max_{0\leq j\leq n/2kh}\xi_j \leq (1-\varepsilon)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\\
&\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\Big(P\Big\{\max_{0\leq j\leq n/2kh}\eta_j \leq \Big(1-\frac{\varepsilon}{2}\Big)\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\\
&\qquad + P\Big\{\tau \geq \frac{\varepsilon}{2}\Big(2\log\frac{n}{kh}\Big)^{1/2}\Big\}\Big)\\
&\leq \sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\Big(\Big(1-\exp\Big\{-\Big(1-\frac{\varepsilon}{2}\Big)\log\frac{n}{kh}\Big\}\Big)^{\frac{n}{2kh}}\\
&\qquad + \exp\Big\{-\frac{\varepsilon^2}{4m_1(kh)}\log\frac{n}{kh}\Big\}\Big)
\end{aligned}$$

$$\le \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} \left(\exp\left\{ -\left(\frac{n}{2kh}\right)^{\varepsilon/2} \right\} + \left(\frac{n}{kh}\right)^{-\varepsilon^2/4m_1(kh)} \right)$$

$$\le h^{-1} n \exp(-(\log n)^2) + \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} n^{-\varepsilon^2 \delta_{kh}/4m_1(kh)}.$$

注意到 $m(T)/\delta_T \to 0$ $(T \to \infty)$, 这就得 (3.3.69). (3.3.58) 证毕. 类似地在条件 (3.3.55′) 和 (3.3.56′) 下我们可证明 (3.3.59). 细节从略.

利用定理 3.3.6 的结论我们可给出下列推论.

推论 3.3.5 设 $\{Y(t); t \ge 0\} = \{X_k(t); t \ge 0\}_{k=1}^{\infty}$ 为独立的 γ 阶分数 O-U 过程, 具有系数 γ_k 和 λ_k, 其中 $0 < \gamma < 1$, $\gamma_k \ge 0$, $\lambda_k > 0$. 假设条件 (3.3.53) 和 (3.3.54) 被满足. 设 a_T 如定理 3.3.5 中定义, 且满足条件 (3.3.55). 那么 (3.3.57)—(3.3.59) 成立.

§3.4 l^∞ 值 Gauss 过程的增量

设 $\{Y(t); t \ge 0\} = \{X_k(t); t \ge 0\}_{k=1}^{\infty}$ 为 Gauss 过程, $EX_k(t) = 0$ 且 $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$ 是非降连续的. 我们将保留上面引进的记号.

3.4.1 连续模

不失一般性, 我们假设, 对每一 $k \ge 1$, 当 $h > 0$ 时 $\sigma_k(h) > 0$. 设 y_h 是下述方程的解

$$\sum_{k=1}^{\infty} (hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)} = h. \tag{3.4.1}$$

下述结果是由 Csörgő, Lin 和 Shao (1994a) 得到的.

定理 3.4.1 假设对某 $\alpha > 0$, $\sigma^{*2}(h)/h^\alpha$ 是拟增的, 且存在正数 A 和 h_0 使得

$$\sum_{k=1}^{\infty} \sigma_k^A(h_0) < \infty. \tag{3.4.2}$$

那么

$$\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\le 1 \quad \text{a.s.} \tag{3.4.3}$$

若 (3.4.2) 代之以条件：对某 $0<h\le h_0$ 使对某个 $c_1>0$ 和每一 $k\ge 1$

$$\inf_{0<s\le h}\frac{\sigma^*(s)}{\sigma_k(s)}\ge c_1\frac{\sigma^*(h)}{\sigma_k(h)} \tag{3.4.4}$$

且

$$\sum_{k=1}^{\infty}\exp\left\{-\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\log\frac{1}{h}\right\}<\infty, \tag{3.4.5}$$

那么 (3.4.3) 对 $y_h=1$ 成立. 若进一步假设 $X_k(\cdot), k=1,2,\ldots,$ 是独立的，且对 $0\le t_1<t_2\le t_3<t_4$

$$E(X_k(t_2)-X_k(t_1))(X_k(t_4)-X_k(t_3))\le 0, \tag{3.4.6}$$

那么

$$\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}=1 \quad \text{a.s.} \tag{3.4.7}$$

且

$$\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}=1 \quad \text{a.s.} \tag{3.4.8}$$

证明 首先，我们列出下列事实. 因为

$$h=\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)}\ge hy_h,$$

所以我们有 $0<y_h\le 1$. 此外通过一些初等计算容易看到，由条件 (3.4.2) 可推出 (3.4.5)，后者保证了方程 (3.4.1) 解的存在唯一性.
我们也有下述性质：存在常数 $d>0$ 使得

$$\sigma^{*2}(h)\ge dh^2. \tag{3.4.9}$$

事实上，注意到 $\sigma_k^2(h)$ 的定义，我们有 $\sigma_k^2(2h)\le 4\sigma_k^2(h)$. 所以，归纳地有

$$\sigma_k^2(h)\ge\frac{1}{4}\sigma_k^2(2h)\ge\cdots\ge\frac{1}{4^l}\sigma_k^2(2^lh)\ge h^2\sigma_k^2\Big(\frac{1}{2}\Big),\qquad 当\frac{1}{2}\le 2^lh\le 1. \tag{3.4.10}$$

这就推得 (3.4.9). 进一步，结合条件 (3.4.2), 我们得

$$\sum_{k=1}^{\infty}\Big(\frac{\sigma_k(h)}{\sigma^*(h)}\Big)^A\le\frac{c_2}{h^A},\qquad 当0<h\le h_0, \tag{3.4.11}$$

其中 $c_2=d^{-A/2}\sum_{k=1}^{\infty}\sigma_k^A(h_0)$.

第一步，我们来证对给定的 $\varepsilon>0$ 存在常数 $C=C(\varepsilon)>0$ 使得

$$P\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1+\varepsilon\Big\}\le Ch^{\varepsilon}y_h^{\varepsilon}. \tag{3.4.12}$$

易知，对 $0<s\le h$, 由 (3.4.1) 有

$$\begin{aligned}
&P\Big\{\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1+\varepsilon\Big\}\\
&\le\sum_{k=1}^{\infty}\exp\Big\{-(1+\varepsilon)^2\Big(\log\frac{1}{hy_h}\Big)\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\Big\}\\
&=\sum_{k=1}^{\infty}(hy_h)^{(1+\varepsilon)^2\sigma^{*2}(h)/\sigma_k^2(h)}\le h^{1+2\varepsilon}y_h^{2\varepsilon}.
\end{aligned} \tag{3.4.13}$$

对任一正数 t 和正整数 $r=r(\varepsilon)$, 令 $r_1=h/2^r$ 和 $t_r=[t/r_1]r_1$. 我们有

$$\begin{aligned}
|X_k(t+s)-X_k(t)|&\le|X_k((t+s)_r)-X_k(t_r)|\\
&\quad+\sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1})-X_k((t+s)_{r+j})|\\
&\quad+\sum_{j=0}^{\infty}|X_k(t_{r+j+1})-X_k(t_{r+j})|.
\end{aligned}$$

那么由 (3.4.13) 有

$$P\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h-r_1}\max_{k\ge 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1+\frac{\varepsilon}{2}\Big\}$$

$$\le \frac{4}{h}2^{2r}h^{1+\varepsilon}y_h^{\varepsilon}\le Ch^{\varepsilon}y_h^{\varepsilon}. \tag{3.4.14}$$

因 $\sigma^{*2}(h)/h^{\alpha}$ 是拟增的，存在 $c_0>0$ 使得

$$\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}\ge c_0 2^{\alpha(r-1)}\frac{\sigma^{*2}(2h/2^r)}{\sigma_k^2(2h/2^r)}\ge c_0 2^{\alpha(r-1)}.$$

若条件 (3.4.2) 被满足，则由 (3.4.11), 对充分大的 r 和充分小的 h, 类似于 (3.4.13), 我们得

$$p_1:=P\Big\{\sup_{0\le t\le 1}\sup_{h-r_1\le s\le h}\max_{k\ge 1}\frac{|X_k((t+s)_r)-X_k((t+h-r_1)_r)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge\frac{\varepsilon}{4}\Big\}$$

$$\le\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{\varepsilon^2\sigma^{*2}(h)/(16\sigma_k^2(2h/2^r))}$$

$$\le 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\exp\Big\{-\Big(\frac{\varepsilon^2}{16}\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}-2-A\Big)\log\frac{1}{hy_h}\Big\}$$

$$\le 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\exp\Big\{-\Big(\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}-1\Big)\frac{A}{2}\Big\}$$

$$\le 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\Big(\frac{\sigma_k(2h/2^r)}{\sigma^*(h)}\Big)^{A}$$

$$\le c_2 2^{r+1}h^{1+A}y_h\Big(\frac{\sigma^*(2h/2^r)}{\sigma^*(h)}\Big)^{A}\Big(\frac{2h}{2^r}\Big)^{-A}$$

$$\le Chy_h. \tag{3.4.15}$$

若条件 (3.4.4) 和 (3.4.5) 被满足，那么注意到 $\sigma^{*2}(h)/h^{\alpha}$ 是拟增的，取 r 充分大和 h 充分小，我们得

$$p_1\le\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)\cdot\varepsilon^2/16\cdot\sigma_k^2(h)/\sigma_k^2(2h/2^r)}$$

$$\le\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)\cdot\varepsilon^2/16\cdot c_1\sigma^{*2}(h)/\sigma^{*2}(2h/2^r)}$$

$$
\begin{aligned}
&\le \frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{2\sigma^{*2}(h)/\sigma_k^2(h)}\\
&\le Chy_h. \qquad (3.4.16)
\end{aligned}
$$

进一步，令 $x_j^2 = 2B\log\frac{1}{hy_h} + 2(1+A)j$, 其中 $B = 2/c_1$. 若条件 (3.4.2) 被满足，那么类似于 (3.4.15), 我们有

$$
\begin{aligned}
p_2 :&= P\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1}) - X_k((t+s)_{r+j})|\\
&\qquad \ge \sum_{j=0}^{\infty}x_j\sigma^*(h/2^{r+j+1})\Big\}\\
&\le \frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{r+j+1}\exp\Big\{-\frac{x_j^2}{2}\frac{\sigma^{*2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})}\Big\}\\
&\le 2\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{r+j+1}e^{-(1+A)j}h^{1+A}y_h\\
&\qquad \cdot\exp\Big\{-\Big(\frac{B\sigma^{*2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})} - 2 - A\Big)\log\frac{1}{hy_h}\Big\}\\
&\le 4\cdot 2^r h^{1+A}y_h\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^j e^{-(1+A)j}\Big(\frac{\sigma_k(h/2^{r+j+1})}{\sigma^*(h/2^{r+j+1})}\Big)^A\\
&\le 4c_2 2^{r(1+A)+A}hy_h\sum_{j=0}^{\infty}2^{(1+A)j}e^{-(1+A)j} \le Chy_h. \qquad (3.4.17)
\end{aligned}
$$

又若条件 (3.4.4) 和 (3.4.5) 被满足，那么类似于 (3.4.16) 和 (3.4.17), 我们有

$$
\begin{aligned}
p_2 &\le \frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{2(r+j+1)}e^{-(1+A)j}\exp\Big\{-B\Big(\log\frac{1}{hy_h}\Big)c_1\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\Big\}\\
&\le Chy_h. \qquad (3.4.18)
\end{aligned}
$$

类似地，在两种情形我们都有

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\sum_{k=1}^{\infty}|X_k(t_{r+j+1}) - X_k(t_{r+j})|\\
&\qquad \ge \sum_{j=0}^{\infty}x_j\sigma^*(h/2^{r+j+1})\Big\} \le Chy_h. \qquad (3.4.19)
\end{aligned}
$$

其次，由于 $\sigma^{*2}(h/2^{r+j+1})/\sigma^{*2}(h) \le c_0^{-1}2^{-\alpha(r+j+1)}$, 对充分大的 r 我们有

$$
\begin{aligned}
&\sum_{j=0}^{\infty} x_j \sigma^*\Big(\frac{h}{2^{r+j+1}}\Big) \\
&\quad = \sigma^*(h)\Big\{(2Bc_0^{-1})^{1/2}\Big(\log\frac{1}{hy_h}\Big)^{1/2}\sum_{j=0}^{\infty}2^{-\alpha(r+j+1)/2} \\
&\qquad + c_0^{-1/2}\sum_{j=0}^{\infty}\frac{(2(1+A)j)^{1/2}}{2^{\alpha(r+j+1)/2}}\Big\} \\
&\quad \le \frac{\varepsilon}{8}\sigma^*(h)\Big(\log\frac{1}{hy_h}\Big)^{1/2},
\end{aligned}
\tag{3.4.20}
$$

综合这些估计，我们得 (3.4.12).

为证 (3.4.3), 我们再次利用 (3.4.9), 由 (3.4.12) 得：当 h 充分小时，

$$
\begin{aligned}
&P\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \ge 1+\varepsilon\Big\} \\
&\quad \le C(hy_h)^{\varepsilon/2}\big(\log\sigma^{*^{-1}}(h)\big)^{-2}.
\end{aligned}
\tag{3.4.21}
$$

设 $\theta > 1$. 定义 $A_i = \{h;\theta^{-i-1} \le \sigma^*(h) < \theta^{-i}\}$, $A_{ij} = \{h;\theta^{-j-1} \le hy_h < \theta^{-j},\ h\in A_i\}$, 和 $h_{ij} = \sup\{h; h\in A_{ij}\}$. 则

$$
\begin{aligned}
&\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \\
&\quad \le \limsup_{i\to\infty}\sup_{j\ge 0}\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{-i-1}(2\log\theta^j)^{1/2}} \\
&\quad \le \limsup_{i\to\infty}\sup_{j\ge 0}\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{\theta^2|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}}.
\end{aligned}
$$

利用 (3.4.21), 我们有

$$
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big\{\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}} \ge 1+\varepsilon\Big\}
$$

$$\le C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(h_{ij}y_{h_{ij}})^{\varepsilon/2}\big(\log\sigma^{*^{-1}}(h_{ij})\big)^{-2}$$

$$\le C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{-j\varepsilon/2}(i\log\theta)^{-2}<\infty.$$

因此, 若能证明在条件 (3.4.4) 和 (3.4.5) 下有 $y_h=1$, 那么由 Borel-Cantelli 引理 (3.4.3) 得证. 为了证明我们可以取 $y_h=1$ 这一事实, 只需证明 $\log\frac{1}{y_h}=o(\log\frac{1}{h})$. 考察方程

$$\sum_{k=1}^{\infty}x^{c_1(\log 2)\sigma^{*^2}(1/2)/\sigma_k^2(1/2)}=1.$$

由条件 (3.4.5) 知, 它的解 $x=x_0>0$ 存在. 于是对任何 $0<h\le 1/2$ 有

$$1=\frac{1}{h}\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)}\le\sum_{k=1}^{\infty}y^{c_1\sigma^{*^2}(1/2)/\sigma_k^2(1/2)}.$$

从而, 对 $0<h<1/2$ 有 $y_h\ge x_0^{\log 2}$, 待证的结论成立.

其次, 在独立性的假设和条件 (3.4.6) 下, 我们来证 (3.4.8). 已有 (3.4.3), 因此只需证明

$$\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1\quad\text{a.s.}\qquad(3.4.22)$$

为此, 只需证明对任何 $h_n\downarrow 0,\ 0<\varepsilon<1$,

$$\lim_{n\to\infty}P\left\{\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h_n)-X_k(t)|}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}\ge 1-\varepsilon\right\}=1.\qquad(3.4.23)$$

事实上, 对充分大的 n, 即 h_n 充分小时, 我们有

$$P\left\{\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h_n)-X_k(t)|}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}<1-\varepsilon\right\}$$

$$\le P\left\{\max_{0\le j\le 1/h_n}\max_{k\ge 1}\frac{X_k((j+1)h_n)-X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}<1-\varepsilon\right\}$$

$$
\begin{aligned}
&\le \prod_{j=0}^{[1/h_n]} \prod_{k=1}^{\infty} P\left\{ \frac{X_k((j+1)h_n) - X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_n y_{h_n})))^{1/2}} < 1-\varepsilon \right\} \\
&\le \prod_{j=0}^{[1/h_n]} \prod_{k=1}^{\infty} \left\{ 1 - \exp\left\{ -\frac{(1-\varepsilon)\sigma^{*^2}(h_n)}{\sigma_k^2(h_n)} \log \frac{1}{h_n y_{h_n}} \right\} \right\} \\
&= \prod_{j=0}^{[1/h_n]} \prod_{k=1}^{\infty} \left\{ 1 - (h_n y_{h_n})^{(1-\varepsilon)\sigma^{*^2}(h_n)/\sigma_k^2(h_n)} \right\} \\
&\le \prod_{j=0}^{[1/h_n]} \exp\left\{ -\sum_{k=1}^{\infty} (h_n y_{h_n})^{(1-\varepsilon)\sigma^{*^2}(h_n)/\sigma_k^2(h_n)} \right\} \\
&\le \exp(-h_n^{-\varepsilon}) \to 0, \quad n \to \infty,
\end{aligned}
$$

在第二个不等式中我们利用了独立性和 Slepian 不等式. 因此 (3.4.8) 得证.

最后, 我们来证明 (3.4.7). 借助 (3.4.3), 只需证明

$$
\liminf_{h \downarrow 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \max_{k \ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \ge 1 \quad \text{a.s.} \tag{3.4.24}
$$

定义 A_{ij} 如上, $h'_{ij} = \inf\{h; h \in A_{ij}\}$. 则

$$
\begin{aligned}
&\liminf_{h \downarrow 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \max_{k \ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \\
&\ge \liminf_{i \to \infty} \inf_{j \ge 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h'_{ij}} \max_{k \ge 1} \frac{|X_k(t+s) - X_k(t)|}{\theta^{-i}(2\log\theta^{j+1})^{1/2}} \\
&\ge \liminf_{i \to \infty} \inf_{j \ge 0} \max_{0 \le l \le 1/h'_{ij}} \max_{k \ge 1} \frac{|X_k((l+1)h'_{ij}) - X_k(lh'_{ij})|}{\theta^2 \sigma^*(h'_{ij})(2\log(1/(h'_{ij} y_{h'_{ij}})))^{1/2}}.
\end{aligned} \tag{3.4.25}
$$

再次利用 Slepian 不等式, 我们有

$$
\begin{aligned}
&P\left\{ \max_{0 \le l \le 1/h'_{ij}} \max_{k \ge 1} \frac{|X_k((l+1)h'_{ij}) - X_k(lh'_{ij})|}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij} y_{h'_{ij}})))^{1/2}} \le 1-\varepsilon \right\} \\
&\quad \le \prod_{l=0}^{[1/h'_{ij}]} \prod_{k=1}^{\infty} \left\{ 1 - \exp\left\{ -\frac{(1-\varepsilon)\sigma^{*^2}(h'_{ij})}{\sigma_k^2(h'_{ij})} \log \frac{1}{h'_{ij} y_{h'_{ij}}} \right\} \right\}
\end{aligned}
$$

$$\leq \exp\left\{-(h'_{ij}y_{h'_{ij}})^{-\epsilon}\right\} \leq \exp\left\{-(h'_{ij}y_{h'_{ij}})^{-\epsilon/2}\log {\sigma^*}^{-1}(h'_{ij})\right\}$$
$$\leq \exp\left\{-\theta^{j\epsilon/2}(i\log\theta)\right\}.$$

倒数第二个不等式是由于 (3.4.9). 所以

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P\left\{\max_{0\leq l\leq 1/h'_{ij}}\max_{k\geq 1}\frac{|X_k((l+1)h'_{ij})-X_k(lh'_{ij})|}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}\leq 1-\epsilon\right\}<\infty. \tag{3.4.26}$$

由 (3.4.25) 和 (3.4.26) 即得 (3.4.24). 定理 3.4.1 证毕.

推论 3.4.1 假设存在常数 $0 < c_1 \leq c_2 < \infty$, 正数列 $\{a_k; k \geq 1\}$ 和非降函数 $\sigma(h)$ 使得对任给的 $h > 0$ 和每一 $k \geq 1$ 有

$$c_1 a_k\sigma(h) \leq \sigma_k(h) \leq c_2 a_k\sigma(h). \tag{3.4.27}$$

此外, 假设对某 $\alpha > 0$, $\sigma^2(h)/h^\alpha$ 是拟增的, 且对某 $A > 0$

$$\sum_{k=1}^{\infty}\exp\{-A{a^*}^2/a_k^2\} < \infty, \tag{3.4.28}$$

其中 $a^* = \max_{k\geq 1} a_k$. 那么 (3.4.3) 对 $y_h = 1$ 成立. 若还假设 $\{X_k(\cdot)\}_{k=1}^{\infty}$ 是独立的且 (3.4.6) 被满足, 那么 (3.4.7) 和 (3.4.8) 对 $y_h = 1$ 成立.

证明 显然 (3.4.27) 蕴含着 (3.4.4). 这样由定理 3.4.1 即得所述的结论.

注 3.4.1 当 (3.4.2) 被满足时, 我们来给出 y_h 的上、下界估计. 注意到对任一 $0 < h \leq e^{-A/2}$,

$$1 \leq y_h\sum_{k=1}^{\infty}h^{{\sigma^*}^2(h)/\sigma_k^2(h)-1} = y_h\sum_{k=1}^{\infty}\exp\left\{-\left(\log\frac{1}{h}\right)\left(\frac{{\sigma^*}^2(h)}{\sigma_k^2(h)}-1\right)\right\}$$
$$\leq y_h\sum_{k=1}^{\infty}\exp\left\{-\frac{A}{2}\left(\frac{{\sigma^*}^2(h)}{\sigma_k^2(h)}-1\right)\right\} \leq y_h\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A,$$

即我们有

$$y_h \geq \left(\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A\right)^{-1}. \tag{3.4.29}$$

其次，注意到对任何 $h > 0$ 和 $\theta > 1$

$$h = \sum_{k=1}^{\infty}(hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)} \geq \sum_{k\in A_\theta(h)}(hy_h)^\theta = Card\{A_\theta(h)\}(hy_h)^\theta,$$

其中 $A_\theta(h) = \{k : \sigma_k^2(h)/\sigma^{*^2}(h) \geq 1/\theta\}$. 因此

$$y_h \leq (\mathrm{Card}\{A_\theta(h)\}/h)^{1/\theta}/h. \tag{3.4.30}$$

结合定理 3.4.1 与 (3.4.29) 可得如下结果.

推论 3.4.2 假设对某 $\alpha > 0$, $\sigma^{*^2}(h)/h^\alpha$ 是拟增的且 (3.4.1) 被满足. 又假设

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A = o\left(\frac{1}{h}\right), \quad h \to 0. \tag{3.4.31}$$

那么 (3.4.3) 对 $y_h = 1$ 成立. 若还假设 $\{X_k(\cdot)\}_{k=1}^{\infty}$ 是独立的且 (3.4.6) 被满足，那么 (3.4.7) 和 (3.4.8) 对 $y_h = 1$ 成立.

作为这个推论的应用，我们有

推论 3.4.3 设 $\{X_k(t); t \geq 0\}_{k=1}^{\infty}$ 是独立 O-U 过程序列，具有系数 γ_k 和 λ_k, $k = 1, 2, \cdots$. 假设对某 $\alpha > 0$, $\sigma^{*^2}(h)/h^\alpha$ 是拟增的且对某 $A \geq 2$

$$\sum_{k=1}^{\infty}\gamma_k^A < \infty. \tag{3.4.32}$$

则 (3.4.7) 和 (3.4.8) 对 $y_h = 1$ 成立.

证明 对 $\{X_k(\cdot)\}$, (3.4.6) 被满足. 由 (3.4.32) 即得

$$\sum_{k=1}^{\infty}\sigma_k^{2A}(h) \leq (2h)^A\sum_{k=1}^{\infty}\gamma_k^A.$$

另一方面，通过回顾 (3.4.9) 的证明并注意到 $E(X_k(t+2h)-X_k(t+h))(X_k(t+h)-X_k(t))\le 0$，易知

$$\liminf_{h\downarrow 0}\sigma^{*^2}(h)/h>0.$$

故我们有

$$\limsup_{h\downarrow 0}\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^{2A}<\infty.$$

这就是说，(3.4.31) 被满足. 由推论 3.4.2 即得推论 3.4.3 的结论.

3.4.2 大增量

首先，我们仍考察存在 $\alpha>0$，$\sigma^{*2}(h)/h^{\alpha}$ 是拟增函数的情形.

设 $0<a_T\le T$，a_T 是 T 的连续函数，满足 $a_T\to\infty$ $(T\to\infty)$. 设 y_T 是下述方程的解：

$$\sum_{k=1}^{\infty}\left(\frac{a_Ty_T}{T\log\sigma^*(a_T)}\right)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)}=\frac{a_T}{T\log\sigma^*(a_T)}. \tag{3.4.33}$$

下列定理是定理 3.4.1 在大增量情形的一个类比 (Lin 1998).

定理 3.4.2 *假设对某 $\alpha>0$，$\sigma^{*^2}(h)/h^{\alpha}$ 是拟增的，且存在正数 h_0，A 和 B 使得对任何 $h\ge h_0$ 满足*

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A<B. \tag{3.4.34}$$

则

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}}\le 1\quad\text{a.s.} \tag{3.4.35}$$

若条件 (3.4.34) 被如下条件代替：存在正数 h_1,c_1,T_0 和 C 使得对任何 $h\ge h_1$ 和每一 $k\ge 1$ 有

$$\inf_{0\le s\le h}\frac{\sigma^*(s)}{\sigma_k(s)}\ge c_1\frac{\sigma^*(h)}{\sigma_k(h)}, \tag{3.4.36}$$

且对任何 $T \geq T_0$

$$\sum_{k=1}^{\infty}\Big(\frac{a_T}{T\log\sigma^*(a_T)}\Big)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)} < C, \tag{3.4.37}$$

则 (3.4.35) 仍正确. 若还假设 $X_k(\cdot), k = 1, 2, \cdots$, 是独立的, 对 $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$$E(X_k(t_2) - X_k(t_1))(X_k(t_4) - X_k(t_3)) \leq 0 \tag{3.4.38}$$

且

$$\lim_{T\to\infty}\frac{\log(T/a_T)}{\log\log\sigma^*(a_T)} = \infty, \tag{3.4.39}$$

则有

$$\lim_{T\to\infty}\sup_{0\leq t\leq T-a_T}\sup_{0\leq s\leq a_T}\max_{k\geq 1}\frac{|X_k(t+s) - X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.4.40}$$

和

$$\limsup_{T\to\infty}\sup_{0\leq t\leq T-a_T}\max_{k\geq 1}\frac{|X_k(t+a_T) - X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.4.41}$$

注 3.4.2 类似于注 3.4.1, 当 T 满足 $\sigma^*(a_T) \geq e^{A/2}$ 时, 我们有

$$y_T \geq \Big(\sum_{k=1}^{\infty}\Big(\frac{\sigma_k(a_T)}{\sigma^*(a_T)}\Big)^A\Big)^{-1}.$$

因此, 由 (3.4.34), 对充分大的 T 有

$$y_T \geq B^{-1} > 0. \tag{3.4.42}$$

此外, 我们可证下列事实: $y_T \leq 1$ 且方程 (3.4.33) 的解在条件 (3.4.37) 下存在且唯一. 而 (3.4.37) 可从 (3.4.34) 推得.

定理 3.4.2 的证明 存在 $d > 0$ 使得对任何 $0 < h \leq 1$ 有 $\sigma^{*2}(h) \geq dh^2$ (参见 (3.4.9)). 若 $h_0 > 1$, 则对任何 $1 < h \leq h_0$,

$\sigma^{*2}(h) \ge \sigma^{*2}(1) \ge h^2\sigma^{*2}(1)/h_0^2$. 因此，若令 $d' = d \wedge (\sigma^{*2}(1)/h_0^2)$，则对任何 $h \le h_0$ 就有 $\sigma^{*2}(h) \ge d'h^2$，由此即得：对 $h \le h_0$ 有

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A \le d'^{-A/2}h^{-A}\sum_{k=1}^{\infty}\sigma_k^A(h_0) =: d_1 h^{-A}, \quad (3.4.43)$$

其中 $d_1 = d'^{-A/2}\sum_{k=1}^{\infty}\sigma_k^A(h_0)$.

设 $\theta > 1$，定义 $A_i = \{T; \theta^{i-1} \le \sigma^*(a_T) < \theta^i\}$，$A_{ij} = \{T; \theta^{j-1} \le T/a_T < \theta^j, T \in A_i\}$，$a_{ij} = \sup\{a_T; T \in A_{ij}\}$，$T_{ij} = \sup\{T; a_T = a_{ij}, T \in A_{ij}\}$，$T'_{ij} = \sup\{T; T - a_T = \sup_{T\in A_{ij}}(T - a_T), T \in A_{ij}\}$ 和 $J = \max\{j; \theta^j \le \max_{T>0} T/a_T\}$. 则 $J \le \infty$ 且

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}}\\
&\le \limsup_{i\to\infty}\max_{1\le j<J}\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}}\max_{k\ge 1}|X_k(t+s)-X_k(t)|/\\
&\qquad [\theta^{i-1}(2\log(\theta^{j-1}\log\theta^{i-1}))^{1/2}]\\
&\le \limsup_{i\to\infty}\max_{1\le j<J}\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\sup_{0\le s\le a_{ij}}\max_{k\ge 1}\theta^2|X_k(t+s)-X_k(t)|/\\
&\qquad [\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij})^{1/2}]. \qquad (3.4.44)
\end{aligned}$$

对任何 $\varepsilon > 0$，设 $r = r(\varepsilon) > 0$ 下面待定. 令 $r_{ij} = a_{ij}/2^r$. 对任何 $t > 0$，令 $t_r := t_{r_{ij}} = [t/r_{ij}]r_{ij}$. 写

$$\begin{aligned}
|X_k(t+s) - X_k(t)| &\le |X_k((t+s)_r) - X_k(t_r)|\\
&\quad + \sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1}) - X_k((t+s)_{r+l})|\\
&\quad + \sum_{l=0}^{\infty}|X_k(t_{r+l+1}) - X_k(t_{r+l})|. \qquad (3.4.45)
\end{aligned}$$

类似于 (3.4.14) 并注意到 (3.4.42)，对充分大的 T，若 $T'_{ij} \le T_{ij}$，

我们有

$$p_0 := P\left\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{0\le s\le a_{ij}-r_{ij}}\ \max_{k\ge1}|X_k((t+s)_r)-X_k(t_r)|/\right.$$

$$\left.\left[\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}\right]\ge 1+\varepsilon\right\}$$

$$\le P\left\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{0\le s\le a_{ij}-r_{ij}}\ \max_{k\ge1}|X_k((t+s)_r)-X_k(t_r)|/\right.$$

$$\left.\left[\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/(a_{ij}y_{T_{ij}})))^{1/2}\right]\ge 1+\varepsilon/2\right\}$$

$$\le \frac{4T'_{ij}}{a_{ij}}2^{2r}\sum_{k=1}^{\infty}\exp\left\{-\left(1+\frac{\varepsilon}{2}\right)^2\left(\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\right)\frac{\sigma^{*2}(a_{ij})}{\sigma_k^2(a_{ij})}\right\}$$

$$\le \frac{4T'_{ij}}{a_{ij}}2^{2r}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^{1+\varepsilon}\le c\left(\frac{T_{ij}}{a_{ij}}\right)^{-\varepsilon}(\log\sigma^*(a_{ij}))^{-1-\varepsilon}$$

$$\le c\theta^{-\varepsilon j}(i\log\theta)^{-1-\varepsilon}. \tag{3.4.46}$$

否则，我们有

$$T_{ij}\le T'_{ij}\le\theta^j a_{T'_{ij}}\le\theta T_{ij},$$

因此 (3.4.46) 也成立. 结合 (3.4.15) 和 (3.4.46) 的证明思路，并用条件 (3.4.34) 代替 (3.4.11), 我们有：当 r 和 T 充分大时,

$$p_1 := P\left\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{a_{ij}-r_{ij}\le s\le a_{ij}}\ \max_{k\ge1}|X_k((t+s)_r)-X_k(t_r)|/\right.$$

$$\left.\left[\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}\right]\ge\varepsilon/2\right\}$$

$$\le c\frac{T'_{ij}}{a_{ij}}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^{2+A}\left(\frac{\sigma^*(2a_{ij}/2^r)}{\sigma^*(a_{ij})}\right)^{A}\left(\frac{2a_{ij}}{2^r}\right)^{-A}$$

$$\le\frac{cT'_{ij}a_{ij}}{(T_{ij}\log\sigma^*(a_{ij}))^{2+A}}\le c\theta^{-j}i^{-2-A}. \tag{3.4.47}$$

在条件 (3.4.36) 和 (3.4.37) 下，我们也有相同的界.

考察 (3.4.45) 右边的第一个级数. 设 $b_{r_{ij}}=\sup\{b;$ $a_{ij}/2^{r+[\log_2 ba_{ij}]+1}\ge h_0\}$ 和 $l_0=[\log_2(b_{r_{ij}}a_{ij})]$. 令 $D=3/c_1$,

$x_l^2 := x_{l_{ij}}^2 = 2D\log(T_{ij}\log\sigma^*(a_{ij})/a_{ij}) + 2(1+A)l$. 若条件 (3.4.34) 被满足, 则由 (3.4.34) 和 (3.4.43) 对充分大的 r 和 T 有

$$
\begin{aligned}
p_2 := P\Big\{ & \sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{0\le s\le a_{ij}}\ \max_{k\ge 1}\sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1}) \\
& - X_k((t+s)_{r+l})| \ge \sum_{l=0}^{\infty} x_l\sigma^*(a_{ij}/2^{r+l+1})\Big\} \\
\le\ & \frac{2T'_{ij}}{a_{ij}}\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}2^{r+l+1}e^{-(1+A)l}\Big(\frac{a_{ij}y_{T_{ij}}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2 \\
& \cdot\exp\Big\{-\Big(\frac{D\sigma^{*2}(a_{ij}/2^{r+l+1})}{\sigma_k^2(a_{ij}/2^{r+l+1})}-2\Big)\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\Big\} \\
\le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2\Big(\sum_{l=0}^{l_0}+\sum_{l=l_0+1}^{\infty}\Big)\sum_{k=1}^{\infty}2^le^{-(1+A)l} \\
& \cdot\Big(\frac{\sigma_k(a_{ij}/2^{r+l+1})}{\sigma^*(a_{ij}/2^{r+l+1})}\Big)^A \\
\le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2\Big(\sum_{l=0}^{l_0}B2^le^{-(1+A)l} \\
& +\sum_{l=l_0+1}^{\infty}d_1 2^le^{-(1+A)l}\cdot(a_{ij}/2^{r+l+1})^{-A}\Big) \\
\le\ & c\cdot\frac{T'_{ij}}{a_{ij}}\Big(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\Big)^2 \le c\theta^{-j}(i\log\theta)^{-2}.
\end{aligned}
\tag{3.4.48}
$$

又若条件 (3.4.36) 和 (3.4.37) 被满足, 我们也有同样的界.

对 (3.4.45) 右边的第二个级数, 我们也有同样的结论.

此外, 类似于 (3.4.20), 当 r 充分大时, 我们有

$$
\sum_{l=0}^{\infty}x_l\sigma^*(a_{ij}/2^{r+l+1}) \le \frac{\varepsilon}{4}\sigma^*(a_{ij})\Big(\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\Big)^{1/2}. \tag{3.4.49}
$$

综合这些估计, 我们得

$$
\sum_{i=1}^{\infty}\sum_{j=1}^{J}P\Big\{\sup_{0\le t\le T'_{ij}-a_{T'_{ij}}}\ \sup_{0\le s\le a_{ij}}\ \max_{k\ge1}|X_k(t+s)-X_k(t)|/
$$

$$\left[\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{1/2}\right] \geq 1+2\varepsilon\Big\}<\infty.$$

所以由 Borel-Cantelli 引理，(3.4.44) 右边概率为 1 地不超过 1. 这就证明了 (3.4.35).

现在来证 (3.4.41). 注意到 (3.4.35), 只需证明

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\max_{k\ge1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}\geq 1 \quad \text{a.s.} \tag{3.4.50}$$

利用 (3.4.39), Slepian 不等式和 $\{X_k(\cdot)\}_{k=1}^{\infty}$ 的独立性，对充分大的 T_n ($\uparrow\infty$), 我们有

$$\begin{aligned}
&P\left\{\sup_{0\le t\le T_n-a_{T_n}}\max_{k\ge1}\frac{|X_k(t+a_{T_n})-X_k(t)|}{\sigma^*(a_{T_n})(2\log(T_n/a_{T_n}))^{1/2}}<1-\varepsilon\right\}\\
&\quad\le P\left\{\max_{0\le j\le T_n/a_{T_n}}\max_{k\ge1}\left[X_k((j+1)a_{T_n})-X_k(ja_{T_n})\right]/\right.\\
&\qquad\qquad\left.\left[\sigma^*(a_{T_n})(2\log((T_n\log\sigma^*(a_{T_n}))/(a_{T_n}y_{T_n})))^{1/2}\right]<1-\varepsilon\right\}\\
&\quad\le\prod_{j=0}^{[T_n/a_{T_n}]}\prod_{k=1}^{\infty}\left\{1-\exp\left\{-\frac{(1-\varepsilon)\sigma^{*2}(a_{T_n})}{\sigma_k^2(a_{T_n})}\log\frac{T_n\log\sigma^*(a_{T_n})}{a_{T_n}y_{T_n}}\right\}\right\}\\
&\quad\le\prod_{j=0}^{[T_n/a_{T_n}]}\exp\left\{-\sum_{k=1}^{\infty}\left(\frac{a_{T_n}y_{T_n}}{T_n\log\sigma^*(a_{T_n})}\right)^{(1-\varepsilon)\sigma^{*2}(a_{T_n})/\sigma_k^2(a_{T_n})}\right\}\\
&\quad\le\exp\left\{-\frac{T_n}{a_{T_n}}\left(\frac{a_{T_n}}{T_n\log\sigma^*(a_{T_n})}\right)^{1-\varepsilon}\right\}\\
&\quad\le\exp\left\{-\left(\frac{T_n}{a_{T_n}}\right)^{\varepsilon/2}\right\}\to 0.
\end{aligned}\tag{3.4.51}$$

因此 (3.4.41) 得证.

最后我们来证明 (3.4.40). 由 (3.4.35), 只需证明 “lim inf” 概率为 1 地不小于 1. 记 $a'_{ij}=\inf\{a_T;T\in A_{ij}\}$, $t'_{ij}=\inf\{T;a_T=a'_{ij},T\in A_{ij}\}$, $t_{ij}=\inf\{T;T-a_T=\inf_{T\in A_{ij}}(T-a_T),T\in A_{ij}\}$. 我们有

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}$$

$$\geq \liminf_{i\to\infty} \min_{1\leq j\leq J} \sup_{0\leq t\leq t_{ij}-a_{t_{ij}}} \sup_{0\leq s\leq a'_{ij}} \max_{k\geq 1} \frac{|X_k(t+s)-X_k(t)|}{\theta^i(2\log\theta^j)^{1/2}}$$

$$\geq \liminf_{i\to\infty} \min_{1\leq j\leq J} \max_{1\leq l\leq t_{ij}/a'_{ij}} \max_{k\geq 1} |X_k((l+1)a'_{ij})-X_k(la'_{ij})|/$$

$$\left[\theta^2\sigma^*(a'_{ij})(2\log((t'_{ij}\log\sigma^*(a'_{ij}))/(a'_{ij}y_{t'_{ij}})))^{1/2}\right].$$

从而，类似于 (3.4.51) 得

$$P\left\{\max_{0\leq l\leq t_{ij}/a'_{ij}} \max_{k\geq 1} \frac{|X_k((l+1)a'_{ij})-X_k(la'_{ij})|}{\sigma^*(a'_{ij})(2\log((t'_{ij}\log\sigma^*(a'_{ij}))/(a'_{ij}y_{t'_{ij}})))^{1/2}} < 1-\varepsilon\right\}$$

$$\leq \exp\left\{-\frac{t_{ij}}{a'_{ij}}\left(\frac{a'_{ij}}{t'_{ij}\log\sigma^*(a'_{ij})}\right)^{1-\varepsilon}\right\}$$

$$\leq \exp\{-(t'_{ij}/a'_{ij})^{\varepsilon/2}\log\sigma^*(a'_{ij})\} \leq \exp\{-\theta^{(j-1)\varepsilon/2}(i\log\theta)\}.$$

这里第二个不等式也是由于 (3.4.39). 因此由 Borel-Cantelli 引理, (3.4.40) 得证. 定理 3.4.2 证毕.

现在我们考察当 $h\to\infty$ 时 $\sigma^{*2}(h)\to\sigma^{*2}<\infty$ 的情形. 设 $\sigma_k^2=\lim_{h\to\infty}\sigma_k^2(h)$ 且 z_T 是下述方程的解:

$$\sum_{k=1}^{\infty}\left(\frac{a_Tz_T}{T}\right)^{\sigma^{*2}/\sigma_k^2}=\frac{a_T}{T}. \tag{3.4.52}$$

继续使用在上节中引入的 $d(m,k)$, L_mx 等记号.

定理 3.4.3 假设当 $h\to\infty$ 时, $\sigma^*(h)\to\sigma^*$ 且存在 $A>0$ 使得

$$\sum_{k=1}^{\infty}\sigma_k^A<\infty. \tag{3.4.53}$$

又假设存在 $2\leq\alpha<e$, $0<\delta<1-1/\alpha$ 和正整数 $m\geq 1$ 使得

$$\int_1^{\infty}\sigma^*(\alpha^{-x^2})dx<\infty, \tag{3.4.54}$$

$$a_T\leq Td(m,(L_mT)^{\delta+1/\alpha})^{-1}. \tag{3.4.55}$$

那么

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}\le 1 \quad \text{a.s.} \tag{3.4.56}$$

若还设 $X_k(\cdot), k=1,2,\cdots$, 独立且 (3.4.38) 被满足, 那么

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1 \quad \text{a.s.}, \tag{3.4.57}$$

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\max_{k\ge 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1 \quad \text{a.s.} \tag{3.4.58}$$

证明 与定理 3.4.2 类似, 条件 (3.4.53) 保证了方程 (3.4.52) 的解存在且唯一. 此外, 存在 $0<b<1$ 使得 $z_T\ge b$, 由条件 (3.4.55) 知 (3.3.62) 仍正确.

对 $\theta>1$, 定义 $A_j=\{T;\theta^{j-1}\le T/a_T<\theta^j\}$, $a_j=\sup\{a_T;T\in A_j\}$, $T_j=\sup\{T;a_T=a_j,T\in A_j\}$ 和 $T_j'=\sup\{T;T-a_T=\sup_{T\in A_j}(T-a_T),T\in A_j\}$. 给定 $r>0$, 对任何 $t>0$ 令 $t_r^j:=t_r(a_j)=[td(m+2,r)/a_j](a_j/d(m+2,r))$. 写

$$\begin{aligned}|X_k(t+s)-X_k(t)|\le{}&|X_k((t+s)_r^j)-X_k(t_r^j)|\\&+\sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1}^j)-X_k((t+s)_{r+l}^j)|\\&+\sum_{l=0}^{\infty}|X_k(t_{r+l+1}^j)-X_k(t_{r+l}^j)|.\end{aligned} \tag{3.4.59}$$

对 (3.4.55) 中的 $\delta>0$, 令 δ_1 满足 $1-1/\alpha-\delta<\delta_1<1-1/\alpha$. 又记 $\varepsilon(a_T)=d(m-1,(L_m a_T)^{1-\delta_1})/L_1 a_T$, $r:=r(a_j)=L_{m+2}a_j^{\varepsilon(a_j)}$, $r':=r'(a_j)=L_{m+2}a_j$. 那么 $d(m+2,r)=a_j^{\varepsilon(a_j)}$ 且 $d(m+2,r')=a_j$, 此外,

$$\begin{aligned}0<r'-r&=L_{m+2}a_j-L_m\Big[(L_2a_j)\Big(1+\frac{L_1\varepsilon(a_j)}{L_2a_j}\Big)\Big]\\&=-L_1(1-\delta_1)<1.\end{aligned} \tag{3.4.60}$$

类似于 (3.4.46), 若 $T_j'/T_j \le 1$, 则我们有

$$
\begin{aligned}
p_0' &:= P\left\{\sup_{0\le t\le T_j'-a_{T_j'}}\ \sup_{0\le s\le a_j}\ \max_{k\ge 1}\frac{|X_k((t+s)_r^j)-X_k(t_r^j)|}{\sigma^*(2\log(T_j/a_j))^{1/2}}\ge 1+\varepsilon\right\}\\
&\le \frac{4T_j'}{a_j}d(m+2,r)^2\left(\frac{a_j}{T_j}\right)^{1+\varepsilon}\le 4a_j^{2\epsilon(a_j)+\varepsilon}\left(\frac{T_j'}{T_j}\right)T_j^{-\varepsilon}\\
&\le 5a_j^{2\epsilon(a_j)+\varepsilon}T_j^{-\varepsilon}.
\end{aligned}\tag{3.4.61}
$$

在 $T_j'/T_j > 1$ 的情形, 我们有

$$
T_j < T_j' \le \theta^j a_{T_j'} \le \theta^j a_j \le \theta T_j.
$$

因此在任何情形, 只要 $\theta < 5/4$, (3.4.61) 总成立. 由条件 (3.4.55) 得

$$
a_j^{2\epsilon(a_j)+\varepsilon/2} \le T_j^{2\epsilon(a_j)+\varepsilon/2} d(m,(L_mT_j)^{\delta+1/\alpha})^{-\varepsilon/2} \le T_j^{\varepsilon/2}.
$$

把上式代入 (3.4.61) 得

$$
p_0' \le 5(T_j/a_j)^{-\varepsilon/2} \le 5\theta^{-(j-1)\varepsilon/2}.
$$

令 $x_l'^2 := x_{lj}'^2 = 4\log(T_j/a_j) + 2(1+A)d(m+1,r+l+1)$. 则有

$$
\begin{aligned}
p_2' &:= P\left\{\sup_{0\le t\le T_j'-a_{T_j'}}\ \sup_{0\le s\le a_j}\ \max_{k\ge 1}\left|\sum_{l=0}^{\infty}\Big(X_k((t+s)_{r+l+1}^j)\right.\right.\\
&\qquad\left. - X_k((t+s)_{r+l}^j)\Big)\right|\\
&\qquad\left.\ge \sum_{l=0}^{\infty}x_l'\sigma^*(a_j/d(m+2,r+l+1))\right\}\\
&\le \frac{4T_j'}{a_j}\left(\frac{a_j}{T_j}\right)^2\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}d(m+2,r+l+1)e^{-(1+A)d(m+1,r+l+1)}\\
&\qquad\cdot\exp\left\{-\Big(\frac{2\sigma^{*2}(a_j/d(m+2,r+l+1))}{\sigma_k^2(a_j/d(m+2,r+l+1))}-2\Big)\log\frac{T_j}{a_j}\right\}\\
&\le \frac{4a_j}{T_j}\frac{T_j'}{T_j}\sum_{l=0}^{\infty}d(m+2,r+l+1)e^{-(1+A)d(m+1,r+l+1)}\\
&\qquad\cdot\sum_{k=1}^{\infty}\left(\frac{\sigma_k(a_j/d(m+2,r+l+1))}{\sigma^*(a_j/d(m+2,r+l+1))}\right)^A.
\end{aligned}\tag{3.4.62}
$$

注意到 $0 < r' - r < 1$ (见 (3.4.60)), 对 $l \geq 0$ 我们有

$$a_j/d(m+2,r+l+1) \leq d(m+2,r')/d(m+2,r+1) \leq 1.$$

从而利用事实：对 $0 < h \leq 1$ 有 $\sigma^{*2}(h) \geq dh^2$ ，我们得

$$\sigma^*(a_j/d(m+2,r+l+1))^{-A} \leq d^{-A/2}(a_j/d(m+2,r+l+1))^{-A},$$

且进一步

$$p_2' \leq \frac{5d^{-A/2}a_j^{1-A}}{T_j}\sum_{l=0}^{\infty}(\alpha/e)^{(1+A)d(m+1,r+l+1)} \leq c\theta^{-j}.$$

对 (3.4.59) 右边的第二个级数，我们有同样的估计.

令 $\beta > 0$ 满足 $r' - r + \beta < 1$. 则当 T 充分大时，由条件 (3.4.54) 有

$$\begin{aligned}
&\sum_{l=0}^{\infty} d(m+1,r+l+1)^{1/2}\sigma^*(a_j/d(m+2,r+l+1))\\
&\leq \sum_{l=0}^{\infty}\left(1-\left(\frac{d(m+1,r+l+1-\beta)}{d(m+1,r+l+1)}\right)^{1/2}\right)^{-1}\\
&\qquad\cdot\int_{d(m+1,r+l+1-\beta)^{1/2}}^{d(m+1,r+l+1)^{1/2}}\sigma^*(a_j\alpha^{-y^2})dy\\
&\leq \left(1-\left(\frac{d(m+1,r+1-\beta)}{d(m+1,r+1)}\right)^{1/2}\right)^{-1}\\
&\qquad\cdot\int_{d(m+1,r+1-\beta)^{1/2}}^{\infty}\sigma^*(\alpha^{d(m+1,r')-y^2})dy\\
&\leq \left(1-\left(\frac{d(m+1,r+1-\beta)}{d(m+1,r+1)}\right)^{1/2}\right)^{-1}\\
&\qquad\cdot\int_{d(m+1,r+1-\beta)^{1/2}-d(m+1,r')^{1/2}}^{\infty}\sigma^*(\alpha^{-y^2})dy\\
&\leq \varepsilon.
\end{aligned}\tag{3.4.63}$$

显然，由 (3.4.63) 可知对充分大的 T 有

$$\sum_{l=0}^{\infty}\sigma^*(a_j/d(m+2,r+l+1)) \leq \frac{\varepsilon}{8}\sigma^*.$$

于是

$$\sum_{l=0}^{\infty} x_l' \,\sigma^*(a_j/d(m+2, r+l+1)) \le \frac{\varepsilon}{2}\sigma^*\Big(2\log\frac{T_j}{a_j}\Big)^{1/2}.$$

综合这些结果我们得

$$\sum_{j=1}^{\infty} P\bigg\{\sup_{j\ge 1}\sup_{0\le t\le T_j'-a_{T_j'}}\sup_{0\le s\le a_j}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T_j/a_j))^{1/2}} \\ \le 1+2\varepsilon\bigg\} < \infty,$$

由此即得 (3.4.56).

若我们用 $(1+\varepsilon)\sigma^*(a_T)$ 代替 σ^* (注意到对充分大的 T, $\sigma^* \le (1+\varepsilon)\sigma^*(a_T)$), (3.4.57) 和 (3.4.58) 的证明分别类似于 (3.4.40) 和 (3.4.41), 故从略. 定理 3.4.3 证毕.

作为定理 3.4.3 的一个应用, 我们对 l^∞ 值 O-U 过程建立大增量结果. 设 $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ 是独立 O-U 过程序列, 具有系数 $\gamma_k \ge 0$ 和 $\lambda_k > 0$.

推论 3.4.4 *假设存在 $A > 0$, 使得*

$$\sum_{k=1}^{\infty}\Big(\gamma_k/\lambda_k\Big)^A < \infty,$$

且存在 $2 \le \alpha < e, 0 < \delta < 1 - 1/\alpha$ 和整数 $m \ge 1$ 使得 (3.4.54) 和 (3.4.55) 被满足. 则

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\max_{k\ge 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.}$$

第四章　Gauss 过程的重对数律和增量的几乎处处下极限

自从 Chung 重对数律 (1948) 和 Strassen 泛函重对数律 (1964) 出现以来, 对某些类型 Gauss 过程的泛函重对数律 (FLIL) 及它们的速度已被 Oodaira (1972), Bolthausen (1978), Grill (1987), Goodman 和 Kuelbs (1988, 1991a, b) 等所讨论. 借助于小球概率估计, 对各种 Gauss 过程和 Gauss 场的 Chung 重对数律已被 Shao (1993), Kuelbs, Li 和 Talagrand (1994), Kuelbs, Li 和 Shao (1995), Monrad 和 Rootzen (1995), Shao 和 Wang (1995a) 等所研究. 对 Gauss 过程的其他重对数律, 小球概率估计, 不可微模也被若干作者所讨论. 在近十年中, 许多作者对所有这些问题, 特别对下极限性质给予了很多关注. 在本章中, 我们将汇集并详细阐述关于各种 Gauss 过程的一系列重对数律和增量的几乎处处下极限的最近结果.

在 §4.1 中, 我们介绍一类 Gauss 过程的 Strassen 泛函重对数律及其精确收敛速度, 以及 Gauss 过程的 Erdös 和 Révész 重对数律. 在 §4.2 中, 建立了 Gauss 过程的小球概率, 由此证明了 Gauss 过程的 Chung 重对数律. 对 Gauss 场的类似结果被介绍在 §4.3 中. 在 §4.4 中, 给出具有平稳增量的 Gauss 过程增量的 a.s. 下极限结果. 在 §4.5 中, 研究了两参数 Gauss 过程的下极限. §4.6 节介绍 Gauss 过程的其他轨道性质, 如 p 变差及其有关的分形性质.

§4.1　Gauss 过程的重对数律

4.1.1　Strassen 重对数律及其收敛速度

设 $\{W(t); t \geq 0\}$ 是 Wiener 过程, $C[0,1]$ 是 $[0,1]$ 上的连续函数空间, 记

$$\mathcal{K}=\left\{f(t)=\int_0^t g(s)ds;\ 0\le t\le 1,\int_0^1 g^2(s)ds\le 1\right\}.$$

那么 $\mathcal{K}$ 是 $C[0,1]$ 的凸对称紧致子集，且 $\mathcal{K}$ 也是具有再生核 (r.k.) 函数 $R(s,t)=s\wedge t$ 的再生核 Hilbert 空间的单位球. 定义

$$f_n(t)=W(nt)/\sqrt{2n\log\log n},\quad 0\le t\le 1.$$

那么 $\{f_n(t);n\ge 3\}$ 是概率为 1 地具有 $C[0,1]$ 中的样本轨道的随机过程序列. Strassen (1964) 证明了下述定理.

定理 S *在 $C[0,1]$ 中，序列 $\{f_n(t)\}$ 概率为 1 地相对紧且它的极限点集与 $\mathcal{K}$ 重合.*

Oodaira (1972) 拓广上述定理到包含 Wiener 过程的 Gauss 过程类上. 设 $\{X(t);t\ge 0\}$ 是一个可分、可测实 Gauss 过程, $X(0)=0,EX(t)=0$ 且协方差为

$$R(s,t)=EX(s)X(t).$$

记 $\sigma^2(t)=R(t,t)$. 假设下列条件被足:

(I) 对任给 $T>0$, 存在正的非降函数 $g(h,T),h>0$, 使得对所有的 $t,t+h\in[0,T]$, 当 $h\to 0$ 时

$$|R(t+h,t+h)-2R(t+h,t)+R(t,t)|\le g(h,T)\to 0,\tag{4.1.1}$$

$$\{g(1,T)\}^{-1/2}\int_1^\infty g^{1/2}(e^{-u^2},T)du\le C<\infty,\tag{4.1.2}$$

且

$$\sigma^2(T)/g(1,T)\uparrow\infty\quad(T\to\infty);\tag{4.1.3}$$

(II) 存在正函数 $v(r),r>0$ 使得 $v(r)\uparrow$, 且对所有的 $r>0,s,t\ge 0$ 有

$$R(rs,rt)=v(r)R(s,t).\tag{4.1.4}$$

在条件 (I) 下，由定理 2.1.3 知，对任一 $T>0$, 过程 $\{X(t);0\le t\le T\}$ a.s. 具有连续样本轨道.

例 具有协方差核

$$R(s,t)=\int_0^{s\wedge t}(s-\lambda)^{\beta}(t-\lambda)^{\beta}d\lambda,\quad -1/2<\beta<\infty$$

的 Gauss 过程满足条件 (I) 和 (II). 这一类过程包含了 Wiener 过程 $\{W(t)\}$(取 $\beta=0$) 和过程 $\left\{\int_0^t W(u)du\right\}$ (取 $\beta=1$). 分数 Wiener 过程是具有平稳增量和协方差核

$$R(s,t)=(s^{2\alpha}+t^{2\alpha}-|s-t|^{2\alpha})/2,\quad 0<\alpha\le 1,\tag{4.1.5}$$

的 Gauss 过程, 它也满足条件 (I) 和 (II).

定义

$$\eta_n(t,\omega)=X(nt,\omega)/(2\sigma^2(n)\log\log n)^{1/2},\quad 0\le t\le 1,\ n\ge 3.\tag{4.1.6}$$

设 $H(\mathcal{R})$ 是具有 r.k. 函数 $R(s,t),0\le s,t\le 1$, 的 r.k. Hilbert 空间. 记

$$\mathbf{K}=\{h\in H(\mathcal{R});\ \|h\|_H\le 1/\sigma(1)\},$$

其中 $\|\cdot\|_H$ 为 $H(\mathcal{R})$ 的范数. Oodaira (1972) 证明了下述结果.

定理 4.1.1 若条件 (I) 和 (II) 被满足, 那么序列 $\{\eta_n(t)\}$ 的极限点集概率为 1 地被包含于集合 $\mathcal{K}$ 中.

Grill (1987) 得到了下述关于标准 Wiener 过程 $\{W(t);0\le t<\infty\}$ 的 Strassen 泛函重对数律收敛于 $\mathcal{K}$ 的 "最佳" 速度:

$$\limsup_{n\to\infty}\inf_{f\in\mathcal{K}}\|f(t)-\eta_n(t)\|(\log\log n)^{\alpha}=\begin{cases}0\ , & \alpha<2/3,\\ \infty\ , & \alpha>2/3.\end{cases}$$

这两个结论等价于

$$P\left\{\eta_n(t)\in\mathcal{K}^{\varepsilon_n}\ \text{最终成立}\right\}=1,\quad \alpha<2/3$$

和

$$P\Big\{\eta_n(t)\notin\mathcal{K}^{\varepsilon_n},\ \text{i.o.}\Big\}=1,\quad \alpha>2/3,$$

其中 $\varepsilon_n=(\log\log n)^{-\alpha},\mathcal{K}^{\varepsilon_n}=\{g;g\in C[0,1],\inf_{f\in\mathcal{K}}\|g(t)-f(t)\|<\varepsilon_n\}$.

称过程 $\{X(t);0\le t<\infty\}$ 是具有指数 α 自相似的，若对每一 $a>0$, 过程 $\{X(at);0\le t<\infty\}$ 和过程 $\{a^{2\alpha}X(t);0\le t<\infty\}$ 同分布. 一个具有平稳增量的零均值的自相似 Gauss 过程 $\{Y(t);0\le t<\infty\}$ 是分数 Wiener 过程. 对 $0\le t<\infty$, $0<\alpha<1$, 我们有

$$Y(t)=V(t)+X(t),\quad t\ge 0$$

其中

$$X(t)=\int_0^t(t-s)^{(2\alpha-1)/2}dW(s),\tag{4.1.7}$$

$$V(t)=\int_{-\infty}^0\Big\{(t-s)^{(2\alpha-1)/2}-(-s)^{(2\alpha-1)/2}\Big\}dW(s).\tag{4.1.8}$$

当 $\alpha=1/2$ 时， $V=0$, 因此 $\{X(t);t\ge 0\}$ 和 $\{Y(t),t\ge 0\}$ 都是 Wiener 过程. Goodman 和 Kuelbs (1991a) 证明了

定理 4.1.2 设 $\{X(t);t\ge 0\}$ 和 $\{Y(t);t\ge 0\}$ 是 (4.1.7) 和 (4.1.8) 中定义的连续的中心化 Gauss 过程，令

$$\mathcal{K}=\Big\{f(t)=\int_0^t(t-u)^{(2\alpha-1)/2}g(u)du,0\le t\le 1,\int_0^1g^2(u)du\le 1\Big\},\tag{4.1.9}$$

其中 $0<\alpha<1$.

(A) 若 $\gamma>0$ 充分大，那么

$$P\Big\{X(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2}\in\mathcal{K}^{\varepsilon_n}\Big\}=1,\tag{4.1.10}$$

其中

$$\varepsilon_n=\begin{cases}\gamma(\log\log\log n/\log\log n)^{2/3}, & \text{当 } \alpha\ge 1/2 \text{ 时},\\ \gamma(\log\log\log n/\log\log n)^{(2\alpha+1)/(2\alpha+2)}, & \text{当 } 0<\alpha<1/2 \text{ 时}.\end{cases}$$

因此

$$\limsup_{n\to\infty}\inf_{f\in\mathcal{K}}\|f(t)-\eta_n(t)\|\Big(\frac{\log\log\log n}{\log\log n}\Big)^{-\theta}=0,$$

其中当 $\alpha\geq 1/2$ 时，$\theta>2/3$; 当 $\alpha<1/2$ 时，$\theta>(2\alpha+1)/2(\alpha+1)$.

(B) 若 $0<\alpha<1, \epsilon_n=\gamma(\log\log n)^{-1/2}$, 且

$$\mathcal{K}=\Big\{f(t)=T_\alpha g(t), 0\leq t\leq 1, \int_{-\infty}^1 g^2(u)du\leq 1\Big\},$$

其中

$$T_\alpha g(t)=\int_0^1(t-u)^{(2\alpha-1)/2}g(u)du+\int_{-\infty}^0((t-u)^{(2\alpha-1)/2}$$
$$-(-u)^{(2\alpha-1)/2})g(u)du,$$

那么对任何 $\gamma>0$

$$P\Big\{Y(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2}\in\mathcal{K}^{\epsilon_n}\text{ 最终成立}\Big\}=1. \qquad (4.1.11)$$

因此对 $\eta_n(\cdot)=Y(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2}$, 当 $\theta<1/2$ 时

$$\limsup_{n\to\infty}\inf_{f\in\mathcal{K}}\|f(t)-\eta_n(t)\|(\log\log n)^\theta=0.$$

Goodman 和 Kuelbs (1991a) 也给出了 p 参数 Wiener 单和 p 维 Wiener 过程的 Strassen 重对数律的收敛速度.

Monrad 和 Rootzen (1995) 给出了分数 Wiener 过程的 Strassen 重对数律的精确收敛速度. 设 $H_\alpha\subseteq C[0,1]$ 是具有 r.k. 函数

$$R(s,t)=\{s^{2\alpha}+t^{2\alpha}-|s-t|^{2\alpha}\}/2, \quad 0\leq s,t\leq 1$$

的 r.k. Hilbert 空间， H_α 中的内积记为 $\langle f,g\rangle_\alpha$. 若 $f\in H_\alpha$, 我们有

$$|f(t)-f(s)|^2\leq|s-t|^\alpha\langle f,f\rangle_\alpha.$$

定理 4.1.3 设 $\langle f,f\rangle_\alpha<1$. 则当 $t\downarrow 0$ 时

$$\liminf\,(\log\log t)^{(2\alpha+1)/2}\|\eta_t-f\|=\gamma(f), \quad \text{a.s.},$$

其中常数 $\gamma(f)$ 满足：存在常数 $0<c<C<\infty$,

$$2^{-1/2}c^{\alpha}(1-\langle f,f\rangle_{\alpha})^{-\alpha}\le\gamma(f)\le 2^{-1/2}C^{\alpha}(1-\langle f,f\rangle_{\alpha})^{-\alpha}.$$

定理 4.1.4 当 $t\downarrow 0$ 或 $t\uparrow\infty$ 时，若 $\langle f,f\rangle_{\alpha}=1$, 则

$$\liminf\,(\log\log t)^{(2\alpha+1)/2}\|\eta_t-f\|=\infty\quad \text{a.s.},$$

而若 $\langle f,f\rangle_{\alpha}<1$，则

$$\liminf\,(\log\log t)^{(2\alpha+1)/2}\|\eta_t-f\|<\infty\quad \text{a.s.}$$

Kuelbs, Li 和 Talagrand (1994) 研究了 Gauss 样本的下极限结果. 并给出了对 Wiener 过程的泛函型 Chung 重对数律的收敛速度的一个应用.

记

$$C_0[0,1]=\{f(x)\in C[0,1]:f(0)=0\},$$

$$Y_{t,T}(x)=\beta_T\big(W(t+a_Tx)-W(t)\big),\quad 0\le x\le 1, 0\le t\le T-a_T,$$

其中 $\beta_T=\{2a_T(\log(T/a_T)+\log\log T)\}^{-1/2}$. Révész (1979) 结合定理 0.2 与定理 S 经详细计算得到

定理 R 设 a_T 是 T 的单调非降函数，满足

(i) $0<a_T\le T$,

(ii) T/a_T 非降.

那么在 $C_0[0,1]$ 中 $\{Y_{t,T}:0\le x\le 1, 0\le t\le T-a_T, T\ge 3\}$ 概率为 1 地相对紧，极限点集为 $\mathcal{K}$.

若 a_T 还满足

(iii) $\lim_{T\to\infty}\big(\log(T/a_T)\big)/\log\log T=\infty$,

那么

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\inf_{f\in\mathcal{K}}\sup_{0\le x\le 1}|Y_{t,T}(x)-f(x)|=0\quad \text{a.s.},$$

且对任一 $f \in \mathcal{K}$, 有

$$\lim_{T\to\infty} \inf_{0\le t\le T-a_T} \sup_{0\le x\le 1} |Y_{t,T}(x) - f(x)| = 0 \quad \text{a.s.}$$

Chen, B. (1998) 指出，借助大偏差估计，上述定理的证明是较简单的，由此还可给出泛函连续模定理. 记

$$M_{t,h}(x) = \frac{W(t+hx) - W(t)}{(2h\log h^{-1})^{1/2}}, \quad 0 \le x \le 1.$$

定理 C 我们有

$$\lim_{h\to 0} \sup_{0\le t\le 1-h} \inf_{f\in\mathcal{K}} \sup_{0\le x\le 1} |M_{t,h}(x) - f(x)| = 0 \quad \text{a.s.},$$

且对任一 $f \in \mathcal{K}$, 有

$$\lim_{h\to 0} \inf_{0\le t\le 1-h} \sup_{0\le x\le 1} |M_{t,h}(x) - f(x)| = 0 \quad \text{a.s.}$$

最近， Wang, W. S. 利用大偏差估计给出了上述定理的精确收敛速度. 设 $f \in C_0[0,1]$, 令

$$I(f) = \begin{cases} \displaystyle\int_0^1 (f'(x))^2\, dx, & 若 f 是绝对连续的, \\ \infty, & 否则. \end{cases}$$

定理 W.1 对充分大的 $\gamma > 0$, 有

$$\mathrm{P}\left\{ \sup_{0\le t\le 1-h} \inf_{f\in\mathcal{K}} \sup_{0\le x\le 1} |M_{t,h}(x) - f(x)| \right.$$
$$\left. \ge \gamma(\log\log h^{-1}/\log h^{-1})^{2/3} \quad \text{i.o.} \right\} = 0,$$

且对每一 $f \in \mathcal{K}$, 有

$$\lim_{h\to 0} \inf_{0\le t\le 1-h} \sup_{0\le x\le 1} |M_{t,h}(x) - f(x)| \log h^{-1}$$
$$= \begin{cases} \dfrac{\pi}{4\sqrt{1-I(f)}}, & I(f) < 1, \\ \infty, & I(f) = 1, \end{cases} \quad \text{a.s.}$$

定理 W.2 设 a_T 如定理 R 所定义. 那么对充分大的 $\gamma > 0$, 有

$$\mathrm{P}\left\{ \sup_{0\le t\le T-a_T} \inf_{f\in\mathcal{K}} \sup_{0\le x\le 1} |Y_{t,T}(x) - f(x)| \right.$$
$$\left. \ge \gamma\big((\log g(T))/g(T)\big)^{2/3} \quad \text{i.o.} \right\} = 0,$$

其中 $g(T) = \log(T/a_T) + \log\log T$, 且对每一 $f \in \mathcal{K}$, 有

$$\liminf_{T\to\infty} \inf_{0\le t\le T-a_T} \sup_{0\le x\le 1} |Y_{t,T}(x) - f(x)| g(T)$$
$$= \begin{cases} \dfrac{\pi}{4\sqrt{1-I(f)}}, & I(f) < 1, \\ \infty, & I(f) = 1, \end{cases} \qquad \text{a.s.}$$

若 a_T 还满足定理 R 中的条件 (iii), 那么上式中的 $\liminf$ 可换成 $\lim$.

Wei, Q.C. (危启才) 讨论了 Hölder 范数下的与定理 R 和 S 类似的结论, 还讨论了 l^p 值 Wiener 过程的同类定理.

4.1.2 Gauss 过程的 Erdös-Révész 重对数律

设 $\{W(t); t \ge 0\}$ 是标准 Wiener 过程, 定义

$$\eta(t) = \sup\{s : 0 \le s \le t, W(s) \ge (2s\log\log s)^{1/2}\}, \quad t \ge 0,$$

$$\eta_\delta(t) = \sup\{s : 0 \le s \le t, W(s)$$
$$\ge (2(1-\delta)s\log\log s)^{1/2}\},\ t \ge 0,\ 0 \le \delta < 1,$$

$$\eta_\delta^{(p)}(t) = \sup\{s : 0 \le s \le t, W(s) \ge s^{1/2}\alpha(\delta, p, s)\}, \quad t \ge 0,$$

其中

$$\alpha(\delta, p, s) = \left(2\left(\log_2 s + \frac{3}{2}\log_3 s + \sum_{j=4}^{p} \log_j s - \delta \log_p s\right)\right)^{1/2}, \ \delta \ge 0,$$

$p = 3, 4, \cdots, \log_j x = \log_1(\log_{j-1} x), \log_1 x = \ln x \ (x > 0)$ 或 $1\ (x \le 0)$. 这里及本节以后均记 $\log x = \ln x$. 由重对数律显然

有

$$\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}\eta_\delta(t)=\lim_{t\to\infty}\eta_\delta^{(p)}(t)=\infty \quad \text{a.s.}$$

和

$$\limsup_{t\to\infty}\frac{\eta(t)}{t}=\limsup_{t\to\infty}\frac{\eta_\delta(t)}{t}=\limsup_{t\to\infty}\frac{\eta_\delta^{(p)}(t)}{t}=1 \quad \text{a.s.}$$

Erdös 和 Révész (1990) 考察了 $\eta(t)$ 的下界，得到一个新的重对数律：对某常数 C_0, $1/4\le C_0\le 2^{14}$,

$$\liminf_{t\to\infty}\frac{(\log_2 t)^{1/2}}{\log_3 t\cdot\log t}\cdot\log\frac{\eta(t)}{t}=-C_0 \quad \text{a.s.}$$

Shao (1994) 求得了 C_0 的精确值和 $\eta_\delta(t)$ 与 $\eta_\delta^{(p)}(t)$ 的精确下界.

定理 4.1.5 对每一 $0<\delta\le 1/2$, 我们有

$$\liminf_{t\to\infty}\frac{(\log_2 t)^{1/2}}{\log_3 t\cdot\log t}\cdot\log\frac{\eta(t)}{t}=-3\sqrt{\pi} \quad \text{a.s.},$$

$$\liminf_{t\to\infty}(\log t)^{\delta-1}(\log_2 t)^{-1/2}\cdot\log\frac{\eta_\delta(t)}{t}=-2\delta\sqrt{\pi/(1-\delta)} \quad \text{a.s.}$$

定理 4.1.6 对 $p=3,4\cdots$, 我们有

$$\liminf_{t\to\infty}\frac{\log_p\eta_0^{(p)}(t)-\log_p t}{\log_{p+1}t}=-2\sqrt{\pi} \quad \text{a.s.},$$

$$\liminf_{t\to\infty}\frac{\log_{p-1}\eta_\delta^{(p)}(t)-\log_{p-1}t}{\log_{p+1}t}=-2\delta\sqrt{\pi} \quad \text{a.s.},\quad 0<\delta<1,$$

$$\liminf_{t\to\infty}\frac{\log_{p-2}\eta_\delta^{(p)}(t)-\log_{p-2}t}{\log_{p-2}t\cdot(\log_{p-1}t)^{1-\delta}\log_p t}=-2\delta\sqrt{\pi} \quad \text{a.s.},\quad \delta>1.$$

注 4.1.1 定理 4.1.5 是说，对任何充分大的 t, 在

$$t^{1-3\sqrt{\pi}\log_3 t\cdot(\log_2 t)^{-1/2}} \quad 和 \quad t$$

之间存在着 s 使得 $W(s) \geq (2s\log\log s)^{1/2}$. 定理 4.1.6 有类似的意义.

令

$$\bar{\eta}(t) = \sup\{s : 1 \leq s \leq t, W(s) \geq (2s\log\log s)^{1/2}\}, \quad 当\ t \geq 1,$$
$$\hat{\eta}(t) = \sup\left\{s : 0 \leq s \leq t, \frac{W(e^s)}{e^{s/2}} \geq (2\log s)^{1/2}\right\}, \quad 当\ t \geq 0.$$

易见对每一 $t > 0$

$$\hat{\eta}(t) = \log \bar{\eta}(e^t) \quad \text{a.s.}$$

因此, 由定理 4.1.5 我们有

$$\liminf_{t\to\infty} \frac{(\log t)^{1/2}}{t \cdot \log_2 t} \cdot (\hat{\eta}(t) - t) = -3\sqrt{\pi} \quad \text{a.s.}$$

显然, $\{W(e^s)/e^{s/2}; s \geq 0\}$ 是 O-U 过程, 它是一个平稳 Gauss 过程. 由此导致去研究一般平稳的 Gauss 过程 $\hat{\eta}(t)$ 的相应问题.

设 $\{X(t); t \geq 0\}$ 是可分平稳 Gauss 过程, $EX(t) = 0$, $EX^2(t) = 1$. 记它的协方差函数

$$r(t) = EX(t+s)X(s), \quad s \geq 0,\ t \geq 0.$$

考察随机过程

$$\xi(t) = \sup\{s : 0 \leq s \leq t, X(s) \geq (2\log s)^{1/2}\}, \quad t \geq 0.$$

在关于 $r(t)$ 的某些条件下, 由重对数律的上类可推得 (参见 Qualls 和 Watanabe 1971)

$$P\{X(s) \geq (2\log s)^{1/2}, \text{i.o.}\} = 1.$$

因此我们有

$$\lim_{t\to\infty} \xi(t) = \infty \quad \text{a.s.}$$

和

$$\limsup_{t\to\infty} (\xi(t) - t) = 0 \quad \text{a.s.}$$

Shao (1992) 对平稳 Gauss 过程 $X(t)$ 得到如下的 Erdös-Révész 重对数律:

定理 4.1.7 假设下述条件被满足:

对某 $C>0, 0<\alpha<2$, 当 $t\to 0$时, $r(t)=1-C|t|^\alpha+o(|t|^\alpha)$,

对某 $\gamma>0$, 当$t\to\infty$ 时, $r(t)=O(t^{-2\gamma})$,

对每一 $s>0, \sup\limits_{t\geq s}|r(t)|<1$.

那么

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t(\log t)^{(\alpha-2)/(2\alpha)}\cdot\log_2 t}=-\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_\alpha(2C)^{1/\alpha}}\quad \text{a.s.}, 0<\alpha<2, \tag{4.1.12}$$

$$\liminf_{t\to\infty}\frac{\log(\xi(t)/t)}{\log_2 t}=-\frac{2\sqrt{\pi}}{H_2\sqrt{2C}}\quad \text{a.s.}, \alpha=2, \tag{4.1.13}$$

其中 $0<H_\alpha:=\lim_{T\to\infty}T^{-1}\int_0^\infty e^s P\{\sup_{0\leq t\leq T}Y(t)>s\}ds<\infty$, $Y(t)$ 是非平稳 Gauss 过程, $EY(t)=-|t|^\alpha$, $Cov(Y(s),Y(t))=-|t-s|^\alpha+|s|^\alpha+|t|^\alpha$.

下面, 我们应用定理4.1.7于两个特殊的Gauss过程: 独立O-U过程的无穷级数和分数 Wiener 过程 (见 Shao 1992). 设 $Y(t)=(X_1(t),X_2(t),\cdots)$, 其中 $\{X_i(t);-\infty<t<\infty\}(i=1,2,\cdots)$ 是独立的 O-U 过程, 具有系数 γ_i 和 λ_i. 假设

$$0<\Gamma_0^2=\sum_{i=1}^\infty\frac{\gamma_i}{\lambda_i}<\infty.$$

定义

$$X(t)=\frac{1}{\Gamma_0}\sum_{i=1}^\infty X_i(t), \tag{4.1.14}$$

$$\sigma^2(t)=\frac{1}{\Gamma_0^2}\sum_{i=1}^\infty\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i t}),\quad t\geq 0. \tag{4.1.15}$$

众所周知 $\{X(t);t\geq 0\}$ 是平稳 Gauss 过程, $EX(t)=0, EX^2(t)=1$ 且协方差函数

$$r(t)=EX(t+s)X(s)=1-\sigma^2(t)\quad s,t\geq 0.$$

定理 4.1.8 设 $\{X(t);t\geq 0\}$ 如 (4.1.14) 所定义. 令

$$\xi(t)=\sup\{s;0\leq s\leq t, X(s)\geq (2\log s)^{1/2}\},\quad t\geq 0.$$

设 $\sigma^2(t)$ 如 (4.1.15). 假设存在 $0<\alpha\leq 1, C>0$ 和 $\delta>0$ 使得

$$\sum_{i=1}^{\infty}\frac{\gamma_i}{\min(\lambda_i,\lambda_i^{1+\delta})}<\infty, \tag{4.1.16}$$

$$\lim_{t\downarrow 0}\frac{\sigma^2(t)}{t^\alpha}=C. \tag{4.1.17}$$

则 (4.1.12) 成立.

设 $\{Z(t);t\geq 0\}$ 是 α 阶的分数 Wiener 过程. 考察

$$\eta(t)=\sup\{s;0\leq s\leq t, Z(s)\geq (2s^{2\alpha}\log\log s)^{1/2}\},\quad t\geq 0.$$

由 $Z(t)$ 的增量的上类 (参见 Grill, 1991), 我们有

$$\lim_{t\to\infty}\eta(t)=\infty\quad \text{a.s.}$$

和

$$\limsup_{t\to\infty}(\eta(t)-t)=0\quad \text{a.s.}$$

下述定理给出了 $\eta(\cdot)$ 的下界.

定理 4.1.9 我们有

$$\liminf_{t\to\infty}\frac{(\log\log t)^{(1-\alpha)/(2\alpha)}\cdot\log(\eta(t)/t)}{\log t\cdot\log\log\log t}=-\frac{(1+\alpha)\sqrt{\pi}}{\alpha H_{2\alpha}}\quad \text{a.s.}$$

§4.2 Gauss 过程的小球概率和 Chung 重对数律

在建立 Chung 重对数律的过程中, 小球概率估计是一个关键. 在本节中, 我们首先讨论 Gauss 过程的小球概率, 然后应用它来获得 Gauss 过程的 Chung 重对数律, 特别地, 我们估计了分

数 Wiener 过程和 O-U 过程无穷级数的小球概率的界并给出了它们的 Chung 重对数律.

4.2.1 Gauss 过程的小球概率

Shao (1993) 及 Monrad 和 Rootzen (1995) 分别独立地对具有平稳增量的 Gauss 过程建立了小球概率. 在此我们介绍 Shao 的结果, 其结论带有较精确的常数.

定理 4.2.1 设 $\{X(t);0\le t\le 1\}$ 是具有平稳增量的 Gauss 过程, $EX(t)=0, X(0)=0$ a.s. 若

$$\sigma^2(h)=E(X(t+h)-X(t))^2,\quad 0\le t\le t+h\le 1, \tag{4.2.1}$$

在 $(0,1)$ 上是非降的凹函数, 那么对每一 $0<x<1$ 有

$$P\left\{\sup_{0\le t\le 1}|X(t)|\le\sigma(x)\right\}\le 2\exp(-0.17/x) \tag{4.2.2}$$

和

$$P\left\{\sup_{0\le t\le 1}|X(t)|\le\sigma(x)+6e\int_0^\infty\sigma(xe^{-y^2})dy\right\}\ge\exp(-1.87/x). \tag{4.2.3}$$

特别地, 若 $\sigma^2(x)$ 是凹函数, 且对某 $\alpha>0$, $\sigma(x)/x^\alpha$ 在 $(0,1)$ 上是非降的, 则

$$P\left\{\sup_{0\le t\le 1}|X(t)|\le c_\alpha\sigma(x)\right\}\ge\exp(-1.87/x), \tag{4.2.4}$$

其中 $c_\alpha=1+3e\sqrt{\pi/\alpha}$.

证明 利用推论 1.2.6 和 $\log(2\Phi(\sqrt{2})-1)\le -0.17$, 我们有

$$\begin{aligned}P\left\{\sup_{0\le t\le 1}|X(t)|\le\sigma(x)\right\}&\le P\left\{\max_{0\le i\le 1/x}|X(ix)|\le\sigma(x)\right\}\\&\le\prod_{i=1}^{[1/x]}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}}e^{-t^2/2}dt\\&=(2\Phi(\sqrt{2})-1)^{[1/x]}\\&\le 2\exp(-0.17/x).\end{aligned}$$

这就证明了 (4.2.2). 由 $\sigma^2(x)$ 凹性, 易知 $\sigma^2(kx) \le k\sigma^2(x)$, $k = 1, 2, \cdots$. 从而除了指数中的常数因子外, (4.2.4) 是下节定理 4.3.1 的特例. 故 (4.2.3) 和 (4.2.4) 的证明从略.

推论 4.2.1 设 $\{Z(t); 0 \le t \le 1\}$ 是阶为 $0 < \alpha < 1/2$ 的分数 Wiener 过程. 那么对每一 $0 < x < 1$ 我们有

$$\exp\Big(-\theta_\alpha x^{-1/\alpha}\Big) \le P\Big\{\sup_{0\le t\le 1} |Z(t)| \le x\Big\} \le 2\exp\Big(-0.17x^{-1/\alpha}\Big), \tag{4.2.5}$$

其中 $\theta_\alpha = 2(1 + 3e\sqrt{\pi/\alpha})^{1/\alpha}$.

注 4.2.1 Monrad 和 Rootzen (1995) 证明了对 α 的所有情形有类似于 (4.2.5) 的不等式成立, 即对阶为 $0 < \alpha < 1$ 的分数 Wiener 过程, 存在与 x 无关的常数 $0 < c \le C < \infty$ 使得

$$\exp\Big(-Cx^{-1/\alpha}\Big) \le P\Big\{\sup_{0\le t\le 1} |Z(t)| \le x\Big\} \le \exp\Big(-cx^{-1/\alpha}\Big). \tag{4.2.6}$$

参见下节的定理 4.3.3.

4.2.2 分数 Wiener 过程的 Chung 重对数律

利用分数 Wiener 过程的小球概率估计, 即 (4.2.6), Monrad 和 Rootzen (1995) 对这类过程建立了如下的 Chung 重对数律.

定理 4.2.2 设 $\{Z(t); t \ge 0\}$ 是阶为 $0 < \alpha < 1$ 的分数 Wiener 过程. 那么存在正常数 c_α, $c_\alpha{}'$ 使得

$$\liminf_{t\to 0} \sup_{0\le s\le t} \frac{|Z(s)|}{t^\alpha(\log\log t^{-1})^{-\alpha}} = c_\alpha \quad \text{a.s.}, \tag{4.2.7}$$

$$\liminf_{t\to \infty} \sup_{0\le s\le t} \frac{|Z(s)|}{t^\alpha(\log\log t)^{-\alpha}} = c_\alpha{}' \quad \text{a.s.} \tag{4.2.8}$$

定理 4.2.2 的证明将在下节给出.

注 4.2.2 Monrad 和 Rootzen (1995) 对更大的一类 Gauss 过程证明了 Chung 重对数律. 设 $\{X(t); -\infty < t < \infty\}$ 是中心化的

具有平稳增量的实连续 Gauss 过程. 假设 $X(0)=0$ 且有连续的协方差函数

$$R(s,t)=\int_{-\infty}^{\infty}(e^{is\lambda}-1)(e^{it\lambda}-1)\Delta(d\lambda),$$

其中对称谱测度 Δ 满足

$$\int_{-\infty}^{\infty}\frac{\lambda^2}{1+\lambda^2}\Delta(d\lambda)<\infty.$$

定理 4.2.3 若

$$\sigma^2(h)=\mathrm{Var}(X(t+h)-X(t))\le c_1h^{2\alpha},\quad 0\le h\le\delta,\ 0\le t\le\delta-h,$$
$$\mathrm{Var}(X(t+h)|X(s),0\le s\le t)\ge c_2h^{2\alpha},\quad 0\le h\le\delta,\ 0\le t\le\delta-h,$$

且对某 $l>0$

$$\liminf_{|\lambda|\to 0}|\lambda|^3\Delta([\lambda,\lambda+l])>0,$$

则存在正常数 c_0 使得

$$\liminf_{t\downarrow 0}\frac{M(t)}{t^\alpha(\log\log t)^{-\alpha}}=c_0\qquad \text{a.s.}$$

注 4.2.3 Li 和 Shao (1999) 的定理 6.9 指出 (参见 Li 和 Linde 1998, Shao 1999): 对阶为 α $(0<\alpha<1)$ 的分数 Wiener 过程 $\{Z(t);t\ge 0\}$ 有精确的小球概率估计, 即存在常数 C_α 使得

$$\lim_{x\to 0}x\log P\Big(\sup_{0\le t\le 1}|Z(t)|\le x^\alpha\Big)=-C_\alpha.$$

从而有如下重对数律

$$\liminf_{t\to\infty}\frac{\sup_{0\le s\le t}|Z(s)|}{t^\alpha(\log\log t)^{-\alpha}}=C_\alpha\qquad \text{a.s.},$$

$$\liminf_{t\to 0}\frac{\sup_{0\le s\le t}|Z(s)|}{t^\alpha(\log\log(1/t))^{-\alpha}}=C_\alpha\qquad \text{a.s.}$$

4.2.3 O-U 过程的无穷级数的 Chung 重对数律

设 $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ 是具有系数 γ_k 和 λ_k 的独立 O-U 过程,

$$X(t) = \sum_{k=1}^{\infty} X_k(t), \qquad -\infty < t < \infty, \tag{4.2.9}$$

为 $Y(\cdot)$ 的无穷级数.

Shao 和 Wang (1995) 对 $\{X(t)\}$ 证明了 Chung 重对数律.

定理 4.2.4 假设

$$0 < \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} < \infty \tag{4.2.10}$$

且对某 $\alpha > 0$, $\sigma(h)/h^{\alpha}$ 在 $[0,1]$ 上是非降的，其中

$$\sigma^2(h) = E(X(h) - X(0))^2 = 2\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}). \tag{4.2.11}$$

那么对某 $0 < C < \infty$

$$\liminf_{h \to 0} \sup_{0 \le t \le h} \frac{|X(t)|}{\sigma(Ch/\log\log(1/h))} = 1 \qquad \text{a.s.}$$

定理 4.2.4 的证明是基于小球概率估计，它将作为定理 4.3.5 的推论在 4.3.3 小节中给出.

Zhang (1995) 对 $X(\cdot)$ 给出了如下的 Chung 型重对数律.

定理 4.2.5 假设 (4.2.10) 被满足且

$$\Gamma_1 = 2\sum_{k=1}^{\infty} \gamma_k < \infty, \tag{4.2.12}$$

则

$$\liminf_{h\to 0}\sup_{0\le s\le h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|=1 \quad \text{a.s.} \tag{4.2.13}$$

它的证明从下节的定理 4.3.5 和下述引理即得.

引理 4.2.1 设 $\{W(t);t\ge 0\}$ 是标准 Wiener 过程, 若 (4.2.13) 被满足, 则对任给的 $\delta>0$, 存在充分小的 $h_0=h_0(\delta)$ 使得对任一 $x>0$, $0<h<h_0$ 有

$$\begin{aligned}
&\frac{2}{\pi}\exp\Big(-\frac{\pi^2}{8(1-\delta)x^2}\Big)\le P\Big\{\sup_{0\le t\le 1}|W(t)|\le x\sqrt{1-\delta}\Big\}\\
&\le P\Big\{\sup_{0\le t\le h}|X(t)-X(0)|\le x(\Gamma_1 h)^{1/2}\Big\}\\
&\le P\Big\{\sup_{0\le t\le 1}|W(t)|\le x\sqrt{1+\delta}\Big\}\\
&\le \frac{4}{\pi}\exp\Big(-\frac{\pi^2}{8(1+\delta)x^2}\Big).
\end{aligned} \tag{4.2.14}$$

证明 易见对任何 $x_1\le x_2\le x_3\le x_4$,

$$E(X(x_4)-X(x_3))((X(x_2)-X(x_1))\le 0,$$

且对所有 $0\le t\le t+s\le T$,

$$\begin{aligned}
&-E\Big(X(T)-X(t+s)\Big)\Big(X(t+s)-X(t)\Big)\\
&-E\Big(X(t+s)-X(t)\Big)\Big(X(t)-X(0)\Big)\\
=&\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}\Big(1-\exp(-\lambda_k s)\Big)\Big(2-\exp(-\lambda_k(T-t-s))-\exp(-\lambda_k t)\Big)\\
\le& 2s\sum_{k=1}^{\infty}\gamma_k\Big(1-\exp\Big(-\lambda_k\frac{T}{2}\Big)\Big)=:s\cdot H(T),
\end{aligned}$$

其中

$$H(T)=2\sum_{k=1}^{\infty}\gamma_k\Big(1-\exp\Big(-\lambda_k\frac{T}{2}\Big)\Big)\to 0\quad(T\to 0).$$

对任意固定的 $\delta>0$, 选充分小的 $\varepsilon>0$. 对任意固定的正整数 n, 对 $i=1,2,\cdots,n$, 令

$$\begin{aligned}
\xi_i&=X\Big(\frac{i}{n}h\Big)-X\Big(\frac{i-1}{n}h\Big),\\
\eta_i&=W\Big(\Gamma_1(1+\varepsilon)\frac{i}{n}h\Big)-W\Big(\Gamma_1(1+\varepsilon)\frac{i-1}{n}h\Big),\\
\eta_i^*&=W\Big(\Gamma_1(1-\varepsilon)\frac{i}{n}h\Big)-W\Big(\Gamma_1(1-\varepsilon)\frac{i-1}{n}h\Big).
\end{aligned}$$

设 $\Sigma_\xi,\Sigma_\eta,\Sigma_{\eta^*}$ 分别是 $(\xi_1,\cdots,\xi_n)',(\eta_1,\cdots,\eta_n)',(\xi_1^*,\cdots,\xi_n^*)'$ 的协方差矩阵. 那么我们有

$$\begin{aligned}
\sum_{j=1 i\neq j}^{n}E\xi_i\xi_j&=E\Big(X(h)-X\Big(\frac{i}{n}h\Big)\Big)\Big(X\Big(\frac{i}{n}h\Big)-X\Big(\frac{i-1}{n}h\Big)\Big)\\
&\quad+E\Big(X\Big(\frac{i}{n}h\Big)-X\Big(\frac{i-1}{n}h\Big)\Big)\Big(X\Big(\frac{i-1}{n}h\Big)-X(0)\Big)\\
&\geq-\frac{1}{n}hH(h).
\end{aligned}$$

由于 $E\xi_i\xi_j\leq 0(i\neq j)$, 我们有

$$\rho_i:=\sum_{j=1 i\neq j}^{n}|E\xi_i\xi_j|=\Big|\sum_{j=1 i\neq j}^{n}E\xi_i\xi_j\Big|\leq\frac{1}{n}hH(h).$$

注意到 $H(h)\to 0,\sigma^2(h)/h\to\Gamma_1(h\to 0)$. 故存在充分小的 h_0 使得对 $0<h\leq h_0$,

$$(1+\varepsilon)\Gamma_1\frac{h}{n}-\sigma^2\Big(\frac{h}{n}\Big)-\rho_i>0,\ \sigma^2\Big(\frac{h}{n}\Big)-(1-\varepsilon)\Gamma_1\frac{h}{n}-\rho_i>0.$$

因此, $\Sigma_\eta-\Sigma_\xi$ 和 $\Sigma_\xi-\Sigma_{\eta^*}$ 是两个具有正的主对角元的矩阵, 且每一行中所有非对角元的绝对值和小于或等于该行的对角元, 从

而 $\Sigma_\eta - \Sigma_\xi, \Sigma_\xi - \Sigma_{\eta^*}$ 是半正定的. 由推论 1.2.4 (Anderson 不等式) 即得

$$P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}\eta_j\right| \le x\Big(\Gamma_1 h\Big)^{1/2}\right\} \le P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}\xi_j\right| \le x\Big(\Gamma_1 h\Big)^{1/2}\right\}$$
$$\le P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}\eta_j^*\right| \le x\Big(\Gamma_1 h\Big)^{1/2}\right\},$$

即

$$P\left\{\max_{1\le i\le n}\left|W\Big((1+\varepsilon)\Gamma_1\frac{i}{n}h\Big)\right| \le x(\Gamma_1 h)^{1/2}\right\}$$
$$\le P\left\{\max_{1\le i\le n}\left|X\Big(\frac{i}{n}h\Big) - X(0)\right| \le x(\Gamma_1 h)^{1/2}\right\}$$
$$\le P\left\{\max_{1\le i\le n}\left|W\Big((1-\varepsilon)\Gamma_1\frac{i}{n}h\Big)\right| \le x(\Gamma_1 h)^{1/2}\right\}.$$

令 $n = 2^m \to \infty$ $(m \to \infty)$, 即得 (4.2.14) 成立.

注 4.2.4 从引理 4.2.1 的证明我们可看到, 若 $\{\xi(t); 0 \le t \le T\}$ 是零均值 a.s. 连续的 Gauss 过程, 满足

(a) 对所有 $0 \le x_1 \le x_2 \le x_3 \le x_4 \le T$,

$$E(\xi(x_4) - \xi(x_3))(\xi(x_2) - \xi(x_1)) \le 0$$

或

$$E(\xi(x_4) - \xi(x_3))(\xi(x_2) - \xi(x_1)) \ge 0;$$

(b) 对某 $0 < A < \infty$ 和所有 $0 \le t \le t + s \le T$

$$E(\xi(t+s) - \xi(t))^2 \le As;$$

(c) 对所有 $x \in [0, T]$ 和任意整数 $m, 1 \le i \le m$,

$$\begin{aligned}&|E(\xi(x) - \xi(ix/m))(\xi(ix/m) - \xi((i-1)x/m))\\&\quad + E(\xi(ix/m) - \xi((i-1)x/m))(\xi((i-1)x/m) - \xi(0))|\\&\quad \le f(x)x/m,\end{aligned}$$

其中 $f(x)$ 是 $(0,\infty)$ 上的实函数， $f(x)\to 0(x\to 0)$. 那么对任一 $\epsilon>0$, 存在充分小的 h_0, 使得对任一 $y>0$, $0<h\le h_0$ 有

$$\begin{aligned}&P\Big\{\sup_{0\le t\le h}|\xi(t)-\xi(0)|\le y(As)^{1/2}\Big\}\\&\ge P\Big\{\sup_{0\le t\le 1}|W(t)|\le y\sqrt{1-\epsilon}\Big\}\ge\frac{2}{\pi}\exp\Big(-\frac{\pi^2}{8(1-\epsilon)y^2}\Big).\end{aligned}$$

注 4.2.5 在 Kuelbs, Li 和 Shao (1995) 中，对具有平稳增量的 Gauss 过程，当小球由各种 Hölder 范数给定时，估计了小球概率. 作为应用，他们在 Hölder 范数下对分数 Wiener 过程建立了 Chung 型泛函重对数律. 在 Li 和 Shao (1995) 中，在 Sobolev 范数下，对某些较一般的 Gauss 过程, 特别地对分数 Wiener 过程给出了精细的小球估计，建立了运用大偏差技术估计小球概率的新方法. 作为应用，他们给出了分数 Wiener 过程的 Chung 重对数律.

§4.3 Gauss 场的小球概率和 Chung 重对数律

Shao 和 Wang (1995) 给出了 d 参数 Gauss 场的小球概率的下界估计, 其中对分数 Lévy-Wiener 场求得了严格的上下界，Wiener 单的小球概率的精确上界和下界被 Talagrand (1994) 所给出. 这些估计被用来研究 Chung 重对数律.

4.3.1 Gauss 场的小球概率

设 $d\ge 1, \mathbf{X}=\{X(\mathbf{t});\mathbf{t}\in R^d\}$ 是实 Gauss 场，均值为零，且 $a\le\mathbf{t}\le b$ 是指对每一个 $i=1,2,\cdots,d$, $a\le t_i\le b$. 在本节中，对 $\mathbf{t}=(t_1,\cdots,t_d)\in R^d, \|\mathbf{t}\|$ 记它的 Euclidean 范数. Shao 和 Wang (1995) 给出了 Gauss 场的小球概率的下界.

定理 4.3.1 假设存在 $[0,1]$ 上的非降函数 $\sigma(x)$ 使得对每一 $\mathbf{s},\mathbf{t}\in[0,1]^d$

$$E|X(\mathbf{t})-X(\mathbf{s})|^2\le\sigma^2(\|\mathbf{t}-\mathbf{s}\|). \tag{4.3.1}$$

又设对某 $\alpha>0$, $\sigma(x)/x^\alpha$ 在 $[0,1]$ 上是非降的, 且对每一 $0\le h\le 1$ 和整数 k, $1\le k\le 1/h$, 满足

$$\sigma(kh)\le k^2\sigma(h). \tag{4.3.2}$$

那么存在正常数 $c=c(\alpha,d)$, 使得对任一 $0<x<1$ 有

$$P\Big\{\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}}|X(\mathbf{t})|\le c\sigma(x)\Big\}\ge\exp\Big(-\frac{c}{x^d}\Big). \tag{4.3.3}$$

令

$$\mathcal{A}=\{(\mathbf{a},\mathbf{b}];\mathbf{a},\mathbf{b}\in[0,1]^d,\ \mathbf{a}\le\mathbf{b}\}.$$

对任一 $A\in\mathcal{A}$, 记 $|A|$ 为 R^d 中 A 的 Lebesgue 测度, 并形式地定义

$$X(A)=\int_{\mathbf{a}}^{\mathbf{b}}dX(\mathbf{t}),$$

其中 $\int_{\mathbf{a}}^{\mathbf{b}}$ 也可理解为 $\mathbf{X}$ 的差分算子.

定理 4.3.2 假设存在 $[0,1]$ 上非降函数 $\sigma(x)$ 使对任何 $A\in\mathcal{A}$ 有

$$E(X(A))^2\le\sigma^2(|A|). \tag{4.3.4}$$

又设对某 $\alpha>0$, $\sigma(x)/x^\alpha$ 是 $[0,1]$ 上的非降函数且满足 (4.3.2). 那么存在仅依赖于 α 和 d 的正常数 c 使得对任何 $0<x<1/2$ 有

$$P\Big\{\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}}|X(\mathbf{t})|\le c\Big(\log\frac{1}{x}\Big)^{d-1}\sigma(x)\Big\}\overset{KS}{\ge}\exp\Big(-\frac{c(\log(1/x))^{d-1}}{x}\Big). \tag{4.3.5}$$

定理 4.3.1 的证明 对整数 $k\ge0$ 和 $\mathbf{t}=(t^{(1)},\cdots,t^{(d)})\in[0,1]^d$, 记

$$\mathbf{t}_k=\frac{[\mathbf{t}2^k]}{2^k}=\Big(\frac{[t^{(1)}2^k]}{2^k},\cdots,\frac{[t^{(d)}2^k]}{2^k}\Big), \tag{4.3.6}$$

$\mathbf{t}_{-1}=\mathbf{0}$. 由条件 (4.3.1) 和 $\sigma(x)/x^\alpha$ 的单调性及定理 2.1.3 易知, $\{X(\mathbf{t});\mathbf{t}\in[0,1]^d\}$ a.s. 连续. 又由条件 (4.3.1) 知 $X(\mathbf{0})=0$ a.s., 从

而

$$|X(\mathbf{t})| = \lim_{k\to\infty} |X(\mathbf{t}_k) - X(\mathbf{0})| \le \sum_{k=1}^{\infty} |X(\mathbf{t}_k) - X(\mathbf{t}_{k-1})|.$$

因此

$$\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} |X(\mathbf{t})| \le \sum_{k=1}^{\infty} \sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} |X(\mathbf{t}_k) - X(\mathbf{t}_{k-1})|, \tag{4.3.7}$$

并且

$$\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} \|\mathbf{t}_k - \mathbf{t}_{k-1}\| \le \sqrt{d}\, 2^{-k}. \tag{4.3.8}$$

不妨设 $0 < x < 1/d$, 否则结论显然. 令整数 n_0 使得

$$1/x \le 2^{n_0} \le 2/x. \tag{4.3.9}$$

记 $N = N(0,1)$ 为标准正态变量. 定义

$$C := C_\alpha = (1 - 2^{-\alpha/2})/(2d^2), \tag{4.3.10}$$

$$x_k = C\sigma\Big(\Big(\frac{3}{2}\Big)^{-|k-n_0|}\sqrt{d}x\Big), \quad k = 1, 2, \cdots. \tag{4.3.11}$$

因 $\sigma(x)/x^\alpha$ 是非降的, 注意到 (4.3.2) 我们有

$$\begin{aligned}
\sum_{k=1}^{\infty} x_k &\le \frac{(1 - 2^{-\alpha/2})}{2d^2} \sum_{k=1}^{\infty} \Big(\frac{3}{2}\Big)^{-\alpha|k-n_0|} \sigma(\sqrt{d}x) \\
&\le \frac{(1 - 2^{-\alpha/2})}{d^2} \sigma(\sqrt{d}x) \sum_{k=1}^{\infty} \Big(\frac{3}{2}\Big)^{-\alpha k} \\
&\le (1 - 2^{-\alpha/2})\sigma(x) \sum_{k=1}^{\infty} \Big(\frac{3}{2}\Big)^{-\alpha k} \\
&\le \sigma(x).
\end{aligned} \tag{4.3.12}$$

注意到 $\mathrm{Card}\{\mathbf{t}_k : \mathbf{0} \le \mathbf{t} \le \mathbf{1}\} \le 2^{kd}$, 由定理 1.2.4 及 (4.3.7),(4.3.8),

(4.3.12) 有

$$
\begin{aligned}
&P\left(\sup_{0\le \mathbf{t}\le 1}|X(\mathbf{t})|\le \sigma(x)\right)\\
&\ge P\left\{\sum_{k=1}^{\infty}\sup_{0\le \mathbf{t}\le 1}|X(\mathbf{t}_k)-X(\mathbf{t}_{k-1})|\le \sigma(x)\right\}\\
&\ge P\left\{\sup_{0\le \mathbf{t}\le 1}|X(\mathbf{t}_k)-X(\mathbf{t}_{k-1})|\le x_k, k=1,2,\cdots\right\}\\
&\ge \prod_{k\ge 1}\left(P\left\{|N|\le \frac{x_k}{\sigma(\sqrt{d}2^{-k})}\right\}\right)^{2^{kd}}\\
&= \prod_{k\ge 1}\left(P\left\{|N|\le \frac{C\sigma\left(\left(\frac{3}{2}\right)^{-|k-n_0|}\sqrt{d}x\right)}{\sigma(\sqrt{d}2^{-k})}\right\}\right)^{2^{kd}}\\
&=: I_1\cdot I_2,
\end{aligned}
\tag{4.3.13}
$$

其中

$$
I_1=\prod_{k=1}^{n_0}P\left\{|N|\le \frac{C\sigma((\frac{3}{2})^{k-n_0}\sqrt{d}x)}{\sigma(\sqrt{d}2^{-k})}\right\}^{2^{kd}},
$$

$$
I_2=\prod_{k=n_0+1}^{\infty}P\left\{|N|\le \frac{C\sigma((\frac{3}{2})^{n_0-k}\sqrt{d}x)}{\sigma(\sqrt{d}2^{-k})}\right\}^{2^{kd}}.
$$

注意到

$$
P\{|N|\le t\}\ge t/2,\quad 若\ 0<t\le 1, \tag{4.3.14}
$$

$$
P\{|N|\le st\}\ge \exp\left(-\frac{1}{1-e^{-s^2/2}}e^{-(st)^2/2}\right),\ 若\ s>0,t\ge 1. \tag{4.3.15}
$$

回顾到 (4.3.9) 及 $\sigma(x)$ 是非降的，我们得

$$
\begin{aligned}
I_1&\ge \prod_{k=1}^{n_0}P\left\{|N|\le C\frac{\sigma((\frac{3}{2})^{k-n_0}\sqrt{d}2^{-n_0})}{\sigma(\sqrt{d}2^{-k})}\right\}^{2^{kd}}\\
&=\prod_{k=1}^{n_0}P\left\{|N|\le C\frac{\sigma(3^{k-n_0}\sqrt{d}2^{-k})}{\sigma(\sqrt{d}2^{-k})}\right\}^{2^{kd}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \prod_{k=1}^{n_0} P\Big\{|N| \leq C3^{2(k-n_0)}\Big\}^{2^{dk}} \qquad (\text{由 } (4.3.2))\\
&\geq \prod_{k=1}^{n_0} \Big(\frac{C}{2}3^{2(k-n_0)}\Big)^{2^{kd}} \qquad (\text{由 } (4.3.14))\\
&= \exp\Big(-\sum_{k=1}^{n_0} 2^{kd}(\log\frac{2}{C} + 2(n_0-k)\log 3)\Big)\\
&\geq \exp\Big(-2^{n_0 d}\sum_{l=0}^{\infty} 2^{-ld}(\log\frac{2}{C} + 2l\log 3)\Big)\\
&= \exp(-2^{n_0 d}D_\alpha) \geq \exp\Big(-2^d D_\alpha/x^d\Big), \qquad (4.3.16)
\end{aligned}
$$

其中 D_α 为仅依赖于 α 和 d 的常数.

现在来估计 I_2. 由于 $2^{-n_0} \leq x \leq 2^{-n_0+1}$ 且 $\sigma(x)/x^\alpha$ 是非降的, 我们有

$$
\begin{aligned}
I_2 &\geq \prod_{k=n_0+1}^{\infty} P\left\{|N| \leq C\frac{\sigma((\frac{3}{2})^{n_0-k}\sqrt{d}x)}{\sigma(2^{-(k-n_0)}\sqrt{d}x)}\right\}^{2^{kd}}\\
&\geq \prod_{k=n_0+1}^{\infty} P\left\{|N| \leq C\Big(\frac{4}{3}\Big)^{(k-n_0)\alpha}\right\}^{2^{kd}}\\
&\geq \prod_{k=n_0+1}^{\infty} \exp\Big(-\frac{2^{kd}}{1-e^{-C^2/2}}\exp\Big(-\frac{C^2}{2}\Big(\frac{4}{3}\Big)^{2(k-n_0)\alpha}\Big)\Big)\\
&\quad (\text{由 } (4.3.15))\\
&= \exp\Big(\frac{-2^{n_0 d}}{1-e^{-C^2/2}}\sum_{k=n_0+1}^{\infty} 2^{(k-n_0)d}\exp\Big(-\frac{C^2}{2}\Big(\frac{4}{3}\Big)^{2(k-n_0)\alpha}\Big)\Big)\\
&= \exp\Big(\frac{-2^{n_0 d}}{1-e^{-C^2/2}}\sum_{l=1}^{\infty} 2^{ld}\exp\Big(-\frac{C^2}{2}\Big(\frac{4}{3}\Big)^{2l\alpha}\Big)\Big)\\
&= \exp(-2^{n_0}D_\alpha) \geq \exp(-2^d D_\alpha/x^d). \qquad (4.3.17)
\end{aligned}
$$

由 (4.3.13), (4.3.16), (4.3.17) 我们得证 (4.3.3) 成立.

注 4.3.1 Talagrand (1993) 证明了如下小球概率估计: 设 $\{Y(\mathbf{t}); \mathbf{t} \in S\}$ 为一实值 Gauss 过程, 其中 $S \subset \mathcal{R}^d$ $(d \geq 1)$ 为

一给定的集合，并赋予距离 $d(\mathbf{s},\mathbf{t}) = \big(E(Y(\mathbf{s}) - Y(\mathbf{t}))^2\big)^{1/2}$. 记 $N_d(S,\epsilon)$ 为以半径为 ϵ 的 d 开球覆盖 S 所需的最小覆盖数. 若存在一函数 $\psi(\epsilon)$ 和一常数 $A > 0$, 使得对 $\forall\epsilon > 0$, $N_d(S,\epsilon) \le \psi(\epsilon)$ 且 $\psi(\epsilon)/A \le \psi(\epsilon/2) \le A\psi(\epsilon)$, 则存在常数 $C > 0$ 使得对任意的 $u \ge 0$, 有

$$P\Big(\sup_{\mathbf{s},\mathbf{t}\in S} |Y(\mathbf{t}) - Y(\mathbf{s})| \le u\Big) \ge \exp\Big(- C\psi(u)\Big).$$

定理 4.3.2 的证明 由定理的条件可得 $\sigma(0) = 0$, 故若 t_i 之一为零，就有 $X(\mathbf{t}) = 0$ a.s. 所以我们可写

$$X(\mathbf{t}) = \int_0^{\mathbf{t}} dX(\mathbf{s}).$$

令

$$X_{\mathbf{i},\mathbf{k}} = \int_{(\mathbf{i}-\mathbf{1})/2^{\mathbf{k}}}^{\mathbf{i}/2^{\mathbf{k}}} dX(\mathbf{s}), \quad \mathbf{1} \le \mathbf{i} \le 2^{\mathbf{k}}, \mathbf{i},\mathbf{k} \in Z^d.$$

由 (4.3.4) 即得

$$Var(X_{\mathbf{i},\mathbf{k}}) \le \sigma^2(2^{-k_1-\cdots-k_d}). \tag{4.3.18}$$

对 $0 \le a \le 1$, 记 $a_k = [a2^k]/2^k$, $k = 0,1,2,\cdots$. 由定理 2.1.3, X 的样本函数是 a.s. 连续的，从而对 $\mathbf{0} \le \mathbf{t} = (t^{(1)},\cdots,t^{(d)}) < \mathbf{1}$, 有

$$\begin{aligned}
|X(\mathbf{t})| &= \Big|\int_0^{t^{(d)}} \cdots \int_0^{t^{(1)}} dX(\mathbf{s})\Big| = \lim_{\mathbf{k}\to\infty} \Big|\int_0^{t^{(d)}_{k_d}} \cdots \int_0^{t^{(1)}_{k_1}} dX(\mathbf{s})\Big| \\
&= \Big|\sum_{k_d=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} \int_{t^{(d)}_{k_d-1}}^{t^{(d)}_{k_d}} \cdots \int_{t^{(1)}_{k_1-1}}^{t^{(1)}_{k_1}} dX(\mathbf{s})\Big| \\
&\le \sum_{k_d=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} \max_{\mathbf{1}\le\mathbf{i}\le 2^{\mathbf{k}}} \Big|\int_{(i_d-1)/2^{k_d}}^{i_d/2^{k_d}} \cdots \int_{(i_1-1)/2^{k_1}}^{i_1/2^{k_1}} dX(\mathbf{s})\Big| \\
&\le \sum_{\mathbf{1}\le\mathbf{k}<\infty} \max_{\mathbf{1}\le\mathbf{i}\le 2^{\mathbf{k}}} |X_{\mathbf{i},\mathbf{k}}| \quad \text{a.s.}
\end{aligned}$$

因此

$$\sup_{0\le \mathbf{t}\le \mathbf{1}} |X(\mathbf{t})| \le \sum_{\mathbf{1}\le \mathbf{k}<\infty} \max_{\mathbf{1}\le \mathbf{i}\le 2^{\mathbf{k}}} |X_{\mathbf{i},\mathbf{k}}|$$

$$\le \sum_{n=d}^{\infty} \sum_{\mathbf{1}\le \mathbf{k}<\infty, \hat{k}=n} \max_{\mathbf{1}\le \mathbf{i}\le 2^{\mathbf{k}}} |X_{\mathbf{i},\mathbf{k}}|, \qquad (4.3.19)$$

其中 $\hat{k} = k_1 + \cdots + k_d$. 设整数 n_0 使得 $1/2^{n_0} \le x \le 2/2^{n_0}$, 且记

$$x_n = \sigma(x(2/3)^{|n-n_0|}), \quad n \ge 1.$$

显然

$$\sum_{n=1}^{\infty} \sum_{\mathbf{1}\le \mathbf{k}<\infty, \hat{k}=n} x_n \le \sum_{n=1}^{\infty} n^{d-1} \sigma(x(2/3)^{|n-n_0|})$$

$$\le \sum_{n=1}^{\infty} n^{d-1} \sigma(x)(2/3)^{|n-n_0|\alpha} \le c n_0^{d-1} \sigma(x) \le c(\log(1/x))^{d-1} \sigma(x).$$

因此, 由定理 1.2.4 和 (4.3.18)

$$P\left\{ \sup_{0\le \mathbf{t}\le \mathbf{1}} |X(\mathbf{t})| \le c\left(\log \frac{1}{x}\right)^{d-1} \sigma(x) \right\}$$

$$\overset{KS}{\ge} \prod_{n=d}^{\infty} \prod_{\mathbf{1}\le \mathbf{k}<\infty, \hat{k}=n} \prod_{\mathbf{1}\le \mathbf{i}\le 2^{\mathbf{k}}} P\{|X_{\mathbf{i},\mathbf{k}}| \le x_n\}$$

$$\ge \prod_{n=d}^{\infty} \prod_{\mathbf{1}\le \mathbf{k}<\infty, \hat{k}=n} \prod_{\mathbf{1}\le \mathbf{i}\le 2^{\mathbf{k}}} P\{|N(0,1)| \le x_n/\sigma(2^{-n})\}$$

$$\ge \prod_{n=d}^{\infty} P\{|N(0,1)| \le x_n/\sigma(2^{-n})\}^{n^{d-1}2^n} =: K_1 \cdot K_2, \qquad (4.3.20)$$

其中

$$K_1 = \prod_{n=d}^{n_0-1} P\{|N(0,1)| \le \sigma(x(2/3)^{n_0-n})/\sigma(2^{-n})\}^{n^{d-1}2^n},$$

$$K_2 = \prod_{n=n_0}^{\infty} P\{|N(0,1)| \le \sigma(x(2/3)^{n-n_0})/\sigma(2^{-n})\}^{n^{d-1}2^n}.$$

由 (4.3.2) 和 (4.3.14), 我们得

$$
\begin{aligned}
K_1 &\geq \prod_{n=d}^{n_0-1} P\{|N(0,1)| \leq \sigma(2^{-n}3^{n-n_0})/\sigma(2^{-n})\}^{n^{d-1}2^n} \\
&\geq \prod_{n=d}^{n_0-1} P\{|N(0,1)| \leq 3^{2(n-n_0)}\}^{n^{d-1}2^n} \\
&\geq \prod_{n=d}^{n_0-1} (0.5\ 3^{n-n_0})^{n^{d-1}2^n} \\
&\geq \exp\left\{-c2^{n_0}\sum_{n=d}^{n_0-1}(n-n_0)n^{d-1}2^{n-n_0}\right\} \\
&\geq \exp\{-cn_0^{d-1}2^{n_0}\} \geq \exp\left\{-\frac{c\log^{d-1}(1/x)}{x}\right\}. \qquad (4.3.21)
\end{aligned}
$$

由 (4.3.15),

$$
\begin{aligned}
K_2 &\geq \prod_{n=n_0}^{\infty} P\{|N(0,1)| \leq \sigma(2^{-n}(4/3)^{n-n_0})/\sigma(2^{-n})\}^{n^{d-1}2^n} \\
&\geq \prod_{n=n_0}^{\infty} P\{|N(0,1)| \leq (4/3)^{(n-n_0)\alpha}\}^{n^{d-1}2^n} \\
&\geq \exp\left\{-3\sum_{n=n_0}^{\infty} n^{d-1}2^n \exp\Big(-(4/3)^{2(n-n_0)\alpha}/2\Big)\right\} \\
&\geq \exp\{-cn_0^{d-1}2^{n_0}\} \geq \exp\left\{-\frac{c\log^{d-1}(1/x)}{x}\right\}. \qquad (4.3.22)
\end{aligned}
$$

由 (4.3.20)—(4.3.22) 就证明了 (4.3.5).

现在让我们应用定理 4.3.1 和定理 4.3.2 来获得关于分数 Lévy-Wiener 场和 Wiener 单的小球概率估计. 我们有

定理 4.3.3 设 $d \geq 1, \{Z(\mathbf{t}); \mathbf{t} \in R^d\}$ 是 α 阶 $(0 < \alpha < 1)$ 分数 Lévy-Wiener 场, 即对所有 $\mathbf{s}, \mathbf{t} \geq 0$, $E(Z(\mathbf{s}) - Z(\mathbf{t}))^2 = \|\mathbf{s} - \mathbf{t}\|^{2\alpha}$. 那么存在仅依赖于 α 和 d 的常数 $0 < c_1 \leq c_2 < \infty$ 使得对任何

$0 < x < 1$ 有

$$\exp\left(-\frac{c_2}{x^{d/\alpha}}\right) \overset{KS}{\leq} P\left\{\sup_{0\leq \mathbf{t}\leq 1}|Z(\mathbf{t})| \leq x\right\} \leq \exp\left(-\frac{c_1}{x^{d/\alpha}}\right). \quad (4.3.23)$$

证明 由定理 4.3.1 即得 (4.3.23) 的下界. 下面我们来建立上界. 对 $\mathbf{i} = (i_1, \cdots, i_d) \in Z^d$, 写 $t_{\mathbf{i}} = \mathbf{i}x^{1/\alpha}$. 显然

$$P\left\{\sup_{0\leq \mathbf{t}\leq 1}|Z(\mathbf{t})| \leq x\right\} \leq P\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}}|Z(t_{\mathbf{i}})| \leq x\right\},$$

且对 $1 \leq \mathbf{j} \leq x^{-1/\alpha}$

$$\begin{aligned} P\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}}|Z(t_{\mathbf{i}})| \leq x\right\} = EI\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}, \mathbf{i}\neq \mathbf{j}}|Z(t_{\mathbf{i}})| \leq x\right\} \\ \cdot P\left\{|Z(t_{\mathbf{j}})| \leq x \middle| Z(t_{\mathbf{i}}), 1 \leq \mathbf{i} \leq x^{-1/\alpha}, \mathbf{i} \neq \mathbf{j}\right\}. \end{aligned}$$

借助于 Pitt (1978) 的引理 7.1, 存在一个正常数 $C = C(\alpha, d)$, 使得

$$\begin{aligned} &\mathrm{Var}(Z(t_{\mathbf{j}})|Z(t_{\mathbf{i}}), 1 \leq \mathbf{i} \leq x^{-1/\alpha}, \mathbf{i} \neq \mathbf{j}) \\ &\quad \geq \mathrm{Var}(Z(t_{\mathbf{j}})|Z(\mathbf{s}), \|\mathbf{s} - t_{\mathbf{j}}\| \geq x^{1/\alpha}\} = Cx^2. \end{aligned}$$

因此

$$\begin{aligned} &P\left\{|Z(t_{\mathbf{j}})| \leq x \middle| Z(t_{\mathbf{i}}), 1 \leq \mathbf{i} \leq x^{-1/\alpha}, \mathbf{i} \neq \mathbf{j}\right\} \\ &\quad \leq P\left\{|N(0,1)| \leq 1/\sqrt{C}\right\} < 1 \end{aligned}$$

且

$$\begin{aligned} &P\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}}|Z(t_{\mathbf{i}})| \leq x\right\} \\ &\quad \leq P\{|N(0,1)| \leq 1/\sqrt{C}\}P\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}, \mathbf{i}\neq \mathbf{j}}|Z(t_{\mathbf{i}})| \leq x\right\}. \end{aligned}$$

这样, 由递推可得

$$\begin{aligned} &P\left\{\max_{1\leq \mathbf{i}\leq x^{-1/\alpha}}|Z(t_{\mathbf{i}})| \leq x\right\} \\ &\quad \leq P\{|N(0,1)| \leq 1/\sqrt{C}\}^{(x^{-1/\alpha}-1)^d} \leq \exp(-c/x^{d/\alpha}), \end{aligned}$$

定理 4.3.3 证毕.

推论 4.3.1 设 $\{W(\mathbf{t}); \mathbf{t} \in R^d\}$ 是标准 Wiener 单. 那么

$$P\left\{\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} |W(\mathbf{t})| \le x\right\} \overset{KS}{\ge} \exp\left(-\frac{c(\log(1/x))^{3(d-1)}}{x^2}\right). \quad (4.3.24)$$

特别地, 若 $d=2$, 则对某常数 $C>0$ 和任何 $0<x<1/2$

$$\exp\left(-\frac{C(\log(1/x))^3}{x^2}\right) \overset{KS}{\le} P\left\{\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} |W(\mathbf{t})| \le x\right\}$$
$$\le \exp\left(-\frac{(\log(1/x))^3}{Cx^2}\right). \quad (4.3.25)$$

证明 由定理 4.3.2 即得 (4.3.24) 式. (4.3.25) 式中的上界的证明从略 (参见 Talagrand, 1994).

注 4.3.2 给出在 (4.3.24) 中的下界与 Bass (1988) 中的结果一致. (4.3.25) 是由 Talagrand (1994) 得到的, 它是不可改进的. Shao 和 Wang (1995) 猜测对 $d>2$, (4.3.24) 的下界也是不可改进的.

注 4.3.3 令人惊奇的是: 从定理 4.3.3 和推论 4.3.1 可见, 分数 Lévy-Wiener 场的小球概率与 Wiener 单的小球概率完全不同. 这表示它们的下极限行为也必定十分不同, 我们将在下一小节中看到这一点.

注 4.3.4 Kôno (1976) 给出了某些 Gauss 场的小球概率的上界. 但对具有平稳增量 (即 $E(X(\mathbf{s})-X(\mathbf{t}))^2=\sigma^2(\|\mathbf{s}-\mathbf{t}\|)$) 的一般 Gauss 场尚未获得. 我们猜测若 $\sigma^2(\cdot)$ 在 $(0,1)$ 上是凹函数, 那么 (4.3.3) 中给出的界是不可改进的.

4.3.2 Gauss 场的 Chung 重对数律

设 $d\ge 1$, $\{X(\mathbf{t}); \mathbf{t}\in[0,1]^d\}$ 是中心化的 Gauss 场且在下述意义下具有平稳增量: 即对任何 $\mathbf{0}\le\mathbf{s},\mathbf{t}\le\mathbf{1}$,

$$E|X(\mathbf{s})-X(\mathbf{t})|^2=\sigma^2(\|\mathbf{s}-\mathbf{t}\|), \quad (4.3.26)$$

其中 $\sigma(\cdot)$ 是 $[0,1]$ 上的非降连续函数. 在此我们来讨论 $\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{h}}|X(\mathbf{t})|$ 的下极限行为. 利用小球概率的上界, 我们可以较容易地导出下极限的下界. 然而, 从小球概率估计导出下极限的上界却不是那么容易. Shao 和 Wang(1995) 发现利用 Monrad 和 Rootzen (1995) 的方法和定理 4.3.1 可建立一个十分一般的结果.

定理 4.3.4 设 $\{X(\mathbf{t});\mathbf{t}\in[0,1]^d\}$ 是中心化的具有平稳增量的 Gauss 过程. 假设 $X(\mathbf{0})=0$, 对某 $\beta>0$, $\sigma(x)/x^\beta$ 在 $[0,1]$ 上是非降的, 且存在 $0<\theta<2$, 使对任何 $0<h<1/2$

$$\sigma(2h)\le\theta\sigma(h). \tag{4.3.27}$$

那么存在 $c>0$ 使得

$$\liminf_{h\to0}\frac{\sup_{\mathbf{0}\le\mathbf{t}\le h}|X(\mathbf{t})|}{\sigma(h(c/\log\log(1/h))^{1/d})}\le1\quad\text{a.s.} \tag{4.3.28}$$

定理 4.3.5 设 $\{X(\mathbf{t});\mathbf{t}\in[0,1]^d\}$ 是满足定理 4.3.4 中的条件的中心化 Gauss 场. 若还存在 $0<c_1\le c_2<\infty$ 使得对某 $0<h_0<1$, 对任何 $0\le x\le h_0h\le h_0^2$ 有

$$\exp\left(-c_2(h/x)^d\right)\le P\left\{\sup_{\mathbf{0}\le\mathbf{t}\le h}|X(\mathbf{t})|\le\sigma(x)\right\}\le\exp\left(-c_1(h/x)^d\right), \tag{4.3.29}$$

那么

$$1\le\liminf_{h\to0}\frac{\sup_{\mathbf{0}\le\mathbf{t}\le h}|X(\mathbf{t})|}{\sigma(h(c_1/\log\log(1/h))^{1/d})}\quad\text{a.s.} \tag{4.3.30}$$

且

$$\liminf_{h\to0}\frac{\sup_{\mathbf{0}\le\mathbf{t}\le h}|X(\mathbf{t})|}{\sigma(h(c_2/\log\log(1/h))^{1/d})}\le1\quad\text{a.s.} \tag{4.3.31}$$

特别地, 由定理 4.3.4, 对分数 Lévy-Wiener 场有下述 Chung 重对数律.

定理 4.3.6 设 $\{Z(\mathbf{t}); \mathbf{t} \in [0.1]^d\}$ 是阶为 α $(0 < \alpha < 1)$ 的分数 Lévy-Wiener 场. 那么对某 $0 < C < \infty$ 有

$$\liminf_{h \to 0} \frac{(\log\log(1/h))^{\alpha/d}}{h^\alpha} \sup_{0 \le \mathbf{t} \le h} |Z(\mathbf{t})| = C \quad \text{a.s.}$$

定理 4.3.4 的证明 记

$$M(h) = \sup_{0 \le \mathbf{t} \le h} |X(\mathbf{t})|.$$

由 (4.3.27) 即得, 存在 $0 < \delta := \delta_{\theta,\beta} \le \beta$ 使得对每一 $0 < h \le 1$ 和整数 k, $1 \le k \le 1/h$ 有

$$\sigma(kh) \le 2k^{1-\delta}\sigma(h). \tag{4.3.32}$$

由 (4.3.26) 和 Minkowski 不等式, 对任意的 $1 \le a < 2$ 和 $0 < h < 1/2$, 我们有

$$\sigma(ah) \le \sigma(h) + \sigma((a-1)h) \le (1 + (a-1)^\beta)\sigma(h). \tag{4.3.33}$$

这样, 定理 4.3.1 的条件被满足, 因此存在 $c_0 > 0$ 使得对任意的 $0 < x \le h \le 1$ 有

$$P\Big\{M(h) \le \sigma(x)\Big\} \ge \exp\Big(-c_0(h/x)^d\Big). \tag{4.3.34}$$

我们来证

$$\liminf_{h \to 0} M(h)/\sigma\big(h(c_0/\log\log(1/h))^{1/d}\big) \le 1 \quad \text{a.s.}$$

对任意的 $0 < \varepsilon < 1$, 令

$$t_k = e^{-k^{1+\varepsilon}}, \quad d_k = e^{k^{1+\varepsilon}+k^\varepsilon}, \quad \sigma_k = \sigma\big(t_k(c_0/\log\log(1/t_k))^{1/d}\big). \tag{4.3.35}$$

只需证明

$$\liminf_{k \to \infty} M(t_k)/\sigma_k \le 1 + \varepsilon^\beta \quad \text{a.s.} \tag{4.3.36}$$

就够了. 为此, 我们利用 X 的谱表示. 下面以 $\mathbf{s}\cdot\mathbf{v}$ 或 $\langle\mathbf{s},\mathbf{v}\rangle$ 记 $\sum_{i=1}^d s_iv_i$. 已知 $EX(\mathbf{s})X(\mathbf{t})$ 有形如如下的唯一 Fourier 表示

$$E\{X(\mathbf{s})X(\mathbf{t})\}=\int_{R^d}(e^{i\mathbf{s}\cdot\mathbf{v}}-1)(e^{-i\mathbf{t}\cdot\mathbf{v}}-1)\Delta(d\mathbf{v})+\langle\mathbf{s},B\mathbf{t}\rangle. \quad (4.3.37)$$

其中 $B=(b_{ij})$ 是半正定矩阵, $\Delta(\cdot)$ 是 $R^d-\{\mathbf{0}\}$ 上的非负测度, 满足

$$\int_{R^d}\frac{\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2}\Delta(d\mathbf{v})<\infty.$$

进一步, 存在一中心化的复值 Gauss 随机测度 $W(\cdot)$ 和与 W 独立的 Gauss 随机向量 $\mathbf{Y}$, 使得

$$X(\mathbf{t})=\int_{R^d}(e^{i\mathbf{t}\cdot\mathbf{v}}-1)W(d\mathbf{v})+\mathbf{Y}\cdot\mathbf{t}. \quad (4.3.38)$$

测度 W 和 Δ 有关系式: 对 R^d 中的一切 Borel 集 A 和 B

$$E\{W(A)\overline{W(B)}\}=\Delta(A\cap B).$$

而且

$$W(-A)=\overline{W(A)}.$$

由 (4.3.37), (4.3.26) 式即得

$$\sigma^2(\|\mathbf{t}-\mathbf{s}\|)=2\int_{R^d}(1-\cos((\mathbf{t}-\mathbf{s})\cdot\mathbf{v}))\Delta(d\mathbf{v})+\langle\mathbf{t},B\mathbf{t}\rangle.$$

特别地, 对 $0<h<1$ 和每一 $i=1,\cdots,d$,

$$\sigma^2(h)=2\int_{R^d}(1-\cos(hv_i))\Delta(d\mathbf{v})+t_i^2b_{ii}\geq 2\int_{R^d}(1-\cos(hv_i))\Delta(d\mathbf{v}). \quad (4.3.39)$$

对 $0<h<1$ 和 $1\leq i\leq d$, 我们有

$$\begin{aligned}\int_{R^d,|v_i|\geq 1/h}\Delta(d\mathbf{v})&\leq\frac{1}{1-\sin 1}\int_{R^d,|v_i|\geq 1/h}\Big(1-\frac{\sin(hv_i)}{hv_i}\Big)\Delta(d\mathbf{v})\\&=\frac{1}{(1-\sin 1)h}\int_{R^d,|v_i|\geq 1/h}\int_0^h(1-\cos(uv_i))du\Delta(d\mathbf{v})\\&=\frac{1}{(1-\sin 1)h}\int_0^h\int_{R^d,|v_i|\geq 1/h}(1-\cos(uv_i))\Delta(d\mathbf{v})du\\&\leq 4\sigma^2(h).\end{aligned}$$

因此

$$\int_{\|\mathbf{V}\|\geq 1/h}\Delta(d\mathbf{v})\leq 4d\sigma^2(dh)\leq 4d^3\sigma(h). \tag{4.3.40}$$

类似地，由 (4.3.39)

$$\begin{aligned}\int_{\|\mathbf{V}\|\leq 1/h}\Delta(d\mathbf{v}) &\leq dh^{-2}\sum_{i=1}^{d}\int_{R^d,|v_i|\leq 1/h}(hv_i)^2\Delta(d\mathbf{v})\\ &\leq 4dh^{-2}\sum_{i=1}^{d}\int_{R^d,|v_i|\leq 1/h}(1-\cos(hv_i))\Delta(d\mathbf{v})\\ &\leq 4d^2h^{-2}\sigma^2(h).\end{aligned} \tag{4.3.41}$$

对 $k=1,2,\cdots$ 和 $\mathbf{0}\leq\mathbf{t}\leq\mathbf{1}$ 定义

$$X_k(\mathbf{t})=\int_{\|\mathbf{v}\|\in(d_{k-1},d_k]}\left(e^{i\mathbf{t}\cdot\mathbf{v}}-1\right)W(d\mathbf{v}),$$

$$\tilde{X}_k(\mathbf{t})=\int_{\|\mathbf{v}\|\notin(d_{k-1},d_k]}\left(e^{i\mathbf{t}\cdot\mathbf{v}}-1\right)W(d\mathbf{v}).$$

显然

$$\begin{aligned}\liminf_{k\to\infty}\frac{M(t_k)}{\sigma_k}\leq &\liminf_{k\to\infty}\frac{\sup_{\mathbf{0}\leq\mathbf{t}\leq t_k}|X_k(\mathbf{t})|}{\sigma_k}\\ &+\liminf_{k\to\infty}\frac{\sup_{\mathbf{0}\leq\mathbf{t}\leq t_k}|\tilde{X}_k(\mathbf{t})|}{\sigma_k}\\ &+d\ \limsup_{k\to\infty}\frac{t_k\|\mathbf{Y}\|}{\sigma_k}.\end{aligned} \tag{4.3.42}$$

由 (4.3.35) 和 (4.3.32) 易知

$$\begin{aligned}\limsup_{k\to\infty}t_k\|\mathbf{Y}\|/\sigma_k &\leq \|Y\|\limsup_{k\to\infty}t_k\Big((\log\log(1/t_k))^{1/d}/(t_kc_0^{1/d})\Big)^{1-\delta}\\ &=0\ \text{ a.s.}\end{aligned} \tag{4.3.43}$$

由 Anderson 不等式 (推论 1.2.4)

$$P\left\{\sup_{\mathbf{0}\leq\mathbf{t}\leq t_k}|X_k(\mathbf{t})|\leq(1+\varepsilon^\beta)\sigma_k\right\}\geq P\left\{M(t_k)\leq(1+\varepsilon^\beta)\sigma_k\right\}.$$

因此由 (4.3.33), (4.3.34) 和 (4.3.35) 得

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\Big\{\sup_{0\le \mathbf{t}\le t_k} |X_k(\mathbf{t})| \le (1+\varepsilon^{\beta})\sigma_k\Big\} \\
&\ge \sum_{k=1}^{\infty} P\Big\{M(t_k) \le \sigma\Big(t_k(1+\varepsilon)(c_0/\log\log(1/t_k))^{1/d}\Big)\Big\} \\
&\ge \sum_{k=1}^{\infty} \exp\Big\{-(1+\varepsilon)^{-d}\log\log(1/t_k)\Big\} = \infty.
\end{aligned}
\tag{4.3.44}
$$

因 $\{\sup_{0\le \mathbf{t}\le t_k} |X_k(\mathbf{t})|; k\ge 1\}$ 是独立的, 由 Borel-Cantelli 引理和 (4.3.44) 即得

$$
\liminf_{k\to\infty} \sup_{0\le \mathbf{t}\le t_k} |X_k(\mathbf{t})|/\sigma_k \le 1+\varepsilon^{\beta} \quad \text{a.s.} \tag{4.3.45}
$$

其次, 我们来估计 (4.3.42) 右边的第二项. 由 (4.3.40),(4.3.41), (4.3.35) 和 (4.3.32), 我们得到对 $\mathbf{0}\le \mathbf{t}\le t_k$

$$
\begin{aligned}
\mathrm{Var}(\tilde{X}_k(\mathbf{t})) &= 2\int_{\|\mathbf{v}\|\notin(d_{k-1},d_k]} (1-\cos(\mathbf{t}\cdot\mathbf{v}))\Delta(d\mathbf{v}) \\
&\le \int_{\|\mathbf{v}\|\le d_{k-1}} \|\mathbf{t}\|^2\|\mathbf{v}\|^2\Delta(d\mathbf{v}) + 4\int_{\|\mathbf{v}\|\ge d_k}\Delta(d\mathbf{v}) \\
&\le 4d^3t_k^2d_{k-1}^2\sigma^2(t_k/(t_kd_{k-1})) + d^3\sigma^2(t_k/(t_kd_k)) \\
&\le 4d^3(t_kd_{k-1})^{2\delta}\sigma^2(t_k) + 4d^3(t_kd_k)^{-2\beta}\sigma^2(t_k) \\
&\le 8d^4e^{-\varepsilon k^{\varepsilon}}\sigma^2(t_k).
\end{aligned}
$$

所以对任意的 $0<\mathbf{s},\mathbf{t}\le t_k$, $\|\mathbf{s}-\mathbf{t}\|\le h\le t_k$ 有

$$
\mathrm{Var}(\tilde{X}_k(\mathbf{s})-\tilde{X}_k(\mathbf{t})) \le \tilde{\sigma}_k^2(h), \tag{4.3.46}
$$

其中 $\tilde{\sigma}_k^2(h)=\min(\sigma^2(h), 16d^4e^{-\varepsilon k^{\varepsilon}}\sigma^2(t_k))$. 注意到

$$
\begin{aligned}
\int_1^{\infty}\tilde{\sigma}_k(t_ke^{-y^2})dy &\le \int_1^{\infty}\min(4d^2e^{-\varepsilon k^{\varepsilon}/2}\sigma(t_k), \sigma(t_ke^{-y^2}))dy \\
&\le \int_1^{\infty}\min(4d^2e^{-\varepsilon k^{\varepsilon}/2}\sigma(t_k), \sigma(t_k)e^{-\beta y^2})dy \le Kke^{-\varepsilon k^{\varepsilon}/2}\sigma(t_k),
\end{aligned}
$$

其中 K 是仅依赖于 d 和 β 的常数. 应用 Fernique 不等式 (定理 1.1.3), 对任意的 $\eta > 0$ 得

$$\begin{aligned}P\left\{\sup_{0\le\mathbf{t}\le t_k}|\tilde{X}_k(\mathbf{t})|>\eta\sigma_k\right\}&\le K\exp\left(-\frac{(\eta\sigma_k)^2}{Kk\exp(-\varepsilon k^\varepsilon)\sigma(t_k)}\right)\\&\le K\exp\left(-\frac{\eta^2(\log\log(1/t_k))^{-2/d}}{Kk\exp(\varepsilon k^\varepsilon)c_0^2}\right)\\&\le K\exp\left(-\frac{\eta^2\exp(\varepsilon k^\varepsilon)}{Kk^2c_0^2}\right).\end{aligned}$$

这样由 Borel-Cantelli 引理

$$\limsup_{k\to\infty}\sup_{0\le\mathbf{t}\le t_k}|\tilde{X}_k(\mathbf{t})|/\sigma_k=0\quad\text{a.s.}\tag{4.3.47}$$

由 (4.3.42), (4.3.43), (4.3.45) 和 (4.3.47) 得证 (4.3.36).

定理 4.3.5 的证明 利用 (4.3.29) 和子序列方法, 易证 (4.3.30). 由定理 4.3.4 的证明可得 (4.3.31).

定理 4.3.6 的证明 由 Pitt 和 Tran (1979) 的 0-1 律知 $Z(\mathbf{t})$ 在 $\mathbf{t}=\mathbf{0}$ 处满足 0-1 律. 从而由定理 4.3.3 和 4.3.5 即得定理的结论.

注 4.3.5 对 $\mathbf{v}\in R^d$ 和 $h>0$, 我们记 $B(\mathbf{v},h)=\{\mathbf{x}\in R^d;\|\mathbf{v}-\mathbf{x}\|\le h\}$ 是以 $\mathbf{v}$ 为中心, h 为半径的闭球. 设 Δ 为 (4.3.37) 所示的谱测度. 若对某 $h>0$

$$\liminf_{\|\mathbf{v}\|\to\infty}\|\mathbf{v}\|^{d+2}\Delta(B(\mathbf{v},h))>0,\tag{4.3.48}$$

则 $X(\mathbf{t})$ 在 $\mathbf{t}=\mathbf{0}$ 处满足 0-1 律 (见 Pitt 和 Tran 1979). 因此若定理 4.3.5 中的条件和 (4.3.48) 被满足, 则存在常数 c 使得 $c_1\le c\le c_2$ 且

$$\liminf_{h\to0}\frac{\sup_{\mathbf{0}\le\mathbf{t}\le h}|X(\mathbf{t})|}{\sigma(h(c/\log\log(1/h))^{1/d})}=1\quad\text{a.s.}$$

4.3.3 应用于分数 Wiener 过程和 O-U 过程无穷级数

设 $Z(t)$ 为阶为 α $(0<\alpha<1)$ 的分数 Wiener 过程，现在我们给出它的 Chung 重对数律的证明.

定理 4.2.2 的证明 (4.2.7) 是定理 4.3.6 的特例. 下证 (4.2.8). 由 (4.2.6) 右边的不等式和子序列方法易证

$$\liminf_{T\to\infty}\Big(\frac{\log\log T}{cT}\Big)^{\alpha}\sup_{0\le t\le T}|Z(t)|\ge 1 \quad \text{a.s.} \tag{4.3.49}$$

另一方面，对任意的 $0<\epsilon<1$, 取 $p>1$ 使得 $(1-\epsilon)p<1$. 令 $T_n=e^{n^p}$. 设 $X_n(t)$ 和 $\tilde{X}_n(t)$ 如 (2.2.19) 所定义. 则 $X_n(t)$ 与 $\tilde{X}_n(t)$ 独立, 且 $\{X_n(\cdot)\}_{n=1}^{\infty}$ 相互独立. 由 Anderson 不等式 (见推论 1.2.4) 和 (4.2.6) 左边的不等式得

$$\begin{aligned}
&P\bigg(\Big(\frac{(1-\epsilon)\log\log T_n}{CT_n}\Big)^{\alpha}\sup_{0\le t\le T_n}|X_n(t)|\ge 1\bigg)\\
&\quad\ge P\bigg(\Big(\frac{(1-\epsilon)\log\log T_n}{CT_n}\Big)^{\alpha}\sup_{0\le t\le T_n}|Z(t)|\ge 1\bigg)\\
&\quad\ge(\log T_n)^{-(1-\epsilon)}=n^{-(1-\epsilon)p}.
\end{aligned}$$

从而由 Borel-Cantelli 引理得

$$\liminf_{n\to\infty}\bigg(\frac{(1-\epsilon)\log\log T_n}{CT_n}\bigg)^{\alpha}\sup_{0\le t\le T_n}|X_n(t)|\le 1 \quad \text{a.s.} \tag{4.3.50}$$

另外，与 (2.2.20) 类似可证

$$\sum_{n=1}^{\infty}P\bigg(\Big(\frac{\log\log T_n}{T_n}\Big)^{\alpha}\sup_{0\le t\le T_n}|\tilde{X}_n(t)|\ge\epsilon\bigg)<\infty.$$

由 Borel-Cantelli 引理得

$$\liminf_{n\to\infty}P\bigg(\Big(\frac{\log\log T_n}{T_n}\Big)^{\alpha}\sup_{0\le t\le T_n}|\tilde{X}_n(t)|\le\epsilon. \tag{4.3.51}$$

结合 (4.3.50) 和 (4.3.51), 并注意到 $Z(t) = X_n(t) + \tilde{X}_n(t)$ 得

$$\liminf_{T\to\infty} \left(\frac{\log\log T}{CT}\right)^{\alpha} \sup_{0\le t\le T} |Z(t)| \le 1 \quad \text{a.s.} \tag{4.3.52}$$

最后, 注意到 $\{Z(t); t > 0\}$ 和 $\{t^{2\alpha} Z(t^{-1}); t > 0\}$ 是两个等价的过程, 故 $Z(t)$ 在 $t = \infty$ 和 $t = 0$ 处都满足 Pitt 和 Tran (1979) 的 0-1 律. 因此, 由 (4.3.49) 和 (4.3.52) 得证 (4.2.8).

对 O-U 过程无穷级数 $X(t) = \sum_{k=1}^{\infty} X_k(t)$ 的 Chung 重对数律已被给出于定理 4.2.4 中.

定理 4.2.4 的证明 事实上, 从 $0 < \sum_{k=1}^{\infty} \gamma_k/\lambda_k < \infty$ 可推得 $X(\cdot)$ 是平稳的 Gauss 过程, 且 $\sigma^2(h)$ 在 $(0,\infty)$ 上是凹函数. 设 $f(v)$ 是 (4.3.37) 中 Δ 的谱密度. 易知

$$f(v) = \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} \cdot \frac{\lambda_k}{\pi(\lambda_k^2 + v^2)}.$$

故 (4.3.48) 成立, 从而 $X(\cdot)$ 满足在 $t = 0$ 处的 0-1 律. 另一方面, 由定理 4.3.1 和定理 4.2.1, (4.3.29) 成立. 因此由定理 4.3.5 就有定理 4.2.4.

§4.4 Gauss 过程增量的下极限

在本节中, 我们讨论有关 Gauss 过程增量的下极限. 我们先介绍 Csörgő 和 Shao (1994) 的结果, 在那里他们讨论了 $\inf_{0\le t\le b_T} \sup_{0\le s\le a_T} \|\Gamma(t+s) - \Gamma(t)\|$ 的下极限的一般准则. 这与 $\Gamma(\cdot)$ 的不可微性有关. 在 4.4.1 小节中, 我们叙述这一准则. 在 4.4.2 小节中, 将它运用于具有平稳增量的 Gauss 过程的研究中. 在 4.4.3 小节中, 我们对独立 O-U 过程的无穷级数建立了下极限的精确界. 由此可知通过一般准则给出的界是不可改进的.

4.4.1 关于下极限的准则

设 B 是具有范数 $\|\cdot\|$ 的可分 Banach 空间, $\{\Gamma(t); -\infty < t < \infty\}$ 是取值于 B 中的随机过程, P 是由 $\Gamma(\cdot)$ 产生的概率测度.

定理 4.4.1 设 a_T 和 b_T 是非负连续函数, $v(t)$ 是非负单调函数. 假设

$$\frac{1+b_T}{a_T}+a_T\to\infty,\quad \text{当}\quad T\to\infty\text{ 时},\tag{4.4.1}$$

又设存在 $C>0$ 和 $d>0$ 使得对每一 $0\le t\le 2b_T$,

$$\frac{da_T}{4(\log(b_T/a_T)+\log\log\tilde{a}_T)}\le x\le\frac{4da_T}{(\log(b_T/a_T)+\log\log\tilde{a}_T)},$$

成立

$$P\Big\{\sup_{0\le s\le a_T/2}\|\Gamma(t+s)-\Gamma(t)\|\le v(t)\Big\}\le c\exp(-da_T/x),\tag{4.4.2}$$

其中 $\tilde{x}=x+1/x$. 那么, 我们有

$$\liminf_{T\to\infty}\inf_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(da_T/2(\log(b_T/a_T)+\log\log\tilde{a}_T))}\ge\frac{1}{2}\quad\text{a.s.}\tag{4.4.3}$$

定理 4.4.1 的证明从略.

容易看出, 若 $a_T\to 0$ 或 $a_T\to\infty$, 或 $b_T\to\infty$ (当 $T\to\infty$), 则 (4.4.1) 被满足, 因此, 作为特殊情形, (4.4.3) 包含了通常的大增量和小增量.

下例说明了上述定理的一般性. 设 $\{W(t);\ t\ge 0\}$ 是标准 Wiener 过程. 众所周知 (参见 Csörgő 和 Révész 1981), 对任何 $t\ge 0$, $a_T>0$, $0<x\le a_T$ 有

$$P\Big\{\sup_{0\le s\le a_T/2}|W(t+s)-W(t)|\le x^{1/2}\Big\}\le 2\exp\Big(-\frac{\pi^2a_T}{16x}\Big),$$

所以, 由定理 4.4.1, 在 (4.4.1) 被满足时

$$\liminf_{T\to\infty}\inf_{0\le t\le b_T}\sup_{0\le s\le a_T}\frac{|W(t+s)-W(t)|}{(\pi^2a_T/32(\log(b_T/a_T)+\log\log\tilde{a}_T))^{1/2}}\ge\frac{1}{2}\quad\text{a.s.}\tag{4.4.4}$$

特别地

$$\liminf_{h\to 0}\sup_{0\le s\le h}\frac{|W(s)|}{(h/\log\log h^{-1})^{1/2}}\ge\frac{\pi}{8\sqrt{2}}\text{ a.s.}\quad (b_T=0,a_T=h),$$

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{|W(t+s)-W(t)|}{(h/\log h^{-1})^{1/2}}\ge\frac{\pi}{8\sqrt{2}}\text{ a.s.}\quad (b_T=1,a_T=h),$$

$$\liminf_{T\to\infty}\sup_{0\le s\le T}\frac{|W(s)|}{(T/\log\log T)^{1/2}}\ge\frac{\pi}{\sqrt{24}}\text{ a.s.}\quad (b_T=0,a_T=T),$$

$$\liminf_{T\to\infty}\inf_{0\le t\le T}\sup_{0\le s\le a_T}\frac{|W(t+s)-W(t)|}{(a_T/(\log T/a_T+\log\log \tilde{a}_T))^{1/2}}$$
$$\ge\frac{\pi}{8\sqrt{2}}\text{ a.s. }(b_T=T),$$

$$\liminf_{T\to\infty}\inf_{0\le t\le T}\sup_{0\le s\le 1}(\log T)^{1/2}|W(t+s)-W(t)|$$
$$\ge\frac{\pi}{8\sqrt{2}}\text{ a.s.}\quad (b_T=T,a_T=1).$$

众所周知，若不计常数因子，所有上述下极限结果是不可改进的 (参见 Csörgő 和 Révész 1981). 这些重要例子说明了定理 4.4.1 中的速度是不可改进的，当然，这并不是说在所有情形下，(4.4.3) 的一般速度函数一定是不可改进的.

4.4.2 具有平稳增量的 Gauss 过程

作为定理 4.4.1 的一个应用，我们来研究实值 Gauss 过程的一般下极限问题. 设 $\{G(t);t\ge 0\}$ 是零均值且具有平稳增量的实 Gauss 过程. 令

$$\sigma^2(h)=E(G(t+h)-G(t))^2,\ t,h\ge 0. \qquad (4.4.5)$$

假设 $\sigma^2(h)$ 在 (0,1) 上是非降的凹函数. 那么由定理 4.2.1, 对所有 $0<x<1$ 我们有

$$P\left\{\sup_{0\le t\le 1}|G(t)-G(0)|\le\sigma(x)\right\}\le 2\exp(-0.17/x). \qquad (4.4.6)$$

结合 (4.4.6) 和定理 4.4.1 我们得下述定理.

定理 4.4.2 设 $\{a_T;t\geq 0\}$ 和 $\{b_T;T\geq 0\}$ 是满足 (4.4.1) 的非负连续函数，$\{G(t);t\geq 0\}$ 是零均值且具有平稳增量的 Gauss 过程. 令 $a^*=\sup_{T>0}a_T$. 假设 $\sigma^2(h)$ 在 $(0,a^*)$ 上是非降的凹函数. 那么

$$\liminf_{T\to\infty}\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\frac{|G(t+s)-G(t)|}{\sigma(a_T/24(\log(b_T/a_T)+\log\log\tilde{a}_T))}\geq\frac{1}{2}\quad \text{a.s.} \tag{4.4.7}$$

对分数 Wiener 过程，定理 4.4.2 的一个直接推论如下.

定理 4.4.3 设 $\{Z(t);t\geq 0\}$ 是阶为 α, $0<\alpha<1/2$, 的分数 Wiener 过程，亦即 $\sigma^2(h)=h^{2\alpha}$. 那么当 (4.4.1) 被满足时有

$$\liminf_{T\to\infty}\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\left(\frac{\log(b_T/a_T)+\log\log\tilde{a}_T}{a_T}\right)^{\alpha}|Z(t+s)-Z(t)|$$
$$\geq 0.1\quad \text{a.s.} \tag{4.4.8}$$

特别地，我们有

$$\liminf_{h\to 0}\sup_{0\leq s\leq h}\left(\frac{\log\log(1/h)}{h}\right)^{\alpha}|Z(s)|\geq 0.1\quad \text{a.s.},$$

$$\liminf_{h\to 0}\inf_{0\leq t\leq 1}\sup_{0\leq s\leq h}\left(\frac{\log(1/h)}{h}\right)^{\alpha}|Z(t+s)-Z(t)|\geq 0.1\quad \text{a.s.},$$

$$\liminf_{T\to\infty}\sup_{0\leq s\leq T}\left(\frac{\log\log T}{T}\right)^{\alpha}|Z(s)|\geq 0.1\quad \text{a.s.},$$

$$\liminf_{T\to\infty}\inf_{0\leq t\leq T}\sup_{0\leq s\leq a_T}\left(\frac{\log(T/a_T)+\log\log\tilde{a}_T}{a_T}\right)^{\alpha}|Z(t+s)-Z(t)|$$
$$\geq 0.1\quad \text{a.s.}$$

4.4.3 独立 O-U 过程的无穷级数

在这一小节中，我们将指出对 O-U 过程的无穷级数，(4.4.7) 中的界是最佳的，进而将指明我们的下极限结果基本上是不可改进的.

回顾坐标过程独立的无穷维 O-U 过程 $Y(\cdot)$ 的无穷级数 $X(\cdot)$ 的定义：

$$\{X(t); -\infty < t < \infty\} = \left\{\sum_{k=1}^{\infty} X_k(t); -\infty < t < \infty\right\}.$$

对 $\{X(\cdot)\}$ 已获得精确的 Lévy 连续模 (见定理 2.2.5, 注 2.2.5, 注 2.2.6 和林正炎，陆传荣 1992 的定理 3.3.2).

本小节的主要目的是讨论 $\{X(\cdot)\}$ 的下极限结果，这里总设

$$0 < \Gamma_0 = \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} < \infty. \tag{4.4.9}$$

$X(\cdot)$ 是一个平稳 Gauss 过程且

$$\sigma^2(h) = E(X(t+h) - X(t))^2 = 2\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}), \quad h \geq 0, t \geq 0. \tag{4.4.10}$$

由于 $\Gamma_0 < \infty$, 易知在 $(0, \infty)$ 上 $\sigma^2(h)$ 是凹函数，所以作为定理 4.4.2 的直接推论，我们有

定理 4.4.4 设 b_h 是 h 的非负连续函数. 那么

$$\liminf_{h\to 0} \inf_{0\leq t\leq b_h} \sup_{0\leq s\leq h} \frac{|X(t+s) - X(t)|}{\sigma\left(\dfrac{h}{24(\log(b_h/h) + \log\log(1/h))}\right)} \geq 0.5 \quad \text{a.s.} \tag{4.4.11}$$

特别地，我们有

$$\liminf_{h\to 0} \inf_{0\leq t\leq 1} \sup_{0\leq s\leq h} \frac{|X(t+s) - X(t)|}{\sigma\left(\dfrac{h}{24\log(1/h)}\right)} \geq 0.5 \quad \text{a.s.} \tag{4.4.12}$$

和

$$\liminf_{h\to 0} \sup_{0\leq s\leq h} \frac{|X(t+s) - X(t)|}{\sigma\left(\dfrac{h}{24\log\log(1/h)}\right)} \geq 0.5 \quad \text{a.s.} \tag{4.4.13}$$

在下一个定理中，我们给出下极限的精确上界.

定理 4.4.5 设 b_h 是 $(0,1)$ 上的非负连续函数，满足

$$\lim_{h\to 0}\frac{\log(b_h/h)}{\log\log(1/h)}=\infty. \tag{4.4.14}$$

假设存在常数 $\theta>1,\alpha>0$ 使得对所有的 $0\le a,h\le 1$ 有

$$\sigma(ah)\le\theta a^{\alpha}\sigma(h). \tag{4.4.15}$$

那么存在仅依赖于 α 的常数 $d>0$ 使得

$$\limsup_{h\to 0}\inf_{0\le t\le b_h}\sup_{0\le s\le h}\frac{|X(t+s)-X(t)|}{\sigma(db_h\log(b_h/h))}\le 4\theta\quad \text{a.s.} \tag{4.4.16}$$

结合定理 4.4.4 和定理 4.4.5 得

推论 4.4.1 设 b_h 是满足 (4.4.14) 的非负连续函数，假设 (4.4.15) 被满足. 那么

$$\begin{aligned}\frac{1}{2}&\le\liminf_{h\to 0}\inf_{0\le t\le b_h}\sup_{0\le s\le h}\frac{1}{\sigma(h/(24\log(b_h/h)))}|X(t+s)-X(t)|\\&\le\limsup_{h\to 0}\inf_{0\le t\le b_h}\sup_{0\le s\le h}\frac{1}{\sigma(dh/\log(b_h/h))}|X(t+s)-X(t)|\\&\le 4\theta\quad \text{a.s.}\end{aligned}$$

特别地

$$\begin{aligned}\frac{1}{2}&\le\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\left(\frac{1}{\sigma(h/(24\log(1/h)))}\right)|X(t+s)-X(t)|\\&\le\limsup_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\left(\frac{1}{\sigma(dh/\log(1/h))}\right)|X(t+s)-X(t)|\\&\le 4\theta\quad \text{a.s.}\end{aligned}$$

推论 4.4.2 假设对某 $0<\alpha<1/2$, $\theta_0>0$

$$\lim_{h\to 0}\frac{\sigma^2(h)}{h^{2\alpha}}=\theta_0. \tag{4.4.17}$$

那么对某 $d=d(\alpha)$ 有

$$\begin{aligned}\frac{1}{2(24)^{\alpha}}&\le\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|\\&\le\limsup_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|\le d\quad \text{a.s.}\end{aligned} \tag{4.4.18}$$

由 (4.4.9) 可知 $\sigma^2(h)/h$ 在 $(0,\infty)$ 上是非增的. 因此对任意的 $0<a\le 1, h>0$ 有

$$\sigma(ah)\ge a^{1/2}\sigma(h), \tag{4.4.19}$$

它和定理 4.4.4 一起可推得

推论 4.4.3 我们有

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|\ge 0.1 \quad \text{a.s.}$$

注 4.4.1 由 (4.4.12), (4.4.13), (4.4.18) 和 Pitt 和 Tran (1979) 的 0-1 律可得: 在条件 (4.4.17) 下, 存在常数 $0<c_1\le\infty$, $0<c_2, c_3<\infty$ 使得

$$\liminf_{h\to 0}\sup_{0\le s\le h}\frac{(\log\log h^{-1})^{\alpha}}{\sigma(h)}|X(s)-X(0)|=c_1 \quad \text{a.s.},$$

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=c_2 \quad \text{a.s.},$$

$$\limsup_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=c_3 \quad \text{a.s.}$$

注 4.4.2 由 (4.4.19) 可得 $\liminf_{h\to 0}\sigma^2(h)/h>0$, 因此

$$\lim_{h\to 0}h(\log h^{-1})^{1/2}/\sigma(h)=0.$$

所以由推论 4.4.3, 过程 $X(\cdot)$ 的几乎所有样本函数是不可微的.

注 4.4.3 需要指出的是关于 $X(\cdot)$ 或 l^p 值 Gauss 过程 (见 §3.3) 的几乎所有有关上极限的已知结果是平行于标准 Wiener 过程的相应结果的 (如参见 Csörgő 和 Révész 1981 的第一章). 而上述结果指出对下极限情形是十分不同的.

张立新 (1995) 建立了下述关于 $X(\cdot)$ 的精确不可微模.

定理 4.4.6 假设

$$\Gamma_1 := 2\sum_{k=1}^{\infty}\gamma_k<\infty. \tag{4.4.20}$$

那么我们有

$$\lim_{h\to 0}\inf_{0\le t\le 1-h}\sup_{0\le s\le h}\Big(\frac{8\log h^{-1}}{\pi^2\Gamma_1 h}\Big)^{1/2}|X(t+s)-X(t)|=1 \quad \text{a.s.} \tag{4.4.21}$$

Csörgő 和 Shao (1994) 指出，存在常数 $C(\alpha)$ 使得 (4.4.18) 中可以等号代替不等号. 他们提出下述猜测:

猜测 假设 (4.4.17) 被满足. 那么存在仅依赖于 α 的常数 $C(\alpha)$ 使得

$$\liminf_{h\to 0}\sup_{0\le s\le h}\frac{(\log\log h^{-1})^{\alpha}}{\sigma(h)}|X(s)|=C(\alpha) \quad \text{a.s.},$$

$$\lim_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=C(\alpha) \quad \text{a.s.}$$

$C(\alpha)$ 的精确的计算看来是困难的.

§4.5 两参数 Gauss 过程的下极限

4.5.1 两参数 Wiener 过程的下极限

设 $\{W(x,y);0\le x,y<\infty\}$ 是两参数 Wiener 过程, $0<a_T\le T$ 和 $b_T\ge T^{1/2}$ 是 T 的非降函数. 设

$$D_T=\{(x,y);xy\le T,0\le x,y\le b_T\},$$

$$D_T^*=\{(x,y);xy=T,0\le x,y\le b_T\}.$$

对矩形 $R=[x_1,x_2]\times[y_1,y_2]$, 定义 $\lambda(R)=(x_2-x_1)(y_2-y_1)$,

$$W(R)=W(x_2,y_2)-W(x_1,y_2)-W(x_2,y_1)+W(x_1,y_1).$$

设 $L_T=\{R;R\subset D_T,\lambda(R)\le a_T\}$ 是矩形集 $R=[x_1,x_2]\times[y_1,y_2]$. 定义

$$\lambda_T=\{2T(\log(1+\log b_T T^{-1/2})-\log\log\log T)\}^{-1/2},$$

$$\beta_T=T^{-1/2}\Big(\frac{\log\log T}{\log b_T T^{-1/2}}\Big)^{1/2}\Big(\log\frac{\log\log T}{\log b_T T^{-1/2}}\Big)^{-3/2}.$$

Lacey (1989) 建立了如下的重对数律：

定理 4.5.1 假设 $\lambda'_T=\{2T\log(1+\log b_T T^{-1/2})\}^{-1/2}$ 满足

$$\lim_{\theta\downarrow 1}\limsup_{k\to\infty}\lambda'_{\theta^k}/\lambda'_{\theta^{k+1}}=1 \tag{a}$$

且 $b'_T=b_T T^{-1/2}$ 是 T 的非降函数，$b'_T\to\infty$. 又设对任何 $0<\epsilon<1$ 及某 $0<a<1$ 有

$$\sum_k \exp\{-(\log b'_{m_k})^{\epsilon}\}<\infty, \tag{4.5.1}$$

其中 $m_k=\exp(k^a)$, $k\in N$. 那么

$$\liminf_{T\to\infty}\sup_{(x,y)\in D_T^*}\lambda'_T W(x,y)=1 \quad \text{a.s.} \tag{4.5.2}$$

特别地

$$\liminf_{T\to\infty}\sup_{\substack{0\le x,y\le T\\ xy=T}}\frac{W(x,y)}{\sqrt{2T\log\log T}}=1 \quad \text{a.s.} \tag{4.5.3}$$

利用推论 4.3.1, 我们可得两参数 Wiener 过程的 Chung 型重对数律 (参见 Talagrand 1994):

定理 4.5.2 我们有

$$0<\liminf_{T\to\infty}\frac{(\log\log T)^{1/2}}{T(\log\log\log T)^{3/2}}\sup_{0\le x,y\le T}|W(x,y)|<\infty \quad \text{a.s.} \tag{4.5.4}$$

Lacey (1989) 问道：对于 (4.5.2), (4.5.1) 式是否是必要的. 进一步的问题是在减弱 (4.5.1) 后，是否存在类似于 (4.5.2) 的下极限. 另外，一个有趣的问题是 :(4.5.3) 型下极限与 (4.5.4) 型下极限之间有什么联系. 张立新 (1996b) 回答了这些问题. 他得到了这两类不同的下极限的分界线，也指出了定理 4.5.1 中的条件 (a) 是不必要的. 其结果如下.

定理 4.5.3 若

$$\Delta_T:=\frac{\log b_T T^{-1/2}}{\log\log T}\to\infty \quad (T\to\infty), \tag{4.5.5}$$

则

$$\begin{aligned}
&\liminf_{T\to\infty} \lambda_T \sup_{R\subset D_T} |W(R)| \\
&\quad = \liminf_{T\to\infty} \lambda_T \sup_{R\subset D_T^*} |W(R)| \qquad (4.5.6)\\
&\quad = \liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T} |W(x,y)| \\
&\quad = \liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T^*} |W(x,y)| \\
&\quad = 1 \quad \text{a.s.}
\end{aligned}$$

若

$$\Delta_T \to 0 \quad (T\to\infty), \tag{4.5.7}$$

则存在正常数 C_1, C_2 使得

$$C_1 \le \liminf_{T\to\infty} \beta_T \sup_{R\subset D_T} |W(R)| \le C_2 \quad \text{a.s.}, \tag{4.5.8}$$

$$C_1 \le \liminf_{T\to\infty} \beta_T \sup_{(x,y)\in D_T} |W(x,y)| \le C_2 \quad \text{a.s.} \tag{4.5.9}$$

介于这两个下极限结果之间的截断双曲线起着联系它们的桥梁作用. 若我们取 $b_T = T^{1/2}$, 则 (4.5.9) 恰为 (4.5.4). 而若取 $b_T = T$, 则 (4.5.6) 就是 Lacey 型重对数律 (4.5.3). 另外, 条件 (4.5.5) 比 Lacey 的条件 (4.5.1) 弱得多. 为验证这一事实, 只需指出在定理 4.5.1 中的条件下, (4.5.1) 等价于

$$\lim_{T\to\infty} \frac{\log\log b_T'}{\log\log\log T} = \infty. \tag{4.5.10}$$

事实上, 若 (4.5.10) 成立, 则

$$\liminf_{k\to\infty} \frac{\log\log b_{m_k}'}{\log\log k} = \liminf_{k\to\infty} \frac{\log\log b_{m_k}'}{\log\log\log m_k} > \frac{2}{\varepsilon}.$$

故对充分大的 k, 有

$$(\log b_{m_k}')^{\varepsilon} \ge (\log k)^2,$$

这就推得 (4.5.1) 成立.

另一方面, 若 (4.5.1) 成立, 注意到 b_T' 是非降的, 我们有

$$k\exp(-(\log b'_{m_k})^{\varepsilon}) \le \sum_{k=1}^{\infty}\exp(-(\log b'_{m_k})^{\varepsilon}) < \infty.$$

故

$$\exp((\log b'_{m_k})^{\varepsilon}) \ge ck,$$

这就推出

$$\liminf_{k\to\infty}\frac{\log\log b'_{m_k}}{\log\log\log m_k} \ge \frac{1}{\varepsilon}.$$

对 $m_k \le T \le m_{k+1}$ 我们有

$$\frac{\log\log b'_T}{\log\log\log T} \ge \frac{\log\log b'_{m_k}}{\log\log\log m_k}\frac{\log\log m_k}{\log\log\log m_{k+1}}.$$

因此 (4.5.10) 成立.

由定理 4.5.1, 我们也可得：若

$$\lim_{T\to\infty}\frac{\log\log b_T T^{-1/2}}{\log\log\log T} = r \ge 1, \tag{4.5.11}$$

那么

$$\liminf_{T\to\infty}\lambda'_T \sup_{(x,y)\in D_T}|W(x,y)| = \left(\frac{r-1}{r}\right)^{1/2} \quad \text{a.s.} \tag{4.5.12}$$

因此，从 (4.5.12) 容易看出，对 (4.5.2) 而言， (4.5.1) 或 (4.5.10) 是十分接近于必要的.

注 4.5.1　记

$$\theta(T) = (2 + 2\log b_T)/\log\log T.$$

Csáki 和 Shi (1998) 研究了当 $\theta(T)$ 趋向于有限极限时两参数 Wiener 过程的下极限性质，得到了如下结果：

假设 $b_T \ge 1$ 是非降的且 $\theta(T)/T$ 是非增的. 又若 $\theta := \lim_{T\to\infty}\theta(T) \in [0,\infty)$. 那么

$$\liminf_{T\to\infty}\sqrt{\frac{\theta(T)}{T}}\sup_{st=T;0\le s,t\le\sqrt{T}b_T} W(s,t)$$
$$= \begin{cases} -2, & \text{当 } \theta = 0 \text{时}, \\ -\sqrt{\theta}\beta(1/\theta), & \text{当 } 0 < \theta < \infty \text{ 时}, \end{cases}$$

$$\liminf_{T\to\infty}\frac{1}{\sqrt{T\theta(T)}}\sup_{st=T;0\le s,t\le\sqrt{T}b_T}|W(s,t)|$$
$$=\begin{cases}\pi/2, & \text{当 } \theta=0 \text{时},\\ \theta^{-1/2}\gamma(1/\theta), & \text{当 } 0<\theta<\infty \text{ 时},\end{cases}$$

其中 $\gamma(u)>0$ 是下述 Kummer 函数 $M(-u/2,1/2,\gamma^2/2)$ 的最大正零点:

$$M(a,b,x):=1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}\cdot\frac{x^n}{n!},$$

$\beta(u)\in(-\infty,\infty)$ 是下述抛物柱函数 $D_u(\cdot)$ 的最大实零点:

$$D_u(x):=2^{\frac{u}{2}}e^{-\frac{x^2}{4}}\Big[\frac{\Gamma(1/2)}{\Gamma((1-u)/2)}M\Big(-\frac{u}{2},\frac{1}{2},\frac{x^2}{2}\Big)$$
$$+\frac{x}{\sqrt{2}}\frac{\Gamma(-1/2)}{\Gamma(-u/2)}M\Big(\frac{1-u}{2},\frac{3}{2},\frac{x^2}{2}\Big)\Big].$$

定理 4.5.1 和 4.5.2 的证明不在这里给出. 定理 4.5.3 的证明基于下述概率不等式.

引理 4.5.1 对任给的 $\varepsilon>0$, 存在常数 $C=C(\varepsilon)>0$, $u_0=u_0(\varepsilon)>0$ $T_0=T_0(\varepsilon)>0$ 使得对任何 $u\ge u_0$, $T\ge T_0$ 有

$$P\Big\{\sup_{(x,y)\in D_T^*}|W(x,y)|\le uT^{1/2}\Big\}$$
$$\le\exp\big(-C(1+\log b_TT^{-1/2})e^{-u^2/(2-\varepsilon)}\big).\qquad(4.5.13)$$

证明 设 $L=L(T)$ 是使下述不等式成立的最大整数:

$$T^{1/2}M^{L+1}<b_T\quad(M>1).$$

定义矩形

$$S_i=S_i(T)=[x_1(i),x_2(i)]\times[y_1(i),y_2(i)]$$
$$=[T^{1/2}M^i,T^{1/2}M^{i+1}]\times[0,T^{1/2}M^{-i-1}],\quad i=0,1,\cdots,L.$$

那么 $S_i \subset D_T$, $\lambda(S_i) = T(1-1/M)$, $i = 0,1,\cdots,L$, 且 $L \geq (\log b_T T^{-1/2})/\log M$. 设

$$\tilde{S}_i = [0, T^{1/2}M^i] \times [0, T^{1/2}M^{-i-1}], \quad i = 0,1,\cdots,L.$$

那么

$$\begin{aligned}
&P\Big\{\sup_{(x,y)\subset D_T^*} |W(x,y)| \leq uT^{1/2}\Big\} \\
&\quad \leq P\Big\{\max_{0\leq i\leq L} |W(x_2(i), y_2(i))| \leq uT^{1/2}\Big\} \\
&\quad = P\Big\{\max_{0\leq i\leq L} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2}\Big\}. \qquad (4.5.14)
\end{aligned}$$

我们使用条件概率化的论证方法. 记 σ 域 $\sigma_i = \sigma\{W(x,y), 0 \leq y \leq b_T, 0 \leq x \leq T^{1/2}M^{i+1}\}$, 那么 $W(S_i) \in \sigma_i$, $W(\tilde{S}_i) \in \sigma_{i-1}$, 且 $W(S_i)$ 与 σ_{i-1} 独立. 故对充分大的 M, 我们有

$$\begin{aligned}
&P\Big\{\max_{0\leq i\leq L} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2}\Big\} \\
&\quad = E\Big[I\Big\{\max_{0\leq i\leq L-1} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2}\Big\} \\
&\qquad \cdot P(|W(\tilde{S}_L) + W(S_L)| \leq uT^{1/2} | \sigma_{L-1})\Big] \\
&\quad \leq E\Big[I\Big\{\max_{0\leq i\leq L-1} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2}\Big\} \cdot P(|W(S_L)| \leq uT^{1/2})\Big] \\
&\quad = P\Big(\max_{0\leq i\leq L-1} |W(\tilde{S}_i) + W(S_i)| \leq uT^{1/2}\Big) \cdot P(|W(S_L)| \leq uT^{1/2}\Big) \\
&\quad \leq \cdots \leq \prod_{i=0}^{L} P(|W(S_i)| \leq uT^{1/2}) \\
&\quad \leq \Big\{1 - 2\Phi\Big(-\Big(\frac{M}{M-1}\Big)^{1/2} u\Big)\Big\}^{L+1} \leq \Big\{1 - \exp\Big(-\frac{u^2}{2-\varepsilon}\Big)\Big\}^{L+1} \\
&\quad \leq \exp\Big\{-c\frac{1}{\log M}\Big(\log b_T T^{-1/2}\Big) e^{-u^2/(2-\varepsilon)}\Big\}. \qquad (4.5.15)
\end{aligned}$$

因此由 (4.5.14) 和 (4.5.15) 有

$$P\left\{\sup_{(x,y)\in D_T^*}|W(x,y)|\le uT^{1/2}\right\}$$

$$\le \exp\left\{-c\frac{1}{\log M}(\log b_T T^{-1/2})e^{-u^2/(2-\varepsilon)}\right\}.$$

这就推得 (4.5.13) 成立.

由推论 4.3.1 即得下述引理.

引理 4.5.2 存在常数 $C>0$ 使得对任何 $0<u\le 1/2$ 和 $T_1,T_2>0$ 有

$$P\left\{\sup_{(x,y)\in[0,T_1]\times[0,T_2]}|W(x,y)|\le (T_1T_2)^{1/2}\left(u\left(\log\frac{1}{u}\right)^3\right)^{1/2}\right\}$$

$$\overset{KS}{\ge}\exp\left(-\frac{C}{u}\right).$$

引理 4.5.3 存在常数 $C_2>0$ 使得对任何 $0<u\le 1/2$ 和 $T>0$ 有

$$\exp\left(-C_2\frac{1}{u}\log b_T T^{-1/2}\right)$$

$$\le P\left\{\sup_{R\subset D_T}|W(R)|\le T^{1/2}\left(u\left(\log\frac{1}{u}\right)^3\right)^{1/2}\right\}$$

$$\le \exp\left(-\frac{1}{C_2u}\log b_T T^{-1/2}\right), \tag{4.5.16}$$

$$\exp\left(-C_2\frac{1}{u}\log b_T T^{-1/2}\right)$$

$$\le P\left\{\sup_{(x,y)\in D_T}|W(x,y)|\le T^{1/2}\left(u\left(\log\frac{1}{u}\right)^3\right)^{1/2}\right\}$$

$$\le \exp\left(-\frac{1}{C_2u}\log b_T T^{-1/2}\right). \tag{4.5.17}$$

证明 注意到

$$\sup_{(x,y)\in D_T}|W(x,y)|\le \sup_{R\subset D_T}|W(R)|\le 4\sup_{(x,y)\in D_T}|W(x,y)|, \tag{4.5.18}$$

我们只要证明 (4.5.16). 首先我们建立下界. 不失一般性可设 $T=1$. 记 $D_T=D$, $b_T=b$ 等等. 取 $R_j=[0,2^j]\times[0,2^{-j+1}]$, $j=-1$

$-[\log_2 b], \cdots, 1+[\log_2 b]$. 注意到每一 R_j 的面积为 2, 且

$$R^* := \bigcup_{j=-1-[\log_2 b]}^{1+[\log_2 b]} R_j \supset D = \{(x,y) : xy \le 1, 0 \le x, y \le b\},$$

那么由引理 4.5.2 和定理 1.2.4 我们有

$$\begin{aligned}
&P\left\{\sup_{(x,y)\in D} |W(x,y)| \le \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad \ge P\left\{\sup_{(x,y)\in R^*} |W(x,y)| \le \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad \overset{KS}{\ge} \prod_{j=-1-[\log_2 b]}^{1+[\log_2 b]} P\left\{\sup_{(x,y)\in R_j} |W(x,y)| \le \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad \overset{KS}{\ge} \exp\left(-\frac{C}{u}\log b\right).
\end{aligned}$$

因此我们求得了下界. 对于上界, 定义

$$S_i = [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}], \quad i = 0, 1, \cdots, L,$$

其中 L 是使 $T^{1/2}M^{L+1} < b_T$ $(M > 1)$ 成立的最大整数. 那么由推论 4.3.1 的 (4.3.25) 式, 我们有

$$\begin{aligned}
&P\left\{\sup_{R\subset D_T} |W(R)| \le T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad \le P\left\{\sup_i \sup_{R\subset S_i} |W(R)| \le T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad = \prod_{i=0}^{L} P\left\{\sup_{R\subset S_i} |W(R)| \le T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad = \prod_{i=0}^{L} P\left\{\sup_{R\subset[0,1]\times[0,1]} |W(R)| \le \left(\frac{M}{M-1}\right)^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\quad \le \prod_{i=0}^{L} P\left\{\sup_{0\le x,y\le 1} |W(x,y)| \le \left(\frac{M}{M-1}\right)^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\}
\end{aligned}$$

$$\leq \exp\Big(-\frac{C}{u}\frac{M-1}{M}L\Big)$$
$$\leq \exp\Big(-\frac{C}{u}\log b_T T^{-1/2}\Big).$$

引理 4.5.3 证毕.

定理 4.5.3 的证明 从定理 2.3.2 的证明的第一步, 令 $a_T = T$ (参见 (2.3.23)) 就得 (4.5.6) 的上界. 利用不等式 (4.5.13) 可验证 (4.5.6) 的下界. 类似地, 利用 (4.5.16) 和 (4.5.17) 可证明 (4.5.8) 和 (4.5.9).

4.5.2 两参数 O-U 过程的不可微模

设 $\{X(t,v); t\geq 0, v\geq 0\}$ 是两参数 O-U 过程 (OUP_2)

$$X(t,v) = e^{-\alpha t-\beta v}\Big\{X_0 + \int_0^t\int_0^v e^{\alpha x+\beta v}dW(x,y)\Big\},\quad t\geq 0,\quad v\geq 0. \tag{4.5.19}$$

Lin (1995d) 给出了 OUP_2 $\{X(t,v)\}$ 的不可微模. 假设存在 $\delta>0$ 使得 $E|X_0|^\delta<\infty$. 记

$$\beta(v) = (1-e^{-2\beta v})/2\beta.$$

定理 4.5.4 对任意的 $v>0$ 我们有

$$\lim_{h\to 0}\inf_{0\leq t\leq 1}\sup_{0\leq s\leq h}\Big(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\Big)^{1/2}|X(t+s,v)-X(t,v)| = 1 \quad \text{a.s.}, \tag{4.5.20}$$

$$\lim_{h\to 0}\inf_{0\leq t\leq 1}\sup_{0\leq s\leq h}\sup_{v\leq u\leq v+h}\Big(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\Big)^{1/2}|X(t+s,u)-X(t,u)| = 1 \quad \text{a.s.} \tag{4.5.21}$$

注 4.5.2 由于 $X(t,v)$ 关于 t 和 v 是对称的, 故对任何 $t>0$ 也有

$$\lim_{h\to 0}\inf_{0\leq v\leq 1}\sup_{0\leq u\leq h}\Big(\frac{8\log h^{-1}}{\pi^2\beta(t)h}\Big)^{1/2}|X(t,v+u)-X(t,v)| = 1 \quad \text{a.s.},$$

$$\lim_{h\to 0}\inf_{0\le v\le 1}\sup_{0\le u\le h}\sup_{t\le s\le t+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(t)h}\right)^{1/2}|X(s,v+u)-X(s,v)|$$
$$=1\quad \text{a.s.}$$

定理 4.5.4 的证明思路如下. 首先, 如同 (2.5.2), 我们写

$$X(t+s,v)-X(t,v)=\xi_1(t,s,v)+\xi_2(t,s,v)+\xi_3(t,s,v),\quad (4.5.22)$$

其中

$$\xi_1(t,s,v)=e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})X_0,$$
$$\xi_2(t,s,v)=e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})\int_0^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y),$$
$$\xi_3(t,s,v)=e^{-\alpha t-\beta v}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y).$$

可证: 对任给的 $\varepsilon>0$, 存在 $h=h(\varepsilon)>0$ 和 $C=C(\varepsilon)>0$ 使得对 $i=1,2$ 有

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}|\xi_i(t,s,u)|\ge\varepsilon\right\}$$
$$\le C(h\log h^{-1})^{\delta/2}.$$

由此易得

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\xi_i(t,s,u)|$$
$$=0\quad \text{a.s.},\quad i=1,2.\quad (4.5.23)$$

为证明

$$\limsup_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t+s,u)-X(t,u)|$$
$$\le 1\quad \text{a.s.},\quad (4.5.24)$$

记

$$\eta_1(t,s,u)=e^{-\beta u}\int_t^{t+s}\int_0^u e^{\beta y}dW(x,y),$$
$$\eta_2(t,s,u)=e^{-\alpha t-\beta u}\int_t^{t+s}\int_0^u (e^{\alpha x}-e^{\alpha t})e^{\beta y}dW(x,y).$$

则 $\xi_3(t,s,u)=\eta_1(t,s,u)+\eta_2(t,s,u)$. 仿照对 $\xi_2(t,s,u)$ 的研究，可得

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_2(t,s,u)|=0 \quad \text{a.s.} \tag{4.5.25}$$

进一步还可证明

$$\limsup_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t,s,u)|\le 1 \quad \text{a.s.} \tag{4.5.26}$$

结合 (4.5.23), (4.5.25) 和 (4.5.26) 即得 (4.5.24).

对任给的 $v>0$

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t+s,v)-X(t,v)|\ge 1 \quad \text{a.s.} \tag{4.5.27}$$

等价于

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t,s,v)|\ge 1 \quad \text{a.s.} \tag{4.5.28}$$

令

$$\eta_v(t)=e^{-\beta v}\int_0^t\int_0^v e^{\beta y}dW(x,y).$$

注意到 $\eta_v(t)/\beta(v)^{1/2}$ 关于 t 是 Wiener 过程且 $\eta_1(t,s,v)$ 是 $\eta_v(t)$ 的增量, 由 Wiener 过程的不可微模可知 (4.5.28) 成立, 因此 (4.5.27) 得证. 结合 (4.5.24), 就完成了定理 4.5.4 的证明.

由类似的讨论, Lu 和 Yu (1997) 对两参数 O-U 过程证明了 Chung 重对数律:

定理 4.5.5 *在定理 4.5.4 的假设下我们有*

$$\liminf_{h\to 0}\sup_{0\le t\le h}\left(\frac{8\log\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t,v)-X(0,v)|=1 \quad \text{a.s.},$$

$$\liminf_{h\to 0}\sup_{0\le t\le h}\sup_{v\le u\le v+h}\left(\frac{8\log\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t,u)-X(0,u)|$$
$$=1 \quad \text{a.s.}$$

§4.6 Gauss 过程的其他轨道性质

4.6.1 Gauss 过程的 p 变差

设 $\{X(t); t \geq 0\}$ 为具有平稳增量、均值为零的 Gauss 过程. 令

$$\sigma^2(h) = E(X(t+h) - X(t))^2. \tag{4.6.1}$$

令 $\pi = \{0 = x_0 < x_1 < \cdots < x_{k_n} = a\}$ 表示 $[0,a]$ 的一个分划且记 $m(\pi) = \max_{1 \leq i \leq k_n}(x_i - x_{i-1})$ 为 π 中的最大区间的长度.

对随机过程的 p 变差的关注始于 Lévy 关于 Wiener 过程 $\{W(t); t \geq 0\}$ 的 2 变差的漂亮结果, 即

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left\{W\left(\frac{i}{2^n}\right) - W\left(\frac{i+1}{2^n}\right)\right\}^2 = 1 \quad \text{a.s.}$$

这一结果有很多推广. Dudley (1973) 指出对 $[0,a]$ 的任何分划序列 $\{\pi(n)\}$, 当 $m(\pi(n)) = o(1/\log n)$ 时,

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} (W(x_i) - W(x_{i-1}))^2 = a \quad \text{a.s.} \tag{4.6.2}$$

而 de la Vega (1974) 指出当关于 $m(\pi(n))$ 的条件减弱为 $m(\pi(n)) = O(1/\log n)$ 时上述结论就不再成立. Taylor (1972) 证明了

$$\lim_{\delta\to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \bar{\psi}(|W(x_i) - W(x_{i-1})|) = 1 \quad \text{a.s.},$$

其中 $\bar{\psi}(x) = |x/\sqrt{2\log\log 1/x}|^2$ 且 $Q_a(\delta) = \{[0,a]$ 的分划 $\pi : m(\pi) \leq \delta\}$. 对一个随机过程, 以 p 代替 2 时, 这类结果称为是关于这个过程的 p 变差结果.

关于 Gauss 过程 p 变差的各种结果已由 Kôno (1969), Kawata 和 Kôno (1973), Giné 和 Klein (1975), Jain 和 Monrad (1983) 及 Adler 和 Pyke (1993) 等人得到. Marcus 和 Rosen (1992b) 证明了下述定理.

定理 4.6.1 若在 $[0,a]$ 上 $\sigma^2(h)$ 是凹函数，且对某 $p \geq 2$ 和 $0 < b < \infty$ 满足 $\lim_{h\to 0}\sigma(h)/h^{1/p} = b$ ，那么对 $[0,a]$ 的任一满足 $m(\pi) = o((1/\log n)^{p/2})$ 的分划序列 $\{\pi(n)\}$ 有

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} |X(x_i) - X(x_{i-1})|^p = E|N(0,1)|^p\, b^p\, a \quad \text{a.s.} \tag{4.6.3}$$

且

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} |X^2(x_i) - X^2(x_{i-1})|^p$$
$$= E|N(0,1)|^p (2b)^p \int_0^a |X(x)|^p dx \quad \text{a.s.} \tag{4.6.4}$$

Shao (1996b) 减弱了加在 $\sigma(h)$ 上的条件且给出了较一般的结果.

定理 4.6.2 设 $p > 1$. 假设 $\sigma^2(h)$ 是非降的. 如果以下两条件之一被满足:

(A1) 在 $[0,a]$ 上 $\sigma^2(h)$ 是凹函数，且

$$m(\pi(n)) = o((\log n)^{-(1\vee(p/2))});$$

(A2) 对某个 $\epsilon_0 > 0$, 在 $[0,a+\varepsilon_0]$ 上 $\sigma^2(h)$ 是凸函数，且

$$\max_{x_i\in\pi(n)} (x_i - x_{i-1})^{\frac{1}{2}+\frac{1}{2}\wedge\frac{1}{p}}/\sigma(x_i - x_{i-1}) = o(\log^{-1/2} n).$$

那么对 $[0,a]$ 的任一分划序列 $\{\pi(n)\}$

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} \frac{(x_i - x_{i-1})|X(x_i) - X(x_{i-1})|^p}{\sigma^p(x_i - x_{i-1})} = aE|N(0,1)|^p \quad \text{a.s.} \tag{4.6.5}$$

定理 4.6.3 假设 $\sigma^2(h)$ 在 $[0,a]$ 上连续，满足

$$\int_1^\infty \sigma(e^{-z^2})dz < \infty.$$

那么在定理 4.6.2 的条件下我们有

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}\frac{(x_i-x_{i-1})|X^2(x_i)-X^2(x_{i-1})|^p}{\sigma^p(x_i-x_{i-1})}$$
$$=2^pE|N(0,1)|^p\int_0^a|X(x)|^pdx\quad \text{a.s.}\tag{4.6.6}$$

4.6.2 Gauss 场的像与图的分形性质

随机分形的研究早在 20 世纪 40 年代就开始了，尽管当时还没有随机分形这个词. Lévy 早在 1940 年左右就开始研究 Wiener 过程的样本轨道性质，随后， Besicovith 和 Taylor 也研究了类似的问题. 综合他们的结果，可知：一维标准 Wiener 过程 $W(\cdot)$ 的零集的 Hausdorff 维数为 1/2, 即

$$\dim\{t\in[0,1]:W(t)=0\}=\frac{1}{2}\quad \text{a.s.}$$

这可能是随机分形的最早的一个漂亮结果. 在 Gauss 场 (包括多参数 Wiener 过程) 的像、图、水平集和多重点的分形性质中有着很多值得重视的有兴趣的问题. 早期的结果是：一维标准 Wiener 过程在 $[0,1]$ 的像集的 Hausdorff 维数为 1, 而 2 维以上 Wiener 过程在 $[0,1]$ 上的像集的 Hausdorff 维数为 2, 即

$$\dim\{W(t):t\in[0,1]\}=\begin{cases}1, & 若 d=1,\\ 2, & 若 d\geq 2,\end{cases}$$

其中 $W(t)=(W_1(t),\cdots,W_d(t))$ 为 d 维 Wiener 过程. Xiao (1995) 研究了指数 α 的 Gauss 场的像和图的 Hausdorff 维数和 packing 维数. 这一节我们介绍他的结果. 其他有关的成果可参见 Adler (1981), Cuzick (1978, 1981), Goldman (1981), Kahane (1985), Taylor 和 Tricot (1985), Talagrand (1995) 等.

设 $X(t)=(X_1(t),\cdots,X_d(t))$ 是 $\mathcal{R}^N$ 上的 $\mathcal{R}^d$ 值 Gauss 向量场，其坐标场 $X_1,\cdots,X_d$ 具有平稳增量. 记

$$\sigma_j^2(t)=E\big(X_j(t)-X_j(0)\big)^2.$$

若对每一 $j=1,2,\cdots,d$, 存在 $0<\alpha_j\le 1$ 使得

$$\begin{aligned}\alpha_j &= \sup\{\alpha>0: \lim_{|t|\to 0}|t|^{-\alpha}\sigma_j(t)=0\}\\ &= \inf\{\alpha>0: \lim_{|t|\to 0}|t|^{-\alpha}\sigma_j(t)=\infty\},\end{aligned}$$

其中 $|\cdot|$ 是 Euclidean 范数, 则称 $X(t)$ 是指数 $\alpha=(\alpha_1,\cdots,\alpha_d)$ 的 (N,d) Gauss 场. 为简单计, 我们将假设, 当 $|t|\ge\epsilon>0$ 时, 在 $[-1,1]^N$ 上, 所有 $|\sigma_j(t)|\ge\delta>0$. 为了避免退化, 我们对坐标场 $X_1,\cdots,X_d$ 作下述限制: 存在常数 $\epsilon>0$ 使得

$$\det\operatorname{cov}(X(t)-X(s))\ge\epsilon\prod_{j=1}^{d}\sigma_j^2(t-s), \tag{4.6.7}$$

其中 $\operatorname{cov}(Y)$ 记为随机向量 Y 的协方差阵, $\det B$ 为矩阵 B 的行列式. 若坐标场是独立的, 这一条件被满足. 分数 Wiener 过程是指数 α 的 Gauss 场的一个特例.

我们来给出 Hausdorff 维数和 packing 维数的定义. 设 $\phi:[0,1]\to[0,\infty)$ 是任一连续的增函数, 满足 $\phi(0)=0$. $E\subset\mathcal{R}^N$ 的 ϕ-Hausdorff 测度定义为

$$\phi\text{-mes}E=\lim_{\delta\to 0}\inf\Big\{\sum_i\phi(2r_i):E\subset\bigcup_i B(x_i,r_i),\ r_i\le\delta\Big\}, \tag{4.6.8}$$

其中 $B(x_i,r_i)$ 表示中心为 x_i, 半径为 r_i 的开球, 此外 $B(x_i,r_i)$ 组成 E 的一个 δ 覆盖 (即半径不超过 δ 的球的类, 它们的并包含了 E). (4.6.8) 中的下确界是在 E 的所有 δ 覆盖上取的. ϕ-mes 是外测度度量且所有的 Borel 集关于它是可测的. 一子集 E 的 Hausdorff 维数定义为

$$\begin{aligned}\dim E &= \inf\{\alpha>0: s^\alpha\text{-mes}E=0\}\\ &= \sup\{\alpha>0: s^\alpha\text{-mes}E=\infty\}.\end{aligned}$$

Taylor 和 Tricot (1985) 用不相交的填充代替最少的覆盖定义

了另一个集函数 ϕ-Pack E :

$$\phi\text{-Pack } E = \lim_{\delta\to 0} \sup\Big\{ \sum_i \phi(2r_i) : B(x_i, r_i) \text{ 互不相交},$$
$$x_i \in E, r_i \le \delta \Big\}.$$

ϕ--Pack 不是可列次可加的, 因而它不是外测度. 然而, ϕ-Pack 是预测度, 因此我们可由它产生一个 $\mathcal{R}^N$ 上的外测度:

$$\phi\text{-pack } E = \inf\Big\{ \sum_i \phi\text{-Pack}E_i : E \subset \bigcup_i E_i \Big\}.$$

ϕ-pack E 称为 E 的 ϕ-packing 测度. E 的 packing 维数定义为

$$\begin{aligned}\text{Dim } E &= \inf\{\alpha > 0 : s^\alpha\text{-pack}E = 0\}\\ &= \sup\{\alpha > 0 : s^\alpha\text{-pack}E = \infty\}.\end{aligned}$$

可证 ϕ-mes $E \le \phi$-pack E (参见 Taylor 和 Tricot 1985), 故

$$0 \le \dim E \le \text{Dim}E \le N. \tag{4.6.9}$$

对每一 $\varepsilon > 0$ 和有界集 $E \subset \mathcal{R}^N$, 设 $M(\varepsilon, E)$ 是覆盖 E 的半径为 ε 的球的最少个数, 令

$$\delta(E) = \liminf_{\varepsilon\to 0} \frac{\log M(\varepsilon, E)}{-\log \varepsilon},$$
$$\Delta(E) = \limsup_{\varepsilon\to 0} \frac{\log M(\varepsilon, E)}{-\log \varepsilon}.$$

δ 和 Δ 分别称为 Kolmogorov 上熵和下熵指数. Tricot (1982) 证明了

$$\text{Dim } E = \widehat{\Delta}(E) := \inf\Big\{ \sup \Delta(E_i) : E \subset \bigcup_i E_i \Big\}. \tag{4.6.10}$$

下面是有关指数为 α 的 (N, d) Gauss 场 $X(t)$ 的像 $X(E) = \{X(t) : t \in E\}$ 和图 $Gr\ X(E) = \{(t, X(t)) : t \in E\}$ 的 Hausdorff 维数的结果, 其中 E 是 $\mathcal{R}^N$ 中的任一紧集.

定理 4.6.4 设 $X(t)$ 是一个指数为 α 的 (N,d) Gauss 场. 其中 α 的坐标分量满足

$$0=\alpha_0<\alpha_1\leq\cdots\leq\alpha_d\leq 1$$

且设 $E\subset\mathcal{R}^N$ 是一紧集. 若对任何 $(s,t)\in E\times E$ (4.6.7) 成立, 则概率为 1 地有

$$\dim X(E)=\min\left\{d;\frac{\dim E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leq j\leq d\right\}$$
$$=\begin{cases}(\dim E+\sum\limits_{i=1}^{k}(\alpha_k-\alpha_i))/\alpha_k, & \text{当 }\sum\limits_{i=0}^{k-1}\alpha_i\leq\dim E\leq\sum\limits_{i=1}^{k}\alpha_i\text{ 时},\\ d, & \text{当 }\dim E>\sum\limits_{i=1}^{d}\alpha_i\text{ 时},\end{cases}\tag{4.6.11}$$

$$\dim GrX(E)$$
$$=\min\left\{\frac{\dim E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leq j\leq d;\ \dim E+\sum_{i=1}^{d}(1-\alpha_i)\right\}$$
$$=\begin{cases}\dim X(E), & \text{当 }\dim X(E)<d\text{ 时},\\ \dim E+\sum\limits_{i=1}^{d}(1-\alpha_i), & \text{当 }\dim X(E)=d\text{ 时}.\end{cases}\tag{4.6.12}$$

若 $E=[0,1]^N$, 我们有下述定理.

定理 4.6.5 设 $X(t)$ 是定理 4.6.4 中定义的 (N,d) Gauss 场, 若对所有的 $s,t\in[0,1]^N$, (4.6.7) 成立, 那么概率为 1 地

$$\dim X([0,1]^N)$$
$$=\mathrm{Dim}\,X([0,1]^N)=\min\left\{d;\frac{N+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leq j\leq d\right\},$$

$$\dim Gr\,X([0,1]^N)=\mathrm{Dim}\,Gr\,X([0,1]^N)$$
$$=\min\left\{N+\sum_{i=1}^{d}(1-\alpha_i);\frac{N+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leq j\leq d\right\}.$$

注 4.6.1 当 $[0,1]^N$ 被任一具有非空的内部的紧集 E 代替时，定理 4.6.5 仍成立.

4.6.3 l^p 值 Gauss 过程增量的分形性质

设 $\{W(t); t \geq 0\}$ 是标准 Wiener 过程. 由重对数律知

$$\limsup_{h \to 0} \frac{|W(t+h) - W(t)|}{(2h \log \log h^{-1})^{1/2}} = 1 \quad \text{a.s.}, \quad \forall t \in [0,1].$$

另一方面，由 Lévy 连续模定理 (定理 0.1) 知

$$\lim_{h \to 0} \sup_{t \in [a,b]} \frac{|W(t+h) - W(t)|}{(2h \log h^{-1})^{1/2}} = 1 \quad \text{a.s.}, \quad \forall 0 \leq a < b \leq 1.$$

我们还可以证明

$$\sup_{t \in [a,b]} \limsup_{h \to 0} \frac{|W(t+h) - W(t)|}{(2h \log h^{-1})^{1/2}} = 1 \quad \text{a.s.}, \quad \forall 0 \leq a < b \leq 1. \tag{4.6.13}$$

它说明：对任何 $0 < \alpha < 1$, a.s. 地存在 $t \in [a,b]$ 使得

$$\limsup_{h \to 0} \frac{|W(t+h) - W(t)|}{(2h \log h^{-1})^{1/2}} \geq \alpha.$$

由 a 和 b 的任意性知：存在许多 (至少可数个) $t \in [0,1]$ 使上式成立. 人们自然要研究这样的 t 组成的集合的性质. Orey 和 Taylor (1974) 研究了集合

$$B(\alpha) := \left\{ t \in [0,1] : \limsup_{h \to 0} \frac{|W(t+h) - W(t)|}{(2h \log h^{-1})^{1/2}} \geq \alpha \right\} \quad (0 \leq \alpha \leq 1)$$

的分形性质. 他们证明了，对每一 $0 \leq \alpha \leq 1$, $B(\alpha)$ 是随机分形，并证明了：

定理 4.6.6 对任给的 $\alpha \in [0,1]$，概率为 1 地有

$$\dim B(\alpha) = 1 - \alpha^2. \tag{4.6.14}$$

如上所述，这一结果对应于 Wiener 过程的重对数律和 Lévy 连续模. 在第二章和第三章中已指明许多 Gauss 过程有着类似于 Wiener 过程的连续模. 例如，在第三章中我们研究了 l^p 值 Gauss 过程的增量, 定理 3.3.3, 3.3.4 给出了 l^p 值 Gauss 过程的连续模. 形如 (4.6.14) 的分形性质是怎样的呢？ Deheuvels 和 Mason (1994, 1995) 也研究了具有独立增量的过程和经验过程的分形性质. 但他们的方法不能用来研究具有相依增量的 Gauss 过程.

设 $\{Y(t),-\infty<t<\infty\}=\{X_k(t),-\infty<t<\infty\}_{k=1}^{\infty}$ 是独立 Gauss 过程序列， $EX_k(t)=0$ 且有平稳增量 $\sigma_k^2(h)=E(X_k(t+h)-X(t))^2$, 这里总设 $\sigma_k(h)$ 是非降连续函数. 回顾前面的记号：

$$\sigma(p,h)=\Big(\sum_{k=1}^{\infty}\sigma_k^p(h)\Big)^{1/p},\qquad \sigma^*(h)=\max_{k\ge1}\sigma_k(h),$$

$$\tilde\sigma(p,h)=\begin{cases}\sigma\big(\dfrac{2p}{2-p},h\big), & \text{若 } 1\le p<2,\\ \sigma^*(h), & \text{若 } p\ge 2,\end{cases}$$

$$\delta_p^p=E|N(0,1)|^p,\quad p\ge 1.$$

若定理 3.3.4 的条件被满足，那么我们有

$$\begin{aligned}&\limsup_{h\to0}\inf_{0\le t\le1}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\\ &\ge\limsup_{h\to0}\min_{0\le n\le h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}\\ &\quad-2\limsup_{h\to0}\sup_{0\le t\le2}\sup_{0\le s\le h^2}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h^2)}\cdot\frac{\sigma(p,h^2)}{\sigma(p,h)}\\ &=\limsup_{h\to0}\min_{0\le n\le h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}.\end{aligned}\tag{4.6.15}$$

类似于 (3.3.33) 的证明，对任何 $1<\theta<2$, 当 h 充分小时我们有

$$\begin{aligned}&P\Big\{\min_{0\le n\le h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}\le 2-\theta\Big\}\\ &\le 2(h^{-2}+1)\exp\Big(-\frac{(\theta-1)^2\delta_p^2\sigma^2(p,h)}{8\,\tilde\sigma^2(p,h)}\Big)\\ &\le 4h^{-2}\exp\Big(-4\log\frac1h\Big)=4h^2\to0\quad h\to0,\end{aligned}$$

它与 (4.6.15) 一起可推出

$$\limsup_{h\to 0}\inf_{0\le t\le 1}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\ge 1 \quad \text{a.s.}$$

从而，如果我们定义一个类似于 $B(\alpha)$ 的随机集合：

$$E(\alpha)=\left\{t\in[0,1]:\limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\ge\alpha\right\},\quad 0\le\alpha\le 1,$$

则对任何 $0\le\alpha\le 1$ 有 $E(\alpha)=[0,1]$ a.s. 因此这时 $E(\alpha)$ 的分形没有什么需要我们考察的.

现在，假设定理 3.3.3 中的条件被满足，并定义类似于 $B(\alpha)$ 的随机集

$$E(\alpha)=\left\{t\in[0,1]:\limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}}\ge\alpha\right\},\quad 0\le\alpha\le 1. \tag{4.6.16}$$

Zhang (1997d) 得到了关于 $E(\alpha)$ 的 Hausdorff 维数的下述结果.

定理 4.6.7 假设对某个 $\Lambda>0$, 在 $[0,\Lambda]$ 上 $\tilde\sigma(p,h)/h^\Lambda$ 是拟增的. 又设

$$\sigma(p,h)=o(\tilde\sigma(p,h)(\log h^{-1})^{1/2}),\qquad h\to 0, \tag{4.6.17}$$

且对每一 $\epsilon>0$

$$\limsup_{h\to 0}\max_{h^{-\epsilon}\le j\le h^{-1}}\max_{k\ge 1}\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{(\log h^{-1})^{-2}\sigma_k^2(h)}\le 0.$$

那么对任何 $\alpha\in[0,1]$, 概率为 1 地有

$$\dim E(\alpha)=1-\alpha^2.$$

对随机集合

$$E^*(\alpha)=\left\{t\in[0,1]:\limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}}=\alpha\right\},\quad 0\le\alpha\le 1, \tag{4.6.18}$$

我们有下述结果:

定理 4.6.8 假设对某个 $\Lambda > 0$, 在 $[0,\Lambda]$ 上 $\tilde{\sigma}(p,h)/h^{\Lambda}$ 是拟增的. 又设 (4.6.17) 成立, 且对某个 $P > 0$

$$\limsup_{h\to 0} \max_{(\log h^{-1})^P \le j \le h^{-1}} \max_{k\ge 1} \Big[E(X_k(h) - X_k(0))(X_k((j+1)h) \\ -X_k(jh))\Big] \Big/ \Big[(\log h^{-1})^{-2}\sigma_k^2(h)\Big] \le 0.$$

那么对任何 $\alpha \in [0,1]$, 概率为 1 地有

$$\dim E^*(\alpha) = 1 - \alpha^2.$$

作为定理 4.6.8 的推论, 我们可以得到关于 l^p 值分数 O-U 过程和分数 O-U 过程的无穷级数的分形结果.

4.6.4 Ornstein-Uhlenbeck 过程的无穷级数的增量与 Chung 重对数律有关的分形性质

设 $\{W(t); t \ge 0\}$ 是标准 Wiener 过程. Orey 和 Taylor (1974) 也研究了集合

$$B_1(\alpha) := \left\{ t \in [0,1] : \liminf_{h\to 0} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} \\ \cdot \sup_{0\le s\le h} |W(t+s) - W(t)| \le \alpha \right\} \quad (\alpha \ge 1)$$

的分形性质. 他们证明了对每一 $\alpha \ge 1$, $B_1(\alpha)$ 是随机分形且有

定理 4.6.9 对任给的 $\alpha \ge 1$, 概率为 1 地有

$$\dim B_1(\alpha) = 1 - \alpha^{-2}.$$

这一结果是对应于 Chung 型重对数律和不可微模的. 在本章的前几节中, 已介绍过对一大类 Gauss 过程, Chung 型重对数律成立. 例如定理 4.2.5 和 4.4.6 分别给出了 O-U 过程的无穷级数的

Chung 型重对数律和精确不可微模. 现在, 我们将对这类过程建立类似于定理 5.4.1 的分形性质.

设 $\{Y(t); -\infty < t < \infty\} = \{X_k(t); -\infty < t < \infty\}_{k=1}^{\infty}$ 是独立 O-U 过程序列, 具有系数 γ_k 和 λ_k. 假设 $\{X(t); -\infty < t < \infty\} = \{\sum_{k=1}^{\infty} X_k(t); -\infty < t < \infty\}$ 是 $\{Y(t); -\infty < t < \infty\}$ 的无穷级数. 令

$$\Gamma_0 = 2\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} < \infty, \quad \Gamma_1 = 2\sum_{k=1}^{\infty} \gamma_k > 0,$$

$$\sigma^2(h) = E(X(t+h) - X(t))^2 = 2\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}).$$

我们定义一个类似于 $B_1(\alpha)$ 的随机集合

$$E_1(\alpha) := \left\{ t \in [0,1] : \liminf \left(\frac{8\log h^{-1}}{\pi^2 \sigma^2(h)} \right)^{1/2} \cdot \sup_{0 \le s \le h} |X(t+s) - X(t)| \le \alpha \right\} \qquad (\alpha \ge 1).$$

Zhang (1998) 证明了下述定理:

定理 4.6.10 假设 $\Gamma_1 = 2\sum_{k=1}^{\infty} \gamma_k < \infty$. 则对任一 $\alpha \ge 1$, 概率为 1 地有

$$\dim E_1(\alpha) = 1 - \alpha^{-2}.$$

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注 * §n.0 表示第 n 章引言部分.

Key Book for the
"Ninth Five-year Plan" of China

PATH PROPERTIES OF GAUSSIAN PROCESSES

Written by Lin Zheng-yan
Lu Chuan-rong
Zhang Li-xin

Foreword

It pleases me very much to have opportunity for introducing this remarkable book on fine analytic path properties of Gaussian and related processes. In a series of papers in the nineteen twenties, Norbert Wiener undertook a mathematical analysis of Brownian motion. He showed that, except for a set of cases of probability zero (with respect to what has been called Wiener measure since), all the Brownian motion paths were continuous non-differentiable curves. In the forties, Paul Lévy proved his famous modulus of continuity theorem that established "the exact rate of continuity" for almost all sample paths of Brownian motion (Wiener process). Ever since, these fundamental contributions have been the principal guidelines in the literature on path properties of general Gaussian and many other related stochastic processes.

Inspired by the approach taken in Chapter 1 of the 1981 M. Csörgő and P. Révész's book to constructing, and proving precise fine analytic path properties of, a Wiener process, in 1987, together with Lin Zheng-yan, we initiated a study of the path behavior of infinite di-

mensional Ornstein-Uhlenbeck processes along similar lines. This line of research has since evolved into similarly studying a wide class of more general Gaussian, as well as other, stochastic processes. In conjunction with the fundamental methods and achievements of the worldwide "French school", the literature on this subject is immense. This book by Lin Zheng-yan, Lu Chuan-rong and Zhang Li-xin is a most timely basic exposition on the global foundations and some of the more recent developments in this ever more beautifully growing complex area of research on the intrinsic path behavior of stochastic processes.

Miklós Csörgő
(a fellow of the Academy of Sciences of the Royal Society of Canada)

Contents

Introduction

Path behavior is one of the basic properties of a stochastic process. Earlier on, the boundedness, the continuity and the non-differentiability were investigated for Gaussian processes. And the study was started with the Brownian motion (Wiener process) which is of a lot of good characteristics.

Let $\{W(t); t \geqslant 0\}$ be a standard Wiener process, i.e., it is a process with the following properties:

(i) $W(t) - W(s)$ is a Gaussian variable with mean zero and variance $t-s$ for all $0 \leqslant s < t < \infty$, and $W(0) = 0$ a.s.;

(ii) $W(t)$ is an independent increment process, i.e., $W(t_2) - W(t_1), W(t_4) - W(t_3), \cdots, W(t_{2i}) - W(t_{2i-1})$ are independent random variables for all $0 \leqslant t_1 < t_2 \leqslant t_3 < t_4 \leqslant \cdots \leqslant t_{2i-1} < t_{2i} < \infty$ $(i = 2, 3, \cdots)$;

(iii) The sample path function $W(t, \omega)$ is continuous in t with probability one.

A standard Wiener process $\{W(t); t \geqslant 0\}$ is a mathematical model of a Brownian motion. Wiener N (1923)

showed that, except for a set of events of probability zero, all paths of $\{W(t); t\geqslant 0\}$ were continuous curves. And later, Wiener and Zygmund (1933) showed that, except for a set of events of probability zero, its all sample paths were non-differentiable. Lévy P (1937, 1948) proved his famous modulus of continuity theorem stated as follows.

Theorem 0.1 *We have*

$$\lim_{h\to 0}\frac{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|W(t+s)-W(t)|}{\sqrt{2h\log(1/h)}}=1 \quad \text{a. s.}$$

and

$$\lim_{h\to 0}\frac{\sup_{0\leqslant t\leqslant 1-h}|W(t+h)-W(t)|}{\sqrt{2h\log(1/h)}}=1 \quad \text{a. s.}$$

The boundedness and the continuity of general Gaussian processes have been studied since 1950's. For a general discuss, one can refer to the work of Adler R J (1990). In Section 2.1 we will introduce some of his main results.

Another important result on the sample path properties of the Wiener process is the functional law of the iterated logarithm proved by Strassen V (1964). In his paper, by using his strong approximation Strassen V also proved that the partial sums of independent identically distributed random variables have the similar property.

In Chapter 1 of the book *Strong Approximations in Probability and Statistics* by Csörgő M and Révész P (1981), fine analytic path properties of a Wiener process were proved. For example, they proved the following theorem on how big the increments of a Wiener process are.

Theorem 0.2 *Let $a_T(T\geqslant 0)$ be a non-decreasing function of T*

for which

(i) $0<a_T\leqslant T$,

(ii) *T/a_T is non-decreasing.*

Then

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=1 \quad \text{a. s.},$$

$$\limsup_{T\to\infty}\beta_T|W(T+a_T)-W(T)|=1 \quad \text{a. s.},$$

$$\limsup_{T\to\infty}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(T+s)-W(T)|=1 \quad \text{a. s.}$$

and

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|=1 \quad \text{a. s.},$$

where $\beta_T=\{2a_T(\log T/a_T+\log\log T)\}^{-1/2}$. If we have also

(iii) $\lim_{T\to\infty}(\log T/a_T)/\log\log T=\infty$,

then

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=1 \quad \text{a. s.}$$

and

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|=1 \quad \text{a. s.}$$

Some authors also studied how large the increments must be when a_T does not satisfy the condition (iii). For example, if (iii) is replaced by

(iv) $\lim_{T\to\infty}(\log T/a_T)/\log\log T=r$, $0<r\leqslant\infty$,

then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=\left(\frac{r}{1+r}\right)^{1/2} \quad \text{a. s.}$$

(c. f. Book S A and Shore T R, 1978); if (iii) is replaced by

(v) $\lim_{T\to\infty}(\log(T/a_T))/\log\log\log T=\infty$,

then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\gamma_1(T)|W(t+s)-W(t)|=1 \quad \text{a. s.},$$

where

$$\gamma_1(T)=\left\{2a_T\log\left(1+\frac{\pi^2}{16a_T}\frac{T}{\log\log T}\right)\right\}^{-1/2}$$

(c. f. Csáki E and Révész P, 1979); and if

(vi) $\lim_{T\to\infty}(T/a_T)/\log\log T=\infty$,

then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\gamma_0(T)|W(t+s)-W(t)|=1 \quad \text{a. s.},$$

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\gamma_0(T)|W(t+a_T)-W(t)|=1 \quad \text{a. s.},$$

where $\gamma_0(T)=\{2a_T(\log(T/a_T)-\log\log\log T)\}^{-1/2}$ (c. f. Shao Q M, 1986).

Obviously, Theorems 0. 1 and 0. 2 are related to Lévy's famous law of the iterated logarithm:

Theorem 0. 3 *(Lévy 1937, 1948) We have*

$$\limsup_{T\to\infty}\frac{|W(T)|}{\sqrt{2T\log\log T}}=1 \quad \text{a. s.},$$

$$\limsup_{h\to 0}\frac{|W(h)|}{\sqrt{2h\log\log 1/h}}=1 \quad \text{a. s.}$$

Csörgő M and Révész P (1981) also studied another type of path properties of a Wiener process $\{W(t);t\geqslant 0\}$. They gave the moduli of non-differentiability of $\{W(t);t\geqslant 0\}$ and showed how small its increments are.

Theorem 0. 4 *We have*

$$\lim_{h\to 0}\inf_{0\leqslant s\leqslant 1-h}\sup_{0\leqslant t\leqslant h}\sqrt{\frac{8\log h^{-1}}{\pi^2 h}}|W(s+t)-W(s)|=1 \quad \text{a. s.}$$

Theorem 0. 5 *Let a_T be a non-decreasing function of T for which conditions* (i) *and* (ii) *in Theorem* 0. 2 *are satisfied. Then*

$$\liminf_{T\to\infty}\gamma_T\inf_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|=1 \quad \text{a. s.},$$

where $\gamma_T=\{8(\log Ta_T^{-1}+\log\log T)/(\pi^2 a_T)\}^{1/2}$. If condition (iii) *in Theorem* 0. 2 *is also satisfied, then*

$$\lim_{T\to\infty}\gamma_T\inf_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|=1 \quad \text{a. s.}$$

Theorem 0. 4 implies the well-known result that almost all sample paths of a Wiener process are nowhere differentiable. Theorems 0. 4 and 0. 5 are related to Chung's laws of the iterated logarithm.

Theorem 0. 6 *We have*

$$\liminf_{T\to\infty}\left(\frac{8\log\log T}{\pi^2 T}\right)^{1/2}\sup_{0\leqslant t\leqslant T}|W(t)|=1 \quad \text{a. s.}$$

$$\liminf_{h\to 0}\left(\frac{8\log\log(1/h)}{\pi^2 h}\right)^{1/2}\sup_{0\leqslant t\leqslant h}|W(t)|=1 \quad \text{a. s.}$$

In the book of Csörgő M and Révész P (1981), it is also investigated how big the increments of a two-parameter Wiener process are.

After the publication of Csörgő M and Révész P (1981), the theory on the strong approximation has been developed quickly. Also, many of the main results in Chapter 1 of the book were improved and were extended to other processes. On an invitation of Csörgő M in 1987, Lin Z Y was a visitor at Carleton University in Canada. During this time Csörgő M suggested a study of some precise path behavior of Ornstein-Uhlenbeck processes. Since then, Csörgő M, Lin Z Y, Shao Q M and various other Hungarian and Chinese scholars have been working on this subject, studying not only some special Gaussian processes but also Gaussian processes in general as well. In their book *Strong Limit Theorems*, Lin Z Y and Lu C R (1992) introduced the related main results up to that time. On the sample path properties, in

their book,

(1) The increments of the Wiener process were studied more precisely and perfectly, for example, the result of Shao Q M (1986) on the liminfs of the increments under the condition (vi) was introduced, the general form of the increments of the Wiener process was discussed, and the limits on the lag increments and their convergence rates were studied;

(2) The result on the increments of the Wiener process under the condition (v) was extended to the two-parameter Wiener process, and, the general form of the lag increments and the general form of the increments of the two-parameter Wiener process were studied;

(3) The result of Ortega J (1984) on the moduli of continuity and how large the increments are of a fractional Wiener process was introduced;

(4) The investigation was initiated of the processes generated by the infinite dimensional Ornstein-Uhlenbeck processes, such as the partial sum process, the infinite series and l^2-norm squared processes.

In 1990's, besides the further study on the increments of the Wiener process, the research has evolved into similar study on many other special Gaussian processes of practical background, as well as a wide class of more general Gaussian processes. The aim of this book is to compile results on various kinds of fine analytic path properties of these Guassian processes, such as continuity, non-differentiability, moduli of continuity and large increments etc., for general Gaussian processes. Some special Gaussian processes (e. g. Ornstein-Uhlenbeck processes) are

considered in more detail. The book extends and deepens some of the results of the two books mentioned above. And it can be regarded as a continuation of them.

In Chapter 1, we introduce some basic tools for the study in the sequel. In particular, a list of important inequalities are presented, such as Borell's inequality, Fernique's inequality, Slepian's inequality, Anderson's inequality, Khatri-Šidák's inequality, etc. As well, an extension of the Borel-Cantelli lemma is given in Section 2. 1, which is another basic tool.

In Chapter 2, some basic results on the continuity and the boundedness of general Gaussian processes are introduced in Section 2. 1 firstly. And then, we start our main topics on the moduli of continuity and limit behavior of large increments of Gaussian processes in Section 2. 2. Many related results, such as the moduli of continuity, the results on how large the increments are and the limit inferior behavior and the general form of the increments, etc., for the Wiener processes are extended in this section to the fractional Wiener processes, and some are also extended to the Gaussian processes with negative correlation. In Section 2. 3, we study the limit inferior behavior of the increments of a two-paramenter Wiener process under a condition similar to (vi). In Sections 2. 4—2. 6, we study moduli of continuity and limit behavior of large increments of a two-parameter Lévy-Wiener process and a two-parameter Ornstein-Uhlenbeck process, as well as more general kernel generated two-parameter Gaussian processes. In the last section of Chapter 2, the local time process of a Gaussian process is investigated, which is also a two-parameter process.

Chapter 3 deals with the infinite dimensional Gaussian processes. Firstly, the conditions for the continuity of l^p-valued Gaussian processes are given in Section 3.1. In particular, the sufficient and necessary condition for the continuity of l^2-valued Ornstein-Uhlenbeck processes is presented. With the estimates of the limsups of a general B-valued separable stochastic process in Section 3.2, we start the study of the moduli of continuity and limit behavior of large increments of l^p-valued Gaussian processes with nearly negative correlation in some sense. In the last section of Chapter 3, the l^∞-valued Gaussian processes are studied.

In Chapter 4, Strassen's law of the iterated logarithm of a class of Gaussian processes with stationary increments and the rate of Strassen's law of the iterated logarithm of a self similar Gaussian process with index α are investigated in Section 4.1. Also, the Strassen's moduli of continuity and Strassen's limit behavior of the large increments of the Wiener process are introduced in this section. In Section 4.2, Erdös-Révész's law of the iterated logarithm of the Wiener process and a class of Gaussian processes with stationary increments are discussed. In particular, Erdös-Révész's law of the iterated logarithm of the infinite series of independent Ornstein-Uhlenbeck processes and the fractional Wiener process is established.

The rest four sections of Chapter 4 deal with Chung's law of the iterated logarithm, the moduli of non-differentiability and how small the increments are of Gaussian processes and Gaussian fields. The small ball probability estimates are the key to study this kind of limit inferior behavior. In Section 4.3, the small ball probability estimates for Gaussian processes with stationary increments, especially for the fractional Wiener processes, are presented, from which Chung's laws of the iterated logarithm of the fractional Wiener processes and the infinite series of Ornstien-Uhlenbeck processes are established. Similar results for Gaussian fields are given in Section 4.4. In Section 4.5, limit inferior behavior of the increments of Gaussian processes are studied. In particular, the exact moduli of non-differentiability of the infinite series of independent Ornstein-Uhlenbeck processes are established. In Section 4.6, the limit inferior behavior for some two-parameter Gaussian processes are investigated. It shall be mentioned that the small ball probability estimate problem as well as its related topics is still an interesting "hot" topic in growing, and there are many new researches on this problem when we prepare the book. One can refer to Li W V and Shao Q M (1999) for more details.

In the last chapter of the book, we introduce some other path properties of Gaussian processes, such as the p-variation and the fractal natures. In Section 5.1, a much general result on the p-variation of a Gaussian process is introduced. In Section 5.2, the Hausdorff dimension and packing dimension of the image and the graph of a Gaussian field are investigated. In Section 5.3, the fractal nature of the increments of the l^p-valued Gaussian process is studied. At last, the fractal nature related to Chung's law of the iterated logarithm of the infinite series of Ornstein-Uhlenbeck processes is discussed in Section 5.4.

The book is dedicated to Miklós Csörgő (a fellow of the Academy of Sciences of the Royal Society of Canada) on the occasion of his 65th birthday.

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Chapter 1

Some Basic Results on Gaussian Variables and Gaussian Processes

Throughout the book, c denotes a positive constant, which may take different values at different places. The letters K, C,… almost always denote the numerical constants. $\mathbf{N}$ is the set of all non-negative integers, $\mathbf{Z}$ is the set of all integers, $\mathbf{R}$ is the set of all real numbers, $\mathbf{Z}_+$ is the set of all positive integers, $\mathbf{R}_+$ is the set of all positive real numbers. We write $A \approx B$ to signify that $C^{-1}A \leqslant B \leqslant CA$ for some positive constant C. We also use the symbol $\sim$ to signify that two sequences (of real numbers) are equivalent, or only of the same order of growth. $(\cdot)^+$, $(\cdot)^-$, $[\cdot]$ denote respectively the positive, negative and integer part functions. Card(A) is the cardinality of a (finite) set A. A^c is the complement of A. I_A is the indicator function of a set A. $(\Omega, \mathscr{F}, P)$ will be a complete probability space.

In this chapter, we present some basic results preparatory to studying sample path properties of Gaussian processes, in-

cluding a list of important inequalities, such as Borell's inequality, Fernique's inequality, Slepian's inequality, Anderson's inequality, Khatri-Šidák's inequality, etc.

Firstly, we recall some basic definitions and classical properties of Gaussian variables.

A real valued random variable X with mean $\mu \in \mathbf{R}$ and variance $\sigma^2 \in \mathbf{R}_+$ is said to be Gaussian (or normal) if its Fourier transform satisfies

$$Ee^{itX} = e^{it\mu - \frac{1}{2}\sigma^2 t^2},$$

or, equivalently, the law of X has the density $\sigma^{-1}\varphi((x-u)/\sigma)$, where

$$\varphi(x) := (2\pi)^{-\frac{1}{2}}\exp(-x^2/2),$$

i.e.

$$\begin{aligned} &P\{(X-\mu)/\sigma \leqslant x\} \\ &= \Phi(x) := \int_{-\infty}^{x} \varphi(t)\mathrm{d}t,\ x \in (-\infty, \infty). \end{aligned}$$

If $\mu = 0$ we call X centered, and if we also have $\sigma = 1$, X is called standard normal. A random vector $\boldsymbol{X} = (X_1, \cdots, X_N)$ in $\mathbf{R}^N$ is (centered) Gaussian if for all real numbers $\alpha_1, \cdots, \alpha_N$, $\sum_{i=1}^N \alpha_i X_i$ is a real valued (centered) Gaussian random variable. The distribution of a centered Gaussian random vector $\boldsymbol{X}$ in $\mathbf{R}^N$ is completely determined by its symmetric (semi-) positive definite covariance matrix $\boldsymbol{\Gamma} = (EX_iX_j)_{1\leqslant i,j\leqslant N}$. Indeed, if $\boldsymbol{\Gamma} = \boldsymbol{A}\boldsymbol{A}'$, the distribution of $\boldsymbol{X}$ is the same as that of $\boldsymbol{A}\boldsymbol{g}$ where $\boldsymbol{g} = (g_1, \cdots, g_N)$ is distributed as the canonical Gaussian distribution γ_N on $\mathbf{R}^N$ with density

$$(2\pi)^{-N/2}\exp(-\|x\|^2/2),$$

where $\|\cdot\|$ is the standard Euclidean norm.

A (centered) Gaussian process is a family $X = \{X_t; t \in T\}$ of random variables, indexed by a parameter set T, such that each finite linear combination $\sum \alpha_t X_t$ is (centered) Gaussian. Throughout the book, T is almost always a subset of $\mathbf{R}$, and sometimes a subset of $\mathbf{R}^k, k > 1$ ("multi-parameter time"), or $[0,1]^k$. If X is a centered Gaussian process, the covariance function $\Gamma(s,t) = EX_sX_t, s,t \in T$, completely determines the distribution of X.

Without further mention we shall always assume that T is a metric space and has a countable dense subset, and shall assume that X is a separable stochastic process, i.e. there exists a negligible set $\Omega_0 \subset \Omega$ and a countable dense set S in T such that, for every $\omega \notin \Omega_0$, every $t \in T$ and $\varepsilon > 0$,

$$X_t(\omega) \in \overline{\{X_s(\omega); s \in S, d(s,t) < \varepsilon\}},$$

where the closure is taken in $\mathbf{R} \cup \{\infty\}$ and $d(\cdot,\cdot)$ is the metric on T. If X is separable, in particular, $\sup_{t\in T}|X_t(\omega)| = \sup_{s\in S}|X_s(\omega)|$, $\sup_{t\in T} X_t(\omega) = \sup_{s\in S} X_s(\omega)$ for every $\omega \notin \Omega_0$. Moreover, every dense sequence S in T can be chosen as a separable set. A process $\{X_t; t \in \mathbf{R}^k\}$ is called to be (strictly) stationary if its distribution is the same as that of $\{X_{t+s}; t \in \mathbf{R}^k\}$ for any $s \in \mathbf{R}^k$. A process $X = \{X_t; t \in T\}$ is almost surely bounded or continuous, or has almost all its trajectories or sample paths bounded or continuous, if for almost all ω, the path $t \to X_t(\omega)$ is bounded or continuous.

Finally, given a Banach space B such that there exists a countable subset D of the unit ball or sphere of the dual space B' such that $\|x\| = \sup_{f\in D} f(x), x \in B$, a random variable X in B is called (centered) Gaussian if $f(X)$ is measurable for every f

in D and if every finite linear combination $\sum_i \alpha_i f_i(X), \alpha_i \in \mathbf{R}, f_i \in D$ is (centered) Gaussian. It is easily seen that X can be regarded as a Gaussian process $\{f(X); f \in D\}$ indexed by D.

There are many modern results on the probability inequalities of a Gaussian vector. We state two of them here. The first is the so-called isoperimetric inequality: for any Borel set A in $\mathbf{R}^N$,

$$\operatorname{inv}\Phi(\gamma_N(A_r)) \geqslant \operatorname{inv}\Phi(\gamma_N(A)) + r, \tag{1.0.1}$$

where A_r is the Euclidean neighborhood of order r of A, and in particular, if $\gamma_N(A) \geqslant 1/2$,

$$1 - \gamma_N(A_r) \leqslant 1 - \Phi(r) \leqslant \frac{1}{2} e^{-r^2/2};$$

the second is Brunn-Minkowski's type inequality: for convex sets A, B in $\mathbf{R}^N$ and $\lambda \in [0,1]$,

$$\begin{aligned} &\operatorname{inv}\Phi(\gamma_N(\lambda A + (1-\lambda)B)) \\ &\geqslant \lambda \operatorname{inv}\Phi(\gamma_N(A)) + (1-\lambda)\operatorname{inv}\Phi(\gamma_N(B)), \end{aligned} \tag{1.0.2}$$

where

$$\lambda A + (1-\lambda)B = \{x \in \mathbf{R}^N; x = \lambda a + (1-\lambda)b, a \in A, b \in B\}.$$

If A is convex, (1.0.1) can be deduced from (1.0.2) by taking B to be the Euclidean ball with center the origin and radius $r/(1-\lambda)$ and letting λ tend to 1.

For the proofs of (1.0.1) and (1.0.2) one can refer to Ledoux and Talagrand (1991), Ehrhard (1983, 1984, 1986) respectively, we do not present them here.

In this chapter, we give two kinds of basic results in theory of Gaussian processes. The first is about the tail behavior of the supremum of a Gaussian process. The second is the comparison theorems.

1.1 Tail Behavior of the Supremum of a Gaussian Process

1.1.1 Borell's inequality

Let X be a centered separable Gaussian random variable with variance σ^2. Then set

$$\Psi(x) := 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.$$

Straightforward approximations give that for all $x > 0$

$$\begin{aligned} &(1 - \sigma^2 x^{-2})(\sigma/\sqrt{2\pi}) x^{-1} e^{-\frac{1}{2}x^2/\sigma^2} \\ &\leqslant P\{X > x\} = \Psi(x/\sigma) \leqslant (\sigma/\sqrt{2\pi}) x^{-1} e^{-\frac{1}{2}x^2/\sigma^2}. \end{aligned} \tag{1.1.1}$$

Assume that $\{X_t; t \in T\}$ has bounded sample paths with probability one, the Borell's inequality tells us that $\sup_{t \in T} X_t$ behaves much like X with $\sigma_T^2 := \sup_{t \in T} EX_t^2$ instead of σ^2, which we shall give as

Theorem 1.1.1 *Let $\{X_t; t \in T\}$ be a centered separable Gaussian process with almost surely bounded sample paths. Let $\|X\| = \sup_{t \in T} X_t$, then for all $\lambda > 0$*

$$P\{|\,\|X\| - E\|X\|\,| > \lambda\} \leqslant 2\exp(-\lambda^2/(2\sigma_T^2)), \tag{1.1.2}$$

where $\sigma_T^2 := \sup_{t \in T} EX_t^2$.

Since a general Banach space valued Gaussian variable can be regarded as a Gaussian process, the process in Theorem 1.1.1

can be taken to be a Banach space valued Gaussian variable. Furthermore, the term $E\|X\|$ in (1.1.2) can be replaced by the median of $\|X\|$. Though (1.1.2), with the median of $\|X\|$ instead of the mean, can be deduced from the isoperimetric inequality (1.0.1) (cf. Ledoux and Talagrand 1991), we present a purely probabilistic proof here, which was given by Maurey and Pisier (Pisier 1986).

Recall that $\{X_t; t\in T\}$ is a separable process, we assume the existence of a countable subset D in T for which

$$\sup_{t\in D} X_t = \sup_{t\in T} X_t.$$

Consequently, to prove Borell's inequality, it suffices to replace $\|X\|$ by $\sup_{t\in D} X_t$ in (1.1.2). Furthermore, if we can show that (1.1.2) holds for D finite, this will also suffice. Taking $D=\{t_1,\cdots,t_k\}, t_i\in T, k<\infty$, the following lemma is the main step in the proof.

Lemma 1.1.1 *Let f: $\mathbf{R}^k\to\mathbf{R}$ have derivatives of up to second order, bounded pointwise by $Ae^{B\|x\|}$ for some $A, B<\infty$, where $\|\cdot\|$ is the usual Euclidean norm, and let $X=(X_{t_1},\cdots,X_{t_k})$ be a k-dimensional centered Gaussian variable with covariance matrix $V_D=(EX_{t_i}X_{t_j})_{1\leqslant i,j\leqslant k}$. If $|f(x)-f(y)|\leqslant\|x-y\|$ for all $x,y\in\mathbf{R}^k$, then for all $\lambda>0$*

$$P\{|f(X)-Ef(X)|>\lambda\}\leqslant 2\exp(-\lambda^2/(2\sigma^2)),\quad(1.1.3)$$

where

$$\sigma^2=\sup_{1\leqslant i\leqslant k} V_D(i,i)=\sup_{1\leqslant i\leqslant k} EX_{t_i}^2.$$

Proof Let $\{B_s; s\geqslant 0\}=\{(B_s^1,\cdots,B_s^k); s\geqslant 0\}$ be a k-dimensional Wiener process, i.e., the B^i are i.i.d. standard, real-valued Wiener process, and choose $0\leqslant s_0<s_1<\cdots<s_n\leqslant 1$. Let

$\mathscr{F}_j$ be the σ-field generated by $\{B_{s_0},\cdots,B_{s_j}\}$, and let $\{V_j; 0\leqslant j\leqslant n\}$ be a sequence of $\mathbf{R}^k$-valued random variables such that V_j is $\mathscr{F}_{j-1}$-measurable for all j. Assume $\|V_j\|<\sigma$ a.s. for all j, and set

$$S_m=\sum_{j=1}^m\langle V_j, B_{s_j}-B_{s_{j-1}}\rangle.\quad(1.1.4)$$

Using the measurability of V_j and the independence of the increments of B_t, we have

$$Ee^{\theta S_n}=E(e^{\theta S_{n-1}}e^{\theta\langle V_n,B_{s_n}-B_{s_{n-1}}\rangle})\leqslant E(e^{\theta S_{n-1}}e^{\frac{1}{2}\theta^2(S_n-S_{n-1})\sigma^2})$$

for all real θ, where the last expression follows from standard Gaussian calculations and the fact that $\|V\|\leqslant\sigma$. Thus

$$Ee^{\theta S_n}\leqslant e^{\frac{1}{2}\theta^2\sigma^2}.$$

A standard Chebychev type argument gives us that

$$\begin{aligned}P\{|S_n|>\lambda\}&=2P\{S_n>\lambda\}\leqslant 2e^{-\theta\lambda}Ee^{\theta S_n}\\&\leqslant 2e^{-\theta\lambda}e^{\frac{1}{2}\theta^2\sigma^2}=2e^{-\frac{1}{2}\lambda^2/\sigma^2},\end{aligned}\quad(1.1.5)$$

the final equality is a consequence of setting $\theta=\lambda/\sigma^2$. Recall now Ito's formula, that for a sufficiently smooth function

$$F=F(x,t):\mathbf{R}^k\times\mathbf{R}_+\to\mathbf{R},$$

$$\begin{aligned}&F(B_t,t)-F(B_s,s)\\&=\int_s^t(\nabla_x F(B_u,u),\mathrm{d}B_u)+\int_s^t\left(\frac{1}{2}\Delta_{xx}F(B_u,u)+F_t(B_u,u)\right)\mathrm{d}u,\end{aligned}\quad(1.1.6)$$

where $\nabla_x=(\partial/\partial x_1,\partial/\partial x_2,\cdots,\partial/\partial x_k)$ and $\Delta_{xx}=\sum_{i,j=1}^k\partial^2/\partial x_i\partial x_j$ denote derivatives of $F(x,t)$ with respect to x, and $F_t(x,t)=\partial F(x,t)/\partial t$. Let $(P_t)_{t\geqslant 0}$ be the Markov semi-group associated with B, so that for smooth $g:\mathbf{R}^k\to\mathbf{R}$

$$(P_tg)(x)=E^xg(B_t)=(2\pi t)^{-k/2}\int_{\mathbf{R}^k}g(y)e^{-\frac{1}{2}\|x-y\|^2/t}\mathrm{d}y,$$

where E^x denotes expectation with respect to the Wiener process B starting at that point $x \in \mathbf{R}^k$ at time zero. Let $\hat{f}: \mathbf{R}^k \to \mathbf{R}$ satisfy the differentiability requirements of the f of the lemma, and assume $|\hat{f}(x) - \hat{f}(y)| \leqslant \sigma \| x - y \|$. Setting $F(x,t) = (P_{1-t}\hat{f})(x)$, the conditions of the lemma imply that F is sufficiently smooth for (1.1.6) to hold. Setting $t=1$, $s=0$ this yields

$$\hat{f}(B_1) - E\hat{f}(B_1) = \int_0^1 \langle \nabla_x (P_{1-u}\hat{f})(B_u), \mathrm{d}B_u \rangle.$$

(Some algebra is necessary.) The fact that $|\hat{f}(x) - \hat{f}(y)| \leqslant \sigma \| x-y \|$ immediately implies that $P_{1-u}\hat{f}$ satisfies the same inequality, and so $\| \nabla_x P_{1-u} \hat{f} \| \leqslant \sigma$ a. s. It then easily follows from (1.1.5) that

$$P\{|\hat{f}(B_1) - E\hat{f}(B_1)| > \lambda\} \leqslant 2\mathrm{e}^{-\frac{1}{2}\lambda^2/\sigma^2}. \qquad (1.1.7)$$

To complete the proof note simply that $f(X) \stackrel{L}{=} f(V_D^{\frac{1}{2}} B_1)$, where $V_D^{\frac{1}{2}}$ satisfies $V_D = V_D^{\frac{1}{2}} (V_D^{\frac{1}{2}})'$, so that (1.1.3) follows from (1.1.7) with $\hat{f} = f(V_D^{\frac{1}{2}} x)$ which satisfies all the requirements placed on it.

Proof of Theorem 1.1.1

Theorem 1.1.1 will follow immediately from Lemma 1.1.1 if only sup(•) is sufficiently smooth function. Unfortunately, it is not, being non-differentiable on the diagonal. Fortunately, however, it can be approximated by smooth functions. Any standard approximation procedure will work.

It is clear that (1.1.2) implies

$$\lim_{\lambda \to \infty} \lambda^{-2} \log P\{\sup_{t \in T} X_t > \lambda\} = -(2\sigma^2)^{-1}.$$

There are many refinements of this equality. In the case of $T = [0,h]$, the following is optimal.

Theorem 1.1.2 *Let X be a centered separable Gaussian process on $\mathbf{R}$ with variance one and covariance function $\Gamma(s,t)$ satisfying*

$$\Gamma(s,t) = 1 - C_0 |s-t|^\alpha + o(|s-t|^\alpha) \quad as \quad |s-t| \to 0,$$

where $0 < \alpha \leqslant 2$ and $C_0 > 0$. Then for any $h > 0$

$$\lim_{\lambda \to \infty} \frac{P\{\max_{t \in [0,h]} X(t) > \lambda\}}{\lambda^{2/\alpha} \Psi(\lambda)} = hC_0^{1/\alpha} H_\alpha,$$

$$\lim_{\lambda \to \infty} \frac{P\{\max_{0 \leqslant j \leqslant [h\lambda^{2/\alpha}/\theta]} X(j\theta\lambda^{-2/\alpha}) > \lambda\}}{\lambda^{2/\alpha} \Psi(\lambda)} = hC_0^{1/\alpha} \frac{H_\alpha(\theta)}{\theta}$$

for each $\theta > 0$, where $H_\alpha(\theta)$ is a positive constant depending only on θ and α satisfying $\lim_{\theta \to 0} H_\alpha(\theta)/\theta = H_\alpha$, $0 < H_\alpha := \lim_{T \to \infty} T^{-1} \int_0^\infty \mathrm{e}^s P\{\sup_{0 \leqslant t \leqslant T} Y(t) > s\} \mathrm{d}s < \infty$, *and $Y(t)$ is a non-stationary Gaussian process with mean $EY(t) = -|t|^\alpha$ and covariance function* $\mathrm{Cov}(Y(s), Y(t)) = -|t-s|^\alpha + |s|^\alpha + |t|^\alpha$.

This result dates back to Pickands (1969a,b) and one can find a full and detailed proof in Leadbetter, Lindgren and Rootzén (1983). We would not present the proof here. There is an extension to random fields on $\mathbf{R}^k$, however, see Qualls and Watanabe (1973). The exact value of H_α is unknown except for two special cases, $H_1 = 1$ and $H_2 = 1/\sqrt{\pi}$, a lower and upper bound as well as two statistical estimators of H_α are given in Shao (1996a), such as:

$$(5.2)^{-1/\alpha} 0.625 \leqslant H_\alpha \leqslant (\alpha \mathrm{e}/\sqrt{\pi})^{2/\alpha} \qquad \text{if } 1 \leqslant \alpha \leqslant 2;$$

$$(\alpha/4)^{1/\alpha}(1 - \mathrm{e}^{-1/\alpha}(1 + 1/\alpha)) \leqslant H_\alpha$$

$$\leqslant (\sqrt{\alpha}(0.77\sqrt{\alpha} + 2.41(8.8 - \alpha\log(0.4 + 2.5/\alpha))^{1/2}))^{2/\alpha}$$

if $0 < \alpha < 1$. In particular, we have

$$0.12 \leqslant H_\alpha \leqslant 3.1 \qquad \text{if } 1 \leqslant \alpha \leqslant 2,$$

$$\lim_{\alpha \to 0} \alpha \log H_\alpha / \log \alpha = 1.$$

1.1.2 Fernique's inequality

Now we consider the case $T=[0,1]^k$ and assume that X is a centered separable Gaussian process with covariance function Γ. For any $(s,t)\in T\times T$, define $d(s,t)=|s-t|=\sup_{1\leqslant i\leqslant k}|s_i-t_i|$. Let φ: $[0,1]\to\mathbf{R}_+$ be a function defined by

$$\begin{aligned}\varphi(h)&=\sup_{\substack{(s,t)\in T\times T\\|s-t|\leqslant h}}\sqrt{E(X(s)-X(t))^2}\\&=\sup_{\substack{(s,t)\in T\times T\\|s-t|\leqslant h}}\sqrt{\Gamma(s,s)-2\Gamma(s,t)+\Gamma(t,t)}.\end{aligned}\tag{1.1.8}$$

Theorem 1.1.3 *Suppose* $\int_1^\infty\varphi(e^{-x^2})dx<\infty$, *and that for some* A, $EX^2(t)\leqslant A^2$ *holds for all* $t\in T$. *Then for* $x\geqslant\sqrt{1+4k\log p}$ *we have*

$$\begin{aligned}&P\left\{\sup_{t\in T}|X(t)|\geqslant x\left(A+(2+\sqrt{2})\int_1^\infty\varphi\left(\frac{1}{2}p^{-u^2}\right)du\right)\right\}\\&\leqslant\frac{5}{2}p^{2k}\int_x^\infty e^{-u^2/2}du,\end{aligned}\tag{1.1.9}$$

where $p\geqslant2$ *is an integer.*

Proof For simplicity, we define the norm:

$$\|f\|=\sup_{s\in S}|f(s)|$$

for an any function f on $S=T$ or $T\times T$. For integer $m>0$, let I_m be the multi-indices integers $\{i=(i_j);\ 1\leqslant j\leqslant k, 0\leqslant i_j<m\}$, and for each $i\in I_m$ define

$$A_i^m=\{x\in[0,1)^k;\forall\ j\in[1,k], i_j\leqslant mx_j<i_j+1\},$$

$$a_i^m=\left(\frac{2i_j+1}{2m},1\leqslant j\leqslant k\right).$$

For each m, define an (unique) approximation X_m of X on $[0,1)^k$

by

$$X_m(x)=X(a_i^m),\quad\forall i\in I_m,\ x\in A_i^m.$$

Then $\|X_m\|$ is the maximum of m^k absolute values of centered Gaussian variables with variances not larger than A, it follows that

$$\forall\ y\in\mathbf{R}_+,\ P\{\|X_m\|\geqslant yA\}\leqslant m^k\sqrt{\frac{2}{\pi}}\int_y^\infty e^{-u^2/2}du.\tag{1.1.10}$$

Given an integer m_1, choose integer $m_2>m_1$ such that m_2/m_1 is also an integer. Then $\{A_i^{m_2};\ i\in I_{m_2}\}$ is a partition of $\{A_i^{m_1};i\in I_{m_1}\}$, and $\|X_{m_1}-X_{m_2}\|$ is the maximum of m_2^k absolute values of centered Gaussian variables with variance not larger than $\varphi\left(\frac{1}{2m_1}\right)$. It follows that

$$\forall\ y\in\mathbf{R}_+,\ P\left\{\|X_{m_1}-X_{m_2}\|\geqslant y\varphi\left(\frac{1}{2m_1}\right)\right\}\leqslant m_2^k\sqrt{\frac{2}{\pi}}\int_y^\infty e^{-u^2/2}du.\tag{1.1.11}$$

Suppose that $\{y_n;n\geqslant0\}$ is a sequence of positive real numbers and $\{m_n;n\geqslant1\}$ a sequence of integers such that for each n, m_{n+1}/m_n is an integer. From (1.1.10) and (1.1.11), it follows that

$$\begin{aligned}&P\left\{\|X_{m_1}\|+\sum_{n=1}^\infty\|X_{m_n}-X_{m_{n+1}}\|\geqslant y_0A+\sum_{n=1}^\infty y_n\varphi\left(\frac{1}{2m_n}\right)\right\}\\&\leqslant\sqrt{\frac{2}{\pi}}\sum_{n=0}^\infty(m_{n+1})^k\int_{y_n}^\infty e^{-u^2/2}du.\end{aligned}\tag{1.1.12}$$

Let $A=\bigcup_n\{a_i^{m_n};i\in I_{m_n}\}$. Then A is a countable dense subset in $[0,1]^k$. Since X is separable, $\|X\|$ is equal to $\sup_{t\in A}|X(t)|$ almost surely, and the latter is less than $\|X_{m_1}\|+\sum_{n=1}^\infty\|X_{m_n}-X_{m_{n+1}}\|$. It follows that

$$P\left\{ \| X \| \geq y_0 A + \sum_{n=1}^{\infty} y_n \varphi\left(\frac{1}{2m_n}\right) \right\}$$

$$\leq \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} (m_{n+1})^k \int_{y_n}^{\infty} e^{-u^2/2} du. \qquad (1.1.13)$$

Now, for integer $p \geq 2$, let

$$m_n = p^{2^n}, y_n = x 2^{n/2}, x \geq \sqrt{1+4k\log p}, \ x_n = 2^{n/2}, \forall n \geq 0.$$

Then for any $n \geq 1$,

$$y_n \varphi\left(\frac{1}{2m_n}\right) \leq x(2+\sqrt{2})(x_n - x_{n-1}) \varphi\left(\frac{1}{2} p^{-x_n^2}\right)$$

$$\leq x(2+\sqrt{2}) \int_{x_{n-1}}^{x_n} \varphi\left(\frac{1}{2} p^{-u^2}\right) du .$$

It follows that

$$\sum_{n=1}^{\infty} y_n \varphi\left(\frac{1}{2m_n}\right) \leq x(2+\sqrt{2}) \int_1^{\infty} \varphi\left(\frac{1}{2} p^{-u^2}\right) du.$$

Also, for any $n \geq 0$

$$(m_{n+1})^k \int_{y_n}^{\infty} e^{-u^2/2} du = \int_x^{\infty} \exp\left\{ k 2^{n+1} \log p + \frac{n}{2} \log 2 - \frac{v^2}{2} 2^n \right\} dv$$

$$\leq \int_x^{\infty} \exp\left\{ -\frac{v^2}{2} + 2k\log p + \frac{1}{2}(n\log 2 + 1 - 2^n) \right\} dv,$$

which implies

$$\sum_{n=0}^{\infty} (m_{n+1})^k \int_{y_n}^{\infty} e^{-u^2/2} du$$

$$\leq p^{2k} \sum_{n=0}^{\infty} 2^{n/2} e^{-\frac{2^n-1}{2}} \int_x^{\infty} e^{-u^2/2} du \leq \frac{5}{2} p^{2k} \int_x^{\infty} e^{-u^2/2} du.$$

This completes the proof.

Corollary 1.1.1 *Let X be a centered separable Gaussian process on $T=[a,b]^k$, with covariance function Γ and $EX^2(t) \leq A$ for all $t \in T$. Let $\varphi(h)$ be defined by* (1.1.8). *Then for $x \geq \sqrt{1+4k\log p}$ we have*

$$P\left\{ \sup_{t\in T} |X(t)| \geq x\left(A + (2+\sqrt{2}) \int_1^{\infty} \varphi\left(\frac{b-a}{2} p^{-u^2}\right) du \right) \right\}$$

$$\leq \frac{5}{2} p^{2k} \int_x^{\infty} e^{-u^2/2} du,$$

where $p \geq 2$ is an integer.

Remark 1.1.1 Let X be a separable process on $T=[a,b]^k$. Suppose that there exists a function $d(\cdot,\cdot)$ on $T\times T$ and C_0, $\gamma, \beta > 0$ such that

$$P\{|X(s)-X(t)| \geq x d(x,t)\} \leq C_0 \exp(-\gamma d^{\beta}(s,t)) \quad (1.1.14)$$

for all $x \geq 0$. Define function φ: $T \to \mathbf{R}_+$ by

$$\varphi(h) = \sup_{\substack{s,t\in T \\ |s-t|\leq h}} d(s,t).$$

Following the proof of Theorem 1.1.3, we also have for $x \geq ((1+4k\log p)/\gamma)^{1/\beta}$,

$$P\left\{ \sup_{t\in T} |X(t)| \geq x\left(\sup_{(s,t)\in T\times T} d(s,t) \right.\right.$$

$$\left.\left. + (1-2^{-1/\beta})^{-1} \int_1^{\infty} \varphi\left(\frac{b-a}{2} p^{-u^2}\right) du \right) \right\}$$

$$\leq \frac{5}{2} C_0 p^{2k} \exp(-\gamma x^{\beta}),$$

where $p \geq 2$ is an integer.

1.2 Comparison Theorems

In this section we investigate the Gaussian comparison theorems which, together with the tail behavior, are very important and useful tools in the theory of Gaussian processes. The first comparison property is the so-called Slepian's inequality (or

Slepian's lemma), without which many of the most basic results in the theory of Gaussian processes would have no proof.

1.2.1 Slepian's inequality

There are a variety of different ways to present Slepian-like inequalities today. We choose the following formulation (from Kahane 1986), which actually includes a number of interesting side results.

Theorem 1.2.1 *Let $\mathbf{X}=(X_1,\cdots,X_N)$ and $\mathbf{Y}=(Y_1,\cdots,Y_N)$ be centered Gaussian random vectors in $\mathbf{R}^N$. Assume that*

$$EX_iX_j\leqslant EY_iY_j \quad if\,(i,j)\in A,$$
$$EX_iX_j\geqslant EY_iY_j \quad if\,(i,j)\in B,$$
$$EX_iX_j=EY_iY_j \quad if\,(i,j)\notin A\cup B,$$

where A and B are subsets of $\{1,\cdots,N\}\times\{1,\cdots,N\}$. If h is a function on $\mathbf{R}^N$ with its second derivatives in the sense of distributions satisfying

$$D_{ij}h\geqslant 0 \quad if\,(i,j)\in A,$$
$$D_{ij}h\leqslant 0 \quad if\,(i,j)\in B,$$

where $D_i=\dfrac{\partial}{\partial x_i}, D_{ij}=\dfrac{\partial^2}{\partial x_i\partial x_j}$, then

$$Eh(\mathbf{X})\leqslant Eh(\mathbf{Y}).$$

Proof We may assume that $\mathbf{X}$ and $\mathbf{Y}$ are independent. Set, for each $t\in[0,1]$, $Z(t)=(1-t)^{1/2}\mathbf{X}+t^{1/2}\mathbf{Y}$ and $\psi(t)=Eh(Z(t))$. We need to show that $\psi(0)\leqslant\psi(1)$, for which it is sufficient to show that $\psi'(t)\geqslant 0$ for all $t\in[0,1]$. We have

$$\psi'(t)=\sum_{i=1}^N E\{D_ih(Z(t))Z_i'(t)\}.$$

Fix t and i, it is easily seen that, for every j

$$EZ_j(t)Z_i'(t)=E(Y_jY_i-X_jX_i)/2.$$

The hypotheses of the theorem then indicate that we can express the Z_j's as

$$Z_j(t)=\alpha_jZ_i'(t)+W_j,$$

where $(W_1,\cdots,W_N)$ is a new sequence of Gaussian vectors, independent of both the X_j's and Y_j's and $\alpha_j\geqslant 0$ if $(i,j)\in A$, $\alpha_j\leqslant 0$ if $(i,j)\in B$, $\alpha_j=0$ if $(i,j)\notin A\cup B$. If we now regard $E\{D_ih(Z(t))\times Z_i'(t)\}$ as a function of the α_j's (for $(i,j)\in A\cup B$), differentiate it with respect to each α_j, and note the hypotheses on h, that the resulting expression is positive if $(i,j)\in A$ and negative if $(i,j)\in B$. That is, $E\{D_ih(Z(t))Z_i'(t)\}$ is an increasing function of α_j for $(i,j)\in A$ and decreasing for $(i,j)\in B$. On the other hand, this function vanishes when all the α_j's are 0, since

$$\begin{aligned}E\{D_ih(Z(t))Z_i'(t)\}&=E\{D_ih(W)Z_i'(t)\}\\&=E\{D_ih(W)\}E\{Z_i'(t)\}=0.\end{aligned}$$

Consequently, $E\{D_ih(Z(t))Z_i'(t)\}\geqslant 0$. Hence $\psi'(t)\geqslant 0$, which is what we have to prove.

The first consequence is Slepian's inequality. It is simply obtained by taking in the theorem $A=\{(i,j);i\neq j\}$, $B=\varnothing$ and $h=I_G$, where G is a product of the intervals $(-\infty,\lambda_j]$.

Corollary 1.2.1 *(Slepian's inequality) If $\mathbf{X}$ and $\mathbf{Y}$ are centered Gaussian vectors in $\mathbf{R}^N$ such that $EX_i^2=EY_i^2$ for all i and*

$$EX_iX_j\leqslant EY_iY_j \quad for\ all\ i\neq j.$$

Then, for all real numbers λ_i, $i\leqslant N$,

$$P\{\bigcup_{i=1}^N(Y_i>\lambda_i)\}\leqslant P\{\bigcup_{i=1}^N(X_i>\lambda_i)\}.$$

In particular, by integration by parts,

$$E \max_{i\leqslant N} Y_i \leqslant E \max_{i\leqslant N} X_i.$$

An interesting extension of Slepian's inequality, which is a result about Gaussian maxima, is a Gordon's result about the minimax of a rectangular array of Gaussian variables.

Corollary 1.2.2 *Let $X=(X_{ij})$ and $Y=(Y_{ij})$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$ be centered Gaussian random vectors such that*

$$EX_{ij}^2 = EY_{ij}^2 \quad \text{for all } i,j;$$

$$EX_{ij}X_{ik} \leqslant EY_{ij}Y_{ik} \quad \text{for all } i,j,k;$$

$$EX_{ij}X_{lk} \geqslant EY_{ij}Y_{lk} \quad \text{for all } i\neq l \text{ and } j,k.$$

Then for all real λ_{ij}

$$P\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(Y_{ij}>\lambda_{ij})\} \leqslant P\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(X_{ij}>\lambda_{ij})\}.$$

This implies, for example, that for any increasing function g on **R**

$$E\{\min_{i\leqslant n}\max_{j\leqslant m} g(Y_{ij})\} \leqslant E\{\min_{i\leqslant n}\max_{j\leqslant m} g(X_{ij})\}$$

and also for all real λ

$$P\{\min_{i\leqslant n}\max_{j\leqslant m} Y_{ij}\geqslant\lambda\} \leqslant P\{\min_{i\leqslant n}\max_{j\leqslant m} X_{ij}\geqslant\lambda\}.$$

Proof Let $N=mn$. For $I\in\{1,\cdots,N\}$ let $i=i(I), j=j(I)$ be the unique $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$ such that $I=m(i-1)+j$. Consider then $\boldsymbol{X}$ and $\boldsymbol{Y}$ as random vectors in $\mathbf{R}^N$ indexed in this way, i.e. $\boldsymbol{X}_I=\boldsymbol{X}_{i(I),j(I)}$. Let

$$A=\{(I,J);i(I)=i(J)\},\ B=\{(I,J);i(I)\neq i(J)\}.$$

Then, the first set of hypotheses of Theorem 1.2.1 is fulfilled. Let h be the indicator function of the set

$$\bigcup_{i=1}^{n}\bigcap_{I;i(I)=i}\{x\in\mathbf{R}^N;\ X_I>\lambda_{i,j(I)}\}.$$

Theorem 1.2.1 implies the conclusion by taking complements.

Slepian's inequality cannot work for $\sup_{t\in T}|X_t|$. To see

this, take $T=\{1,2\}$, with X_1 and X_2 standard normal with correlation ρ. Writing $P_\rho(\lambda)$ for the probability under correlation ρ that $\max(X_1,X_2)>\lambda$ and Ψ as usual for the right hand tail of the standard normal distribution function, we see that

$$P_{-1}(\lambda)=P_{-1}\{X_1\vee X_2>\lambda\}=P\{|X|>\lambda\}=2\Psi(\lambda),$$

$$P_0(\lambda)=2\Psi(\lambda)-\Psi^2(\lambda),$$

$$P_1(\lambda)=P_1\{X_1\vee X_2>\lambda\}=P\{X>\lambda\}=\Psi(\lambda).$$

Hence $P_{-1}(\lambda)\geqslant P_0(\lambda)\geqslant P_1(\lambda)$ as Slepian requires. But if $\hat{P}_\rho(\lambda)$ is the probability that $\max(|X_1|,|X_2|)>\lambda$, then $\hat{P}_{-1}(\lambda)=\hat{P}_1(\lambda)=2\Psi(\lambda)$, $\hat{P}_0(\lambda)=4\{\Psi(\lambda)-\Psi^2(\lambda)\}$, so that for all $\lambda>0$

$$\hat{P}_{-1}(\lambda)\leqslant\hat{P}_0(\lambda),\ \hat{P}_0(\lambda)\geqslant\hat{P}_1(\lambda),$$

and the monotonicity required by Slepian's lemma breaks down.

One of the inequalities which can work for $\sup_{t\in T}|X(t)|$ is the Anderson's inequality.

1.2.2 Anderson's inequality

Theorem 1.2.2 *Let E be a convex set in $\mathbf{R}^N$, symmetric about the origin. Let $f(x)\geqslant 0$ be a function such that*

(i) $f(x)=f(-x)$,

(ii) $K_u:=\{x;f(x)\geqslant u\}$ *is convex for every* $u(0<u<\infty)$,

(iii) $\int_E f(x)\mathrm{d}x<\infty$ *(in the Lebesgue sense).*

Then

$$\int_E f(x+hy)\mathrm{d}x \geqslant \int_E f(x+y)\mathrm{d}x$$

for all $0\leqslant h\leqslant 1$.

Before we prove this theorem, here are some consequences:

Corollary 1. 2. 3 *Let X be a centered Gaussian vector in $\mathbf{R}^N$. If E is a convex set, symmetric about the origin, x is a point in $\mathbf{R}^N$. Then*

$$P\{X+x\in E\}\leqslant P\{X+hx\in E\}$$

for all $0\leqslant|h|\leqslant1$.

Proof Since $P\{X-hx\in E\}=P\{-X+hx\in -E\}=P\{X+hx\in E\}$ by symmetry, we can assume that $h\geqslant 0$. Let $f(x)=(2\pi)^{-N/2}\exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\}$ be the density of X, where Σ is a positive definite matrix. The conclusion follows by Theorem 1. 2. 2.

Remark 1. 2. 1 Corollary 1. 2. 3 is also a consequence of (1. 0. 2). To see this, it is enough to consider the case that X has the distribution γ_N. Then in (1. 0. 2) by taking $A=E+x$, $B=E-x$ and $\lambda=(h+1)/2$, it follows that

$$\begin{aligned}&\mathrm{inv}\Phi(\gamma_N(E-hx))\geqslant\mathrm{inv}\Phi(\gamma_N(\lambda A+(1-\lambda)B))\\&\geqslant\lambda\mathrm{inv}\Phi(\gamma_N(E+x))+(1-\lambda)\mathrm{inv}\Phi(\gamma_N(E-x)).\end{aligned}$$

By symmetry, we have

$$\mathrm{inv}\Phi(\gamma_N(E-hx))\geqslant\mathrm{inv}\Phi(\gamma_N(E-x)),$$

which implies $\gamma_N(E-hx)\geqslant\gamma_N(E-x)$, as required.

Corollary 1. 2. 4 *Let X_1 and X_2 be two centered Gaussian vectors in $\mathbf{R}^N$ with covariance matrices Σ_1 and Σ_2 respectively. If $\Sigma_2-\Sigma_1$ is positive semi-definite and E is a convex set, symmetric about the origin, then*

$$P\{X_1\in E\}\geqslant P\{X_2\in E\}.$$

Proof Let Y be a centered Gaussian vector in $\mathbf{R}^N$ with covariance matrix $\Sigma_2-\Sigma_1$ and independent of X_1. Then X_2 has the same distribution as X_1+Y. By Corollary 1. 2. 3, it follows that

$$P\{X_2\in E\}=P\{X_1+Y\in E\}=\int P\{X_1+y\in E\}\mathrm{d}P_Y(y)$$

$$\leqslant\int P\{X_1\in E\}\mathrm{d}P_Y(y)=P\{X_1\in E\}.$$

The following is a direct consequence of Corollary 1. 2. 4.

Corollary 1. 2. 5 *Let $\{X_i(t);0\leqslant t\leqslant T\}$ be a centered Gaussian process with covariance function $\Gamma_i(t,s)\,(i=1,2)$. Suppose that $\Gamma_2(t,s)-\Gamma_1(t,s)$ is a positive function. Then*

$$P\left\{\int_0^T X_1^2(t)\mathrm{d}t\leqslant x\right\}\geqslant P\left\{\int_0^T X_2^2(t)\mathrm{d}t\leqslant x\right\}.$$

If $X_i(t)$ is a separable process, then

$$P\{\sup_{0\leqslant t\leqslant T}|X_1(t)|\leqslant x\}\geqslant P\{\sup_{0\leqslant t\leqslant T}|X_2(t)|\leqslant x\}.$$

Now, we give the proof of Theorem 1. 2. 2.

Proof of Theorem 1. 2. 2 Equivalently we have to prove

$$\int_{E+hy}f(x)\mathrm{d}x\geqslant\int_{E+y}f(x)\mathrm{d}x.$$

The theorem follows almost directly once we prove that for every u

$$\mathrm{vol}\{(E+hy)\cap K_u\}\geqslant\mathrm{vol}\{(E+y)\cap K_u\},$$

where vol{ • } denotes the volume of the set. Let $\alpha=(h+1)/2$, so that $\alpha y+(1-\alpha)(-y)=hy$. Then $(E+hy)\cap K_u\supset\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}$, because K_u is convex and $E+hy\supset\alpha(E+y)+(1-\alpha)(E-y)=\{\alpha E+(1-\alpha)E\}+hy$. Thus

$$\begin{aligned}&\mathrm{vol}\{(E+hy)\cap K_u\}\\&\geqslant\mathrm{vol}\{\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}\}.\end{aligned}$$

$(E+y)\cap K_u$ is the mirror image through the origin of $(E-y)\cap K_u$, and therefore these two sets have the same volume. Then

$$\begin{aligned}&\mathrm{vol}\{\alpha\{(E+y)\cap K_u\}+(1-\alpha)\{(E-y)\cap K_u\}\}\\&\geqslant\mathrm{vol}\{(E+y)\cap K_u\}\end{aligned}$$

by the Brunn-Minkowski theorem (cf. Bonnsen and Fenchel 1948), which states that

$$\mathrm{vol}^{1/N}\{(1-\theta)E_0+\theta E_1\}\geqslant(1-\theta)\mathrm{vol}^{1/N}(E_0)+\theta\ \mathrm{vol}^{1/N}(E_1)$$

(E_0 and E_1 nonempty, $0\leqslant\theta\leqslant 1$). Thus

$$H(u):=\mathrm{vol}\{(E+hy)\cap K_u\}\geqslant\mathrm{vol}\{(E+y)\cap K_u\}=:H^*(u).$$

Definitions of Lebesgue and Lebesgue-Stieltjes integrals show

$$\int_{E+hy}f(x)\mathrm{d}x-\int_{E+y}f(x)\mathrm{d}x=-\int_0^\infty u\mathrm{d}H(u)+\int_0^\infty u\mathrm{d}H^*(u)$$
$$=\int_0^\infty u\mathrm{d}\{H^*(u)-H(u)\}.$$

Integration by parts shows for $b>a\geqslant 0$

$$\int_a^b u\mathrm{d}\{H^*(u)-H(u)\}$$
$$=b\{H^*(b)-H(b)\}-a\{H^*(a)-H(a)\}$$
$$+\int_a^b\{H(u)-H^*(u)\}\mathrm{d}u$$
$$\geqslant b\{H^*(b)-H(b)\}.$$

Since the integral of $f(x)$ over E is finite, $bH(b)\to 0$ as $b\to\infty$ and hence $bH^*(b)\to 0$ as $b\to\infty$. Therefore $\int_0^\infty u\mathrm{d}\{H^*(u)-H(u)\}\geqslant 0$, which completes the proof.

Using Anderson's inequality (Corollary 1.2.3), we can get the following inequality for $\max_i|X_i|$, which was given by Marcus (1968).

Theorem 1.2.3 *Let $X=(X_1,\cdots,X_N)$ be a centered Gaussian vector in $\mathbf{R}^N$ with a positive definite covariance matrix Σ. Let Σ_i be the determinant of the i^{th} principle minor of Σ. Define $\rho_i=\Sigma_{i-1}/\Sigma_i$, $i=1,\cdots,N,\Sigma_0=1$. Then*

$$P\{\max_{1\leqslant i\leqslant N}|X_i|\leqslant a\}\leqslant\prod_{i=1}^N\sqrt{\frac{2}{\pi}}\int_0^{a(\rho_i)^{1/2}}\mathrm{e}^{-t^2/2}\mathrm{d}t.$$

Proof Let $A=\Sigma^{-1}$, and $A=P'P$, where $P=(P_{ij})$ is a symmetric positive definite matrix. Define $Y_i=\sum_j P_{ij}X_j$. Then $Y_1,\cdots,Y_N$ are independent standard normal variables, and the functions Y_i have the following form:

$$Y_i=(\rho_i)^{1/2}X_i+f_i(Y_{i-1},\cdots,Y_1),\ i=1,\cdots,N.$$

Then by Corollary 1.2.3, it follows that

$$P\{\max_{1\leqslant i\leqslant N}|X_i|\leqslant a\}=P\{\bigcap_{i=1}^N(|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\leqslant a(\rho_i)^{1/2})\}$$
$$=E(I\{\bigcap_{i=1}^{N-1}\{|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\leqslant a(\rho_i)^{1/2}\}\}$$
$$\times P\{|Y_N-f_N(Y_{N-1},\cdots,Y_1)|\leqslant a(\rho_N)^{1/2}|Y_{N-1},\cdots,Y_1|\})$$
$$\leqslant E(I\{\bigcap_{i=1}^{N-1}\{|Y_i-f_i(Y_{i-1},\cdots,Y_1)|\leqslant a(\rho_i)^{1/2}\}\}$$
$$\times P\{|Y_N|\leqslant a(\rho_N)^{1/2}\})$$
$$\leqslant\cdots\leqslant\prod_{i=1}^N P\{|Y_i|\leqslant a(\rho_i)^{1/2}\}=\prod_{i=1}^N\sqrt{\frac{2}{\pi}}\int_0^{a(\rho_i)^{1/2}}\mathrm{e}^{-t^2/2}\mathrm{d}t.$$

Corollary 1.2.6 *Let $X(t)$ be a real valued stationary Gaussian process with covariance function $\gamma(h)=EX(t)X(t+h)$ which is convex for $h\in[0,\delta]$ for some $\delta>0$. Let $t_0<t_1<\cdots<t_N$, $t_N-t_0\leqslant\delta$. Then*

$$P\{\max_{1\leqslant i\leqslant N}|\xi_i|\leqslant a\}\leqslant\prod_{i=1}^N\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}a/a_{ii}}\mathrm{e}^{t^2/2}\mathrm{d}t,$$

where ξ_i can be either $X(t_i)-X(t_{i-1})$ or $X(t_i)-X(t_0)$, $a_{ii}=\sigma(t_i-t_{i-1})$, $i=1,\cdots,N$ and $\sigma^2(t)=E(X(t)-X(0))^2$.

Proof Let $\xi_i=X(t_i)-X(t_{i-1})$, $i=1,\cdots,N$ and $A=(a_{ij})=(E\xi_i\xi_j)$ be the covariance matrix of $(\xi_1,\cdots,\xi_N)$. By the convexity of $\gamma(h)$, it follows that $a_{ij}\leqslant 0$ for $i\neq j$. For each i, Let Σ_i be the determinant of the i^{th} principal minor of A,

$$S_u^i=\sum_{\substack{v=1\\v\neq u}}^i|a_{uv}|=a_{uu}-\sum_{v=1}^i a_{uv}:=\sigma_u^i a_{uu},\ 1\leqslant u\leqslant i,$$

and

$$t_i^i = \max_{1 \leqslant k \leqslant i-1} \sigma_k^i.$$

By the convexity of $\gamma(h)$, it is easily seen that

$$S_u^n = a_{uu} - E(X(t_u) - X(t_0))(X(t_u) - X(t_{u-1})) < a_{uu}, \ 1 \leqslant u < n$$

and

$$S_i^i = \gamma(0) - \gamma(t_i - t_{i-1}) + \gamma(t_i - t_0) - \gamma(t_{i-1} - t_0)$$
$$\leqslant \gamma(0) - \gamma(t_i - t_{i-1}) = \frac{1}{2} a_{ii}.$$

Then it follows that $\boldsymbol{A}$ is a matrix with positive diagonal elements such that the sum of the absolute values of all the off diagonal elements in given row is less than the diagonal element in that row, and hence $\boldsymbol{A}$ is positive definite. Also, we have

$$(a_{ii} + t_i^i S_i^i)\Sigma_{i-1} \geqslant \Sigma_i \geqslant (a_{ii} - t_i^i S_i^i)\Sigma_{i-1}.$$

Since $S_k^i \leqslant S_k^n \leqslant a_{kk}$, it follows that $0 \leqslant t_i^i \leqslant 1$. Then

$$\frac{\Sigma_{i-1}}{\Sigma_i} \leqslant \frac{2}{a_{ii}} = \frac{2}{\sigma^2(t_i - t_{i-1})}.$$

To complete the proof, it is sufficient to note that the functions Σ_{i-1}/Σ_i for the covariance matrices of the random variables $X(t_j) - X(t_{j-1})$, $j = 1, \cdots, i$ and the functions Σ_{i-1}/Σ_i for the covariance matrices of the random variables $X(t_j) - X(t_0)$, $j = 1, \cdots, i$ are equal.

1.2.3 Khatri-Šidák's inequality

Let $\boldsymbol{X} = (X_1, \cdots, X_N)$ be a centered Gaussian vector with $EX_iX_j \leqslant 0 (i \neq j)$. Slepian's inequality (Corollary 1.2.1) tells us that

$$P\{\bigcap_{i=1}^N (X_i \leqslant \lambda_i)\} \leqslant \prod_{i=1}^N P\{X_i \leqslant \lambda_i\}$$

for any numbers λ_i, $i \leqslant N$.

But, unfortunately, Slepian's elegant proof does not extend to the case of a two-side barrier. For the latter case, the following theorem gives a two-side analogue to this inequality.

Theorem 1.2.4 *Let $(X_1, \cdots, X_N)$ be a centered Gaussian vector in $\mathbf{R}^N$ with an arbitrary covariance matrix, then for any positive numbers λ_i, $i \leqslant N$*

$$P\{\bigcap_{i=1}^N (|X_i| \leqslant \lambda_i)\} \geqslant P\{\bigcap_{i=1}^{N-1} (|X_i| \leqslant \lambda_i)\} P\{|X_N| \leqslant \lambda_N\}$$
$$\geqslant \prod_{i=1}^N P\{|X_i| \leqslant \lambda_i\}.$$

We can rewrite Theorem 1.2.4 in the following way.

Theorem 1.2.4′ *Let $\{X(t); t \in T\}$ be a centered separable Gaussian process, $\{\lambda(t); t \in T\}$ be a real positive function. Then for any $t_0 \in T$ we have*

$$P\left\{\sup_{t \in T} \frac{|X(t)|}{\lambda(t)} \leqslant 1\right\} \geqslant P\left\{\sup_{t \in T \setminus \{t_0\}} \frac{|X(t)|}{\lambda(t)} \leqslant 1\right\} P\left\{\frac{|X(t_0)|}{\lambda(t_0)} \leqslant 1\right\}.$$

Theorem 1.2.4 is due to Khatri (1967) and Šidák (1968), so it is called Khatri-Šidák's inequality. We present Khatri's proof here, which seems simpler. Actually, Theorem 1.2.4 is a consequence of the following proposition.

Proposition 1.2.1 *Let $\boldsymbol{X} = (\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)})$ be a centered Gaussian vector in $\mathbf{R}^{m+n}$, where $\boldsymbol{X}^{(1)} = (X_1^{(1)}, \cdots, X_m^{(1)})$, $\boldsymbol{X}^{(2)} = (X_1^{(2)}, \cdots, X_n^{(2)})$ are centered Gaussian vectors in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Let D_1 and D_2 be two convex sets in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, symmetric about the origin. Then*

$$P\{\boldsymbol{X}^{(1)} \in D_1, \boldsymbol{X}^{(2)} \in D_2\} \geqslant P\{\boldsymbol{X}^{(1)} \in D_1\} P\{\boldsymbol{X}^{(2)} \in D_2\}$$

provided that the rank of $\mathrm{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) := E(X_1^{(1)}, \cdots, X_m^{(1)})' \times (X_1^{(2)}, \cdots, X_n^{(2)})$ *is at most one.*

To prove Proposition 1.2.1, we need

Lemmal 1.2.1 *Let $g(\boldsymbol{X})$ and $h(\boldsymbol{X})$ be two functions of random vector $\boldsymbol{X}$ in $\mathbf{R}^N$. Then*

$$Eg(\boldsymbol{X})h(\boldsymbol{X}) \geqslant Eg(\boldsymbol{X})Eh(\boldsymbol{X})$$

provided $(g(x_1)-g(x_2))(h(x_1)-h(x_2)) \geqslant 0$ for any two point x_1 and x_2 in $\mathbf{R}^N$, while $Eg(\boldsymbol{X})h(\boldsymbol{X}) \leqslant Eg(\boldsymbol{X})Eh(\boldsymbol{X})$ provided $(g(x_1)-g(x_2))(h(x_1)-h(x_2)) \leqslant 0$ for any two point x_1 and x_2 in $\mathbf{R}^N$.

Proof Let $\boldsymbol{Y}$ be an independent copy of $\boldsymbol{X}$. Then $(g(\boldsymbol{X}) - g(\boldsymbol{Y}))(h(\boldsymbol{X}) - h(\boldsymbol{Y})) \geqslant 0$, which implies $E(g(\boldsymbol{X}) - g(\boldsymbol{Y})) \times (h(\boldsymbol{X}) - h(\boldsymbol{Y})) \geqslant 0$. Hence the conclusion follows.

Proof of Proposition 1.2.1 Let Σ_1 and Σ_2 be the covariance matrices of $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ respectively. By the fact that the rank of $\mathrm{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)})$ is at most one, there exist two vectors $\boldsymbol{a} = (a_1, \cdots, a_m)$ in $\mathbf{R}^m$ and $\boldsymbol{b} = (b_1, \cdots, b_n)$ in $\mathbf{R}^n$ such that $\mathrm{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \boldsymbol{a}'\boldsymbol{b}$ and we can write

$$\boldsymbol{X}^{(1)} = \boldsymbol{Y}^{(1)} + \boldsymbol{a}g, \quad \boldsymbol{X}^{(2)} = \boldsymbol{Y}^{(2)} + \boldsymbol{b}g,$$

where g is a standard normal variable, $\boldsymbol{Y}^{(1)}(\boldsymbol{Y}^{(2)})$ is a Gaussian vector with covariance matrix $\Sigma_1 - \boldsymbol{a}'\boldsymbol{a}(\Sigma_2 - \boldsymbol{b}'\boldsymbol{b})$, and $\boldsymbol{Y}^{(1)}$, $\boldsymbol{Y}^{(2)}$ and g are independent. Noting Corollary 1.2.3 implies that $P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}y \in D_1\}$ and $P\{\boldsymbol{Y}^{(2)} + \boldsymbol{b}y \in D_2\}$ and both increasing in $|y|$, by Lemma 1.2.1 we have

$$\begin{aligned}
P\{\boldsymbol{X}^{(1)} \in D_1, \boldsymbol{X}^{(2)} \in D_2\} &= P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}g \in D_1, \boldsymbol{Y}^{(2)} + \boldsymbol{b}g \in D_2\} \\
&= \int P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}y \in D_1, \boldsymbol{Y}^{(2)} + \boldsymbol{b}y \in D_2\} \mathrm{d}P_g(y) \\
&= \int P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}y \in D_1\} P\{\boldsymbol{Y}^{(2)} + \boldsymbol{b}y \in D_2\} \mathrm{d}P_g(y)
\end{aligned}$$

$$\begin{aligned}
&\geqslant \int P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}y \in D_1\} \mathrm{d}P_g(y) \int P\{\boldsymbol{Y}^{(2)} + \boldsymbol{b}y \in D_2\} \mathrm{d}P_g(y) \\
&= P\{\boldsymbol{Y}^{(1)} + \boldsymbol{a}g \in D_1\} P\{\boldsymbol{Y}^{(2)} + \boldsymbol{b}g \in D_2\} \\
&= P\{\boldsymbol{X}^{(1)} \in D_1\} P\{\boldsymbol{X}^{(2)} \in D_2\},
\end{aligned}$$

which is what we have to prove.

Proof of Theorem 1.2.4 In Proposition 1.2.1, choosing $\boldsymbol{X}^{(1)} = (X_1, \cdots, X_{N-1})$, $\boldsymbol{X}^{(2)} = X_N$, $D_1 = \bigcap_{i=1}^{N-1} \{|x_i| \leqslant \lambda_i\}$ and $D_2 = \{|x_N| \leqslant \lambda_N\}$ implies

$$P\{\bigcap_{i=1}^{N} (|X_i| \leqslant \lambda_i)\} \geqslant P\{\bigcap_{i=1}^{N-1} (|X_i| \leqslant \lambda_i)\} P\{|X_N| \leqslant \lambda_N\}.$$

Then the conclusion follows by induction.

Theorem 1.2.5 *Let $\boldsymbol{X} = (X_1, \cdots, X_N)$ be a centered Gaussian vector in $\mathbf{R}^N$ with covariance matrix $\boldsymbol{\Gamma}$ satisfying that $a_{ij} = \alpha_i \alpha_j (a_{ii} \times a_{jj})^{1/2}$ for $|\alpha_i| \leqslant 1$, $i \neq j$, and $a_{ii} > 0$ for all i. Then for any positive numbers λ_i, $i \leqslant N$*

$$P\{|X_i| \geqslant \lambda_i; i = 1, \cdots, N\} \geqslant \prod_{i=1}^{N} P\{|X_i| \geqslant \lambda_i\}.$$

Proof Let $\sigma_i^2 = a_{ii}$. By the hypothesis, we can write $\boldsymbol{\Gamma} = \boldsymbol{T} + \boldsymbol{\alpha}'\boldsymbol{\alpha}$, **where** $\boldsymbol{T}$ is a $N \times N$ diagonal matrix with diagonal elements $\sigma_i^2(1 - \alpha_i^2)$, and $\boldsymbol{\alpha} = (\sigma_1\alpha_1, \cdots, \sigma_N\alpha_N)$. Then we can write

$$\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{\alpha}g,$$

where $\boldsymbol{Y} = (Y_1, \cdots, Y_N)$ is centered Gaussian vector in $\mathbf{R}^N$ with covariance matrix $\boldsymbol{T}$, g is a standard normal variable independent of $\boldsymbol{Y}$. It follows that $Y_1, \cdots, Y_N, g$ are independent. Noting that Corollary 1.2.3 implies that $P\{|Y_i + \sigma_i\alpha_i y| \geqslant \lambda_i\}$ is non-decreasing in $|y|$ for each i, by Lemma 1.2.1 we have

$$P\{|X_i| \geqslant \lambda_i; i = 1, \cdots, N\} = \int \prod_{i=1}^{N} P\{|Y_i + \sigma_i\alpha_i y| \geqslant \lambda_i\} \mathrm{d}P_g(y)$$

$$\geqslant \prod_{i=1}^{N} \int P\{|Y_i + \sigma_i \alpha_i y| \geqslant \lambda_i\} \mathrm{d}P_g(y) = \prod_{i=1}^{N} P\{|X_i| \geqslant \lambda_i\},$$

which complets the proof.

Finally, we give an extension of Theorem 1.2.4′.

Theorem 1.2.4″ *Let* $\{Y(t); t \in T\} = \{X_k(t); t \in T\}_{k=1}^{\infty}$ *be a sequence of independent centered separable Gaussian processes,* $\{\lambda(t); t \in T\}$ *be a positive real function. Then for any* $t_0 \in T$

$$P\left\{\sup_{t \in T} \frac{\| Y(t) \|_{l_p}}{\lambda(t)} \leqslant 1\right\}$$
$$\geqslant P\left\{\sup_{t \in T \setminus \{t_0\}} \frac{\| Y(t) \|_{l_p}}{\lambda(t)} \leqslant 1\right\} P\left\{\frac{\| Y(t_0) \|_{l_p}}{\lambda(t_0)} \leqslant 1\right\},$$

where $p \geqslant 1$ *and* $\| Y(t) \|_{l_p}^{p} = \sum_{k=1}^{\infty} |X_k(t)|^p$.

This is an immediate consequence of the following proposition.

Proposition 1.2.2 *Let* $X = \{X_i(t); t \in T \cup \{t_0\}\}_{i=1}^{N}$ *be a sequence of independent centered separable Gaussian processes,* D_1 *be a convex set in the function space* $\mathbf{R}^{N \times T} = \{x(t) = (x_1(t), \cdots, x_N(t)); t \in T\}$, *symmetric in the sense of that* $(x_1(\cdot), \cdots, x_N(\cdot)) \in D_1$ *implies* $(\varepsilon_1 x_1(\cdot), \cdots, \varepsilon_N x_N(\cdot)) \in D_1$ *for all* $\varepsilon_i = \pm 1$, $i = 1, \cdots, N$, *and* D_2 *be a convex set in* $\mathbf{R}^N$, *symmetric in the sense of that* $(x_1, \cdots, x_N) \in D_2$ *implies* $(\varepsilon_1 x_1, \cdots, \varepsilon_N x_N) \in D_2$ *for all* $\varepsilon_i = \pm 1$, $i = 1, \cdots, N$. *Then*

$$P\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\}$$
$$\geqslant P\{\{X(t); t \in T\} \in D_1\} P\{X(t_0) \in D_2\}.$$

Proof We can assume that T is finite. Let $N_j(x_j)$, $N_{j,T}(x_j^{(T)})$ and $N_{j,0}(x_j^{(0)})$ be the densities of $\{X_j(t); t \in T \cup \{t_0\}\}$, $\{X_j(t); t \in T\}$ and $X_j(t_0)$ respectively. By noting that for fixed $\{X_2(t), \cdots, X_N(t); t \in T \cup \{t_0\}\}$, the sets $D_1' = \{x_1(t); \{x(t); t \in T\} \in D_1\}$ and $D_2' = \{x_1(t_0); x(t_0) \in D_1\}$ are convex and symmetric about the origin in $\mathbf{R}^{1 \times T}$ and $\mathbf{R}^1$ respectively, it follows from Proposition 1.2.1 that

$$P\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\} = \int_{D_1 \times D_2} \prod_{j=1}^{N} N_j(x_j) \mathrm{d}x_j$$
$$= \int_{D_1 \times D_2} N_1(x_1) \mathrm{d}x_1 \prod_{j=2}^{N} N_j(x_j) \mathrm{d}x_j$$
$$\geqslant \int_{D_1 \times D_2} N_{1,T}(x_1^{(T)}) \mathrm{d}x_1^{(T)} N_{1,0}(x_1^{(0)}) \mathrm{d}x_1^{(0)} \prod_{j=2}^{N} N_j(x_j) \mathrm{d}x_j.$$

Iterating the arguments for x_2, then $x_3, \cdots$, and finally x_N we get

$$P\{\{X(t); t \in T\} \in D_1, X(t_0) \in D_2\}$$
$$\geqslant \int_{D_1 \times D_2} \prod_{j=1}^{N} \{N_{j,T}(x_j^{(T)}) \mathrm{d}x_j^{(T)} N_{j,0}(x_j^{(0)}) \mathrm{d}x_j^{(0)}\}$$
$$= P\{\{X(t); t \in T\} \in D_1\} P\{X(t_0) \in D_2\},$$

as desired.

Remark 1.2.2 We can rewrite Khatri-Šidák's inequality (Theorems 1.2.4 or 1.2.4′) in the following way: if $(X_1, \cdots, X_n)$ is a centered Gaussian vector, then

$$P\{\max_{1 \leqslant i \leqslant n} |X_i| \leqslant 1\} \geqslant P\{|X_1| \leqslant 1\} P\{\max_{2 \leqslant i \leqslant n} |X_i| \leqslant 1\}. \tag{1.2.1}$$

This inequality is a special case of the Gaussian correlation conjecture which states that for any centered Gaussian vector $(X_1, \cdots, X_n)$ and each $1 \leqslant k \leqslant n$,

$$P\{\max_{1 \leqslant i \leqslant n} |X_i| \leqslant 1\} \geqslant P\{\max_{1 \leqslant i \leqslant k} |X_i| \leqslant 1\} P\{\max_{k+1 \leqslant i \leqslant n} |X_i| \leqslant 1\}. \tag{1.2.2}$$

An equivalent formulation of (1.2.2) is as follows: let A and B be two symmetric convex sets in a separable Banach space $\mathscr{E}$,

$(\boldsymbol{X},\boldsymbol{Y})$ be a centered Guassian vetcor in $\mathscr{E}\times\mathscr{E}$, then

$$P\{\boldsymbol{X}\in A,\boldsymbol{Y}\in B\}\geqslant P\{\boldsymbol{X}\in A\}P\{\boldsymbol{Y}\in B\}. \tag{1.2.3}$$

For early history of Gaussian correlation conjecture we refer to Gupta el. (1972), Tong (1980), Schechtman, Schumprecht and Zinn (1998), et al. (1.2.1) tells us that the conjecture (1.2.2) holds for $k=1$. By Proposition 1.2.1, it is known that (1.2.3) holds, if $\mathscr{E}$ is an Euclidean space and the rank of $\mathrm{Cov}(X,Y)$ is at most 1. It is also known that the Gaussian correlation conjecture is true for some other special cases. For example, Pitt (1977) showed that (1.2.2) holds for $n=4$ and $k=2$; Schechtman, Schumprecht and Zinn (1998) showed that the conjecture is true whenever the sets A and B are symmetric and ellipsoid or the sets are not too large; Harge (1998) proved that (1.2.3) holds if one set is symmetric ellipsoid and the other is simply symmetric convex, etc. In the general case, whether the conjecture is true or not is still an open problem. Recently, Shao (1999) gave an inequality approximated to (1.2.2) as follows:

$$P\{\max_{1\leqslant i\leqslant n}|X_i|\leqslant 1\}\geqslant 2^{-k\vee(n-k)}P\{\max_{1\leqslant i\leqslant k}|X_i|\leqslant 1\}P\{\max_{k+1\leqslant i\leqslant n}|X_i|\leqslant 1\} \tag{1.2.4}$$

for each $1\leqslant k\leqslant n$. And, Li (1999) established a weak form of the conjecture which states that for any $0<\lambda<1$, any two symmetric convex sets A and B in $\mathscr{E}$ and any centered Gaussian vector $(\boldsymbol{X},\boldsymbol{Y})$ in $\mathscr{E}\times\mathscr{E}$,

$$P\{\boldsymbol{X}\in A,\boldsymbol{Y}\in B\}\geqslant P\{\boldsymbol{X}\in\lambda A\}P\{\boldsymbol{Y}\in(1-\lambda^2)^{1/2}B\}. \tag{1.2.5}$$

(1.2.5) is a consequence of Anderson's inequality. In fact, if we let $a=(1-\lambda^2)^{1/2}/\lambda$, and $(\boldsymbol{X}^*,\boldsymbol{Y}^*)$ be an independent copy of $(\boldsymbol{X},\boldsymbol{Y})$, then $\boldsymbol{X}-a\boldsymbol{X}^*$ and $\boldsymbol{Y}+\boldsymbol{Y}^*/a$ are independent. Thus, by

Anderson's inequality (Corollary 1.2.3),

$$\begin{aligned}P\{\boldsymbol{X}\in A,\boldsymbol{Y}\in B\}&\geqslant P\{(\boldsymbol{X},\boldsymbol{Y})+(-a\boldsymbol{X}^*,\boldsymbol{Y}^*/a)\in A\times B\}\\&=P\{\boldsymbol{X}-a\boldsymbol{X}^*\in A,\boldsymbol{Y}+\boldsymbol{Y}^*/a\in B\}\\&=P\{\boldsymbol{X}-a\boldsymbol{X}^*\in A\}P\{\boldsymbol{Y}+\boldsymbol{Y}^*/a\in B\}\\&\geqslant P\{\boldsymbol{X}\in\lambda A\}P\{\boldsymbol{Y}\in(1-\lambda^2)^{1/2}B\}.\end{aligned}$$

Chapter 2

Moduli of Continuity and Limit Behavior of Large Increments for Gaussian Processes

With this chapter, we begin the subject of path properties of Gaussian processes and related processes. In this chapter, we shall be interested in real valued Gaussian processes, and special interest will be in multi-parameter processes.

One of the most important path properties of a Gaussian process is the continuity of its sample paths. Some basic and general results on the continuity of not only real Gaussian processes but also general Gaussian processes will be presented in the first section of this chapter. These basic results will be used in other sections and chapters occasionally. We start our main topics on the moduli of continuity and limit behavior of large increments for Gaussian processes in Section 2. 2 and with a simple but important Gaussian process, i. e. , the fractional Wiener process. Sections 2. 3, 2. 4 and 2. 5 deal with the moduli of continuity and limit behavior of large increments of some special two-

parameter Gaussian processes. In Section 2. 3, what we will study is the "simplest" two-parameter Gaussian process — the two-parameter Wiener process, whose increments are independent and stationary. The process studied in Section 2. 4 is the two-parameter fractional Lévy-Wiener process which is a Gaussian process with stationary but not independent increments. It is an extension of the two-parameter Lévy-Wiener process. In Section 2. 5, we will study the two-parameter Ornstein-Uhlenbeck process whose increments are neither independent nor stationary. It is an extension of the famous one-parameter Ornstein-Uhlenbeck process. Section 2. 6 deals with a kind of more general two-parameter Gaussian processes. In the last section, we investigate the local time $L(x,t)$ of a Gaussian process, which is also a two-parameter stochastic process. Moduli of continuity of the process $L(x,t)$ in t for fixed x and the suprema in $x\in \mathbf{R}$ of moduli of continuity of $L(x,t)$ in t are discussed.

2. 1 The Continuity of Gaussian Processes

2. 1. 1 Boundedness and continuity

Let (T,d) be a pseudo-metric space. For each $\varepsilon>0$, denote by $N(T,d;\varepsilon)$ the entropy number, that is, the smallest number of open balls of radius ε in metric d which form a covering of T. It is clear that T is totally bounded for d if and only if $N(T,d;$

$\varepsilon)<\infty$ for every $\varepsilon>0$, a property which will always be satisfied under all the conditions we will deal with. Denote further by $D=D(T)$ the diameter of (T,d), i.e., $D=\sup\{d(s,t);s,t\in T\}$.

Let $X=\{X_t;t\in T\}$ be a centered Gaussian process and define the canonical metric on T by $d_X(s,t)=\|X_t-X_s\|_2, s,t\in T$. It is easily seen that

$$E\sup_{s,t\in T}|X_s-X_t|=E\sup_{s,t}(X_s-X_t)=2E\sup_{t\in T}X_t.$$

It follows that for every $t_0\in T$,

$$E\sup_{t\in T}X_t\leqslant E\sup_{t\in T}|X_t|\leqslant E|X_{t_0}|+E\sup_{s,t\in T}|X_s-X_t|$$
$$\leqslant E|X_{t_0}|+2E\sup_{t\in T}X_t.$$

This inequality tells us that as far as almost sure boundedness is concerned it is irrelevant whether we work with $\sup_t X_t$ or $\sup_t |X_t|$.

The next result tells us that a centered Gaussian process is bounded if and only if the moment of its supremum is bounded.

Theorem 2.1.1 *Suppose that X is a centered Gaussian process on T and (T,d_X) is bounded, then*

$$P\{\sup_{t\in T}X_t<\infty\}=1\Leftrightarrow E\sup_{t\in T}X_t<\infty.$$

Proof Write, as usual, $\|X\|$ for $\sup_{t\in T}X_t$. Since $E\|X\|<\infty$ obviously implies $\|X\|<\infty$ a.s., we need only prove the alternative implication. In fact, we shall prove the somewhat stronger result that for $\alpha>0$ sufficiently small (where "sufficiently small" is a function of $\sup_{t\in T}EX_t^2$)

$$Ee^{\alpha\|X\|^2}<M<\infty, \tag{2.1.1}$$

where M is a universal constant.

Let Y and Z be two independent copies of X. Then, since $(Y+Z)/\sqrt{2}$ and $(Y-Z)/\sqrt{2}$ are also independent copies of

X, it follows that both $(\|Y\|,\|Z\|)$ and $(\|Y+Z\|/\sqrt{2}, \|Y-Z\|/\sqrt{2})$ are pairs of independent copies of $\|X\|$. Consequently, for every pair (a,b) of reals

$$P\{\|X\|\leqslant a\}P\{\|X\|>b\}$$
$$=P\{\|Y+Z\|\leqslant\sqrt{2}a, \|Y-Z\|>\sqrt{2}b\}$$
$$\leqslant P\{\|Y\|>(b-a)\sqrt{2}, \|Z\|>(b-a)\sqrt{2}\}$$
$$\leqslant(P\{\|Y\|>(b-a)\sqrt{2}\})^2.$$

We now choose $a>0$ such that $q:=P\{\|X\|\leqslant a\}\in(\frac{1}{2},1)$. Iterating the above inequality with

$$b=b_n=(\sqrt{2^{n+1}}-1)(\sqrt{2}+1)a$$

easily yields that, for each $x\geqslant a$,

$$P\{\|X\|\geqslant x\}\leqslant\exp\left(-\frac{x^2}{24a^2}\log\frac{q}{1-q}\right),$$

it follows that

$$Ee^{\alpha\|X\|^2}\leqslant e^{\alpha a^2}+\int_a^\infty\exp\left\{-\left(\frac{1}{24a^2}\log\frac{q}{1-q}-\alpha\right)x^2\right\}dx.$$

Taking $\alpha<\left(\log\frac{q}{1-q}\right)\Big/(24a^2)$ yields (2.1.1).

Remark 2.1.1 Note that we do not use the fact that $\|\cdot\|$ was the supremum norm. The above proof actually works if $\|\cdot\|$ is any measurable semi-norm, and X takes values in a separable Banach space. Also, if we are asked the boundedness properties of a centered Gaussian process X with respect to another metric d for which T is bounded, we need simply assume in addition that d_X is bounded on (T,d).

The next theorem relates continuity closely to boundedness.

Theorem 2.1.2 *Let $X=\{X_t;t\in T\}$ be almost surely bounded and*

let d be a metric on T such that the canonical metric d_X is d-uniformly continuous. Then X is d-uniformly continuous with probability one if and only if

$$\lim_{\eta\to 0} E \sup_{d(s,t)<\eta} (X_s - X_t) = 0.$$

To prove Theorem 2. 1. 2, we need the well-known Borel-Cantelli lemma, which will be used frequently throughout the book. We present it in the following version.

Lemma 2. 1. 1 *Let $\{A_n; n \geq 1\}$ be a sequence of events, if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n, i.o.) = 0$, where $\{A_n, i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. If $\sum_{n=1}^{\infty} P(A_n) = \infty$ and*

$$\liminf_{n\to\infty} \sum_{1\leqslant j<k\leqslant n} (P(A_j A_k) - P(A_j)P(A_k)) \Big/ \Big(\sum_{j=1}^{n} P(A_j) \Big)^2 = 0,$$

then $P(A_n, i.o.) = 1$.

Proof The proof can be found in Petrov (1995). Since the proof is simple, we give details for the sake of completeness. The first part of Lemma 2. 1. 1 follows easily from

$$P(A_n, i.o.) = P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) \leqslant P(\bigcup_{k=n}^{\infty} A_k)$$
$$\leqslant \sum_{k=n}^{\infty} P(A_k) \to 0 \quad \text{as } n \to \infty,$$

since $\sum_{n=1}^{\infty} P(A_k)$ converges. For the second part of Lemma 2. 1. 1, we let $I_n = I_{A_n}$. Then $EI_n = P(A_n)$. It follows that

$$\liminf_{n\to\infty} P\left\{ \Big| \sum_{k=1}^{n} I_k - \sum_{k=1}^{n} P(A_k) \Big| \geqslant \frac{1}{2} \sum_{k=1}^{n} P(A_k) \right\}$$
$$\leqslant \liminf_{n\to\infty} \frac{4\mathrm{Var}(\sum_{k=1}^{n} I_k)}{(\sum_{k=1}^{n} P(A_k))^2}$$
$$= \liminf_{n\to\infty} \frac{8\sum_{1\leqslant k<l\leqslant n} (P(A_k A_l) - P(A_k)P(A_l))}{(\sum_{k=1}^{n} P(A_k))^2} = 0,$$

which implies

$$\liminf_{n\to\infty} P(B_n) = 0,$$

where $B_n = \left\{ \sum_{k=1}^{n} I_k \leqslant \frac{1}{2} \sum_{k=1}^{n} P(A_k) \right\}$. Hence there exists an increasing sequence of integers $\{n_m\}$ such that $\sum_{m=1}^{\infty} P(B_{n_m}) < \infty$. By the first part of Lemma 2. 1. 1 which has just been showed, we conclude that $\sum_{k=1}^{n_m} I_k \geqslant \frac{1}{2} \sum_{k=1}^{n_m} P(A_k)$ for all but finitely many m's with probability one, which together with $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies that $P(\sum_{k=1}^{\infty} I_k \text{ diverges}) = 1$. It follows that $P(A_n, i.o.) = 1$ as required.

Proof of Theorem 2. 1. 2 Let $\varphi_d(\eta) = E\sup_{d(s,t)<\eta}(X_s - X_t)$. We start with necessity. Let $U_\varepsilon = \{(s,t) \in T\times T; d(s,t) < \varepsilon\}$, $Y_{t,s} = X_t - X_s$. Then $Y_{t,s}$ is a centered almost surely bounded Gaussian process on $(T\times T, d_Y)$, where

$$d_Y((s,t),(s',t')) = \| (X_s - X_t) - (X_{s'} - X_{t'}) \|_2$$
$$\leqslant d_X(s,t) + d_X(s',t').$$

Since d_X is d-uniformly continuous, we can find a $\varepsilon > 0$ such that $(s,t),(s',t') \in U_\varepsilon$ implies $d_Y((s,t),(s',t')) \leqslant 1$. It follows that (U_ε, d_Y) is bounded. Thus, Theorem 2. 1. 1 implies

$$E \sup_{d(s,t)<\varepsilon} |X_t - X_s| < \infty.$$

Also, for almsot all ω we have

$$\lim_{\eta\to 0} \sup_{d(s,t)<\eta} |X_t(\omega) - X_s(\omega)| = 0,$$

so that the fact $\lim_{\eta\to\infty} \varphi_d(\eta) = 0$ follows from dominated convergence theorem.

Conversely, we can find a sequence η_n with $\varphi_d(\eta_n) \leqslant 4^{-n}$. Consider the event $A_n = \{\sup_{d(s,t)<\eta_n} | X_s - X_t | > 2^{-n}\}$. Since

$\sum_{n=1}^{\infty} P(A_n) \leqslant \sum_{n=1}^{\infty} 2^{-n} < \infty$, the Borel-Cantelli lemma gives us that X is almost surely d-uniformly continuous as required.

2.1.2 Fernique's conditions

Now, we consider the case of $T=[0,1]^k$ and assume that X is a centered Gaussian process with covariance function Γ. For any $(s,t)\in T\times T$ define $d(s,t)=|s-t|=\sup_{1\leqslant i\leqslant k}|s_i-t_i|$. Let $\varphi:[0,1]\to\mathbf{R}_+$ be a function defined by (1.1.8) i.e.,

$$\varphi(h)=\sup_{\substack{(s,t)\in T\times T\\ d(s,t)\leqslant h}}\|X_s-X_t\|_2. \tag{2.1.2}$$

Theorem 2.1.3 *There exists a constant K such that*

$$E\sup_{t\in T}|X_t|\leqslant K\left\{\sup_{t\in T}\|X_t\|_2+\int_1^{\infty}\varphi(\mathrm{e}^{-x^2})\mathrm{d}x\right\}. \tag{2.1.3}$$

Furthermore, if $\int_1^{\infty}\varphi(\mathrm{e}^{-x^2})\mathrm{d}x<\infty$, then X is uniformly continuous almost surely (here, continuity is to mean d-continuity).

Proof By Theorem 1.1.3 and Remark 1.1.1, we have

$$\begin{aligned}E\sup_{t\in T}|X_t| &\leqslant \left\{\sup_{t\in T}\|X_t\|_2+(2+\sqrt{2})\int_1^{\infty}\varphi\left(\frac{1}{2}p^{-u^2}\right)\mathrm{d}u\right\}\\ &\quad\times\left\{\sqrt{1+4k\log p}+\frac{5}{2\sqrt{\mathrm{e}}}\right\}\\ &\leqslant K\left\{\sup_{t\in T}\|X_t\|_2+\int_1^{\infty}\varphi(\mathrm{e}^{-u^2})\mathrm{d}u\right\},\end{aligned}$$

and for any $t_0\in T$

$$\begin{aligned}E\sup_{d(t,t_0)\leqslant h}|X_t| &\leqslant\left\{\varphi(h)+(2+\sqrt{2})\int_1^{\infty}\varphi(hp^{-u^2})\mathrm{d}u\right\}\\ &\quad\times\left\{\sqrt{1+4k\log p}+\frac{5}{2\sqrt{\mathrm{e}}}\right\}\\ &\leqslant K\left\{\varphi(h)+\int_1^{\infty}\varphi(h\mathrm{e}^{-u^2})\mathrm{d}u\right\}.\end{aligned} \tag{2.1.4}$$

Hence the first part of the theorem is proved, and we conclude that for any $t_0\in T$, X is almost surely continuous at t_0. But what we want to prove is that X is uniformly continuous. To this end, we need some more lemmas.

Lemma 2.1.2 *Let $\{X_i(t);t\in T_i\}$, $i=1,2,\cdots,N$ be centered Gaussian processes. Then*

$$\begin{aligned}&E\max_{i\leqslant N}\sup_{t\in T_i}X_i(t)\\ &\leqslant 2\max_{i\leqslant N}E\sup_{t\in T_i}X_i(t)+3(\log N)^{1/2}\max_{i\leqslant N}\sup_{t\in T_i}\|X_i(t)\|_2.\end{aligned}$$

In particular, for any real centered Gaussian variables X_i, $i=1,2,\cdots,N$ we have

$$E\max_{i\leqslant N}X_i\leqslant 5(\log N)^{1/2}\max_{i\leqslant N}(EX_i^2)^{1/2}.$$

Proof Write $\|X_i\|$ for $\sup_{t\in T_i}X_i(t)$, $\sigma(X_i)$ for $\sup_{t\in T_i}(EX_i^2(t))^{1/2}$. We can assume that the right hand of the inequality is finite, otherwise there is nothing to prove. Assume that $\max_{i\leqslant N}\sigma(X_i)\leqslant 1$. For any $\delta\geqslant 0$, we have by integration by parts and Borell's inequality (cf. Threorem 1.1.1) that

$$\begin{aligned}&E\max_{i\leqslant N}|\,\|X_i\|-E\|X_i\|\,|\\ &\leqslant\delta+\sum_{i=1}^{N}\int_{\delta}^{\infty}P\{|\,\|X_i\|-E\|X_i\|\,|>u\}\mathrm{d}u\\ &\leqslant\delta+N\int_{\delta}^{\infty}\exp(-u^2/2)\mathrm{d}u.\end{aligned}$$

Then, simply let $\delta=(2\log N)^{1/2}$ so that we have obtained, by homogeneity,

$$E\max_{i\leqslant N}\|X_i\|\leqslant 2\max_{i\leqslant N}E\|X_i\|+3(\log N)^{1/2}\max_{i\leqslant N}\sigma(X_i),$$

as required.

Lemma 2.1.3 *Let $Y=\{Y_t;t\in T\}$ be a centered Gaussian process on a metric space (T,d). Let $d_Y(s,t)=\|Y_s-Y_t\|_2$ be, as usual,*

the canonical metric. Then for any $\eta>0$

$$E\sup_{d(s,t)<\eta}|Y_s-Y_t|$$
$$\leqslant K\{\sup_{t\in T}E\sup_{d(s,t)<\eta}|Y_s-Y_t|+\sup_{d(s,t)<3\eta}d_Y(s,t)(\log N(T,d;\eta))^{1/2}\},$$

where $K>0$ *is a constant.*

Proof Given $\eta>0$, let $N=N(T,d;\eta)$ (assumed to be finite and larger than 2). Let $U=\{u_1,\cdots,u_N\}$ in T be such that the d-balls of radius $\eta>0$ and center u_i cover T. Clearly

$$\sup_{d(s,t)<\eta}|Y_s-Y_t|\leqslant 2\max_{u\in U}(\sup_{d(t,u)<\eta}|Y_t-Y_u|)+\max_{\substack{u,v\in U\\ d(u,v)<3\eta}}|Y_u-Y_v|.$$

By Lemma 2.1.2, we have

$$E\max_{u\in U}(\sup_{d(t,u)<\eta}|Y_t-Y_u|)=E\max_{u\in U}(\sup_{d(t,u)<\eta}(Y_t-Y_u))$$
$$\leqslant 2\max_{u\in U}E(\sup_{d(t,u)<\eta}(Y_t-Y_u))+3\max_{d(s,t)<\eta}d_Y(s,t)(\log N)^{1/2};$$

similarly,

$$E\max_{\substack{u,v\in U\\ d(u,v)<3\eta}}|Y_u-Y_v|\leqslant 5\max_{d(s,t)<3\eta}d_Y(s,t)(\log N^2)^{1/2},$$

and the lemma is proved.

Continue the proof of Theorem 2.1.3 By (2.1.4) and Lemma 2.1.3, we have

$$E\sup_{d(s,t)\leqslant h}|X_t-X_s|$$
$$\leqslant K\left\{\varphi(h)+\int_1^\infty\varphi(he^{-u^2})du+\varphi(3h)(\log h^{-1})^{1/2}\right\}. \quad (2.1.5)$$

Since $\int_1^\infty\varphi(e^{-u^2})du<\infty$, we have $u\varphi(e^{-u^2})\to 0$ as $u\to\infty$ which implies $\varphi(h)(\log h^{-1})^{1/2}\to 0$ as $h\to 0$. Thus, we conclude that

$$\lim_{h\to 0}E\sup_{d(s,t)\leqslant h}|X_t-X_s|=0.$$

Theorem 2.1.2 gives that X is almost surely uniformly continuous. This completes the proof.

If $X=\{X(t);t\in[0,1]\}$ is a centered Gaussian process with

stationary increments and $\varphi(h)=\|X(t+h)-X(t)\|_2$ is non-decreasing on h. Then, the Fernique condition that $\int_1^\infty\varphi(e^{-u^2})du<\infty$ is also necessary for X to be bounded or continuous. We will prove this fact later on.

2.1.3 Majorizing measure conditions

As usual, T is the parameter space of a centered Gaussian process X, equipped with the canonical metric d_X. Now, let m be a probability measure on T, and let $g:[0,1]\to\mathbf{R}_+$ be the function defined by

$$g(t)=(\log t^{-1})^{1/2},\ 0\leqslant t\leqslant 1.$$

Let $B(t,\varepsilon)$ be an ε-ball in the d_X-metric about the point $t\in T$.

Definition 2.1.1 *A probability measure* m *is called a majorizing measure (for* (T,d_X)*) if*

$$\sup_{t\in T}\int_0^\infty g(m(B(t,\varepsilon)))d\varepsilon<\infty. \quad (2.1.6)$$

The following general result on the boundedness and continuity of a Gaussian process dates back to Fernique (1978) and was obtained by Talagrand (1987) in the general case. Since it is not an easy result, we would rather not present the proof here. One can refer to Ledoux and Talagrand (1991).

Theorem 2.1.4 *Let* $X=\{X_t;t\in T\}$ *be a centered Gaussian process. Then, for some constant* $K>0$ *and any probability measure* m *on* (T,d_X),

$$E\sup_{t\in T}X_t\leqslant K\sup_{t\in T}\int_0^\infty g(m(B(t,\varepsilon)))d\varepsilon, \quad (2.1.7)$$

and there exists a probability measure m on (T, d_X) such that

$$\sup_{t\in T}\int_0^\infty g(m(B(t,\varepsilon)))\mathrm{d}\varepsilon \leqslant KE\sup_{t\in T}X_t. \qquad (2.1.8)$$

i.e., X is almost surely bounded on (T, d_X) if and only if (T, d_X) admits a majorizing measure.

Furthermore, X is almost surely bounded and (uniformly) continuous on (T, d_X) if and only if (T, d_X) is totally bounded and there exists a probability measure m on (T, d_X) such that

$$\lim_{\eta\to 0}\sup_{t\in T}\int_0^\eta g(m(B(t,\varepsilon)))\mathrm{d}\varepsilon = 0. \qquad (2.1.9)$$

Remark 2.1.2 Note that, if $\varepsilon > 2D$ where $D = D(T)$ is the diameter of (T, d_X) then $m(B(t,\varepsilon)) = 1$, the upper limit of the integrals in (2.1.7) and (2.1.8) is really $D/2$. Also, if we are asked the continuity properties of a centered Gaussian process X with respect to another metric d for which T is compact, we need simply assume in addition that d_X is continuous on (T, d), in other words that X is continuous in L_2(or in probability). Actually, if (T, d) is any compact metric space, a centered Gaussian process $X = \{X_t; t\in T\}$ is continuous on (T, d) if and only if it is continuous on (T, d_X) and d_X is continuous on (T, d). Sufficiency is obvious. If X is d-continuous, so is d_X. By the compactness, X and d_X are both d-uniformly continuous. Theorem 2.1.2 gives

$$\lim_{\varepsilon\to 0}E\sup_{d(s,t)\leqslant\varepsilon}|X_s - X_t| = 0.$$

For $\eta > 0$, let $A_\eta = \{(s,t)\in T\times T;\ d_X(s,t)\leqslant\eta\}$. This is a closed set in $T\times T$ and $\bigcap_\eta A_\eta = A_0$. It is easily seen that $\{(s,t)\in T\times T;\ d(s,s')\leqslant\varepsilon, d(t,t')\leqslant\varepsilon, (s',t')\in A_0\}$ is a closed set and contains A_0. Fix $\varepsilon > 0$. By compactness, there exists $\eta > 0$ such that, whenever $(s,t)\in A_\eta$, there exists $(s',t')\in A_0$ with $d(s,s')\leqslant\varepsilon$, $d(t,t')\leqslant\varepsilon$. Then

$$|X_s - X_t| \leqslant |X_s - X_{s'}| + |X_{s'} - X_{t'}| + |X_{t'} - X_t|.$$

Since $(s',t')\in A_0$, $X_{s'} = X_{t'}$ with probability one. It follows that

$$E\sup_{d_X(s,t)\leqslant\eta}|X_s - X_t| \leqslant 2E\sup_{d(s,t)\leqslant\varepsilon}|X_s - X_t|.$$

Then

$$\lim_{\eta\to 0}E\sup_{d_X(s,t)\leqslant\eta}|X_s - X_t| = 0,$$

which implies the almost sure d_X-uniform continuity of X by Theorem 2.1.2 again.

The following corollary is due to Dudley (1973).

Corollary 2.1.1 *Let $X = \{X_t; t\in T\}$ be a centered Gaussian process. Then*

$$E\sup_{t\in T}X_t \leqslant K\int_0^\infty (\log N(T, d_X; \varepsilon))^{1/2}\mathrm{d}\varepsilon, \qquad (2.1.10)$$

where $K > 0$ is a constant. Furthermore, if this entropy integral converges, X is almost surely bounded and (uniformly) continuous on (T, d_X).

Note that the upper limit of the entropy integral in (2.1.10) is really D.

Proof We may assume that the diameter $D = D(T)$ is finite, otherwise, $N(t, d_X; \varepsilon)$ is infinite for some $\varepsilon > 0$ and there is nothing to prove. We will show that there exists a probability measure m on T such that

$$\sup_{t\in T}\int_0^D g(m(B(t,\varepsilon)))\mathrm{d}\varepsilon \leqslant K\int_0^D (\log N(T, d_X; \varepsilon))^{1/2}\mathrm{d}\varepsilon, \qquad (2.1.11)$$

where K is some constant. Let l_0 be the largest integer with $2^{-l_0} \geqslant D$. For every $l \geqslant l_0$, let $T_l \subset T$ denote the set of the centers of a

minimal family such that the balls $B(t,2^{-l})(t\in T_l)$ cover T. By definition, Card $T_l=N(T,d_X;2^{-l})$. Consider the probability measure m on T given by

$$m=\sum_{l>l_0}2^{-l+l_0}N(T,d_X;2^{-l})^{-1}\sum_{t\in T_l}\delta_t,$$

where δ_t is the Dirac measure at t. Clearly, for every t and $l>l_0$, $m(B(t,2^{-l}))\geqslant 2^{-l+l_0}N(T,d_X;2^{-l})^{-1}$. Hence

$$\begin{aligned}\int_0^D g(m(B(t,\varepsilon)))\,d\varepsilon &\leqslant \sum_{l>l_0}2^{-l}g(m(B(t,2^{-l})))\\
&\leqslant \sum_{l>l_0}2^{-l}(\log(2^{l-l_0}N(T,d_X;2^{-l})))^{1/2}\\
&\leqslant \sum_{l>l_0}2^{-l}(\log 2^{l-l_0})^{1/2}+\sum_{l>l_0}2^{-l}(\log N(T,d_X;2^{-l}))^{1/2}\\
&\leqslant 2^{-l_0+1}\int_0^{1/2}(\log x^{-1})^{1/2}dx+2\int_0^D(\log N(T,d_X;\varepsilon))^{1/2}d\varepsilon\\
&\leqslant 4D\int_0^{1/2}(\log x^{-1})^{1/2}dx+2\int_0^D(\log N(T,d_X;\varepsilon))^{1/2}d\varepsilon\\
&\leqslant 2\left(1+2\int_0^{1/2}(\log x^{-1})^{1/2}dx\right)\int_0^D(\log N(T,d_X;\varepsilon))^{1/2}d\varepsilon\end{aligned}$$

and the announced claim follows. Similarly, for any $\eta\leqslant D$, if let l_0 be the largest integer with $2^{-l_0}>\eta$ we get

$$\begin{aligned}\int_0^\eta g(m(B(t,\varepsilon)))\,d\varepsilon\leqslant 2\left(1+2\int_0^{1/2}\left(\log\frac{1}{x}\right)^{1/2}dx\right)\\
\times\int_0^\eta(\log N(T,d_X;\varepsilon))^{1/2}d\varepsilon.\end{aligned}\tag{2.1.12}$$

By Theorem 2.1.4, (2.1.10) follows. When the entropy integral converges, m is a majorizing measure which satisfies (2.1.9), it follows that X is almost surely bounded and (uniformly) continuous on (T,d_X). This completes the proof.

When X is stationary, there is a converse to the above corollary. To state the result, we only consider the case that T is a subset of $\mathbf{R}_+^k$, and write $T_1+T_2=\{t+s;\ t\in T_1,\ s\in T_2\}$ for T_1 and $T_2\subset\mathbf{R}_+^k$, $T'=T+T$, $T''=T+T+T$, and $t+T=\{t+s;\ s\in T\}$ for $t\in\mathbf{R}_+^k$ and $T\subset\mathbf{R}_+^k$.

Theorem 2.1.5 *Let $X=\{X_t;\ t\in\mathbf{R}_+^k\}$ be a centered Gaussian process indexed by $\mathbf{R}_+^k$. Assume that X is with stationary increments, i.e. d_X is translation invariant in the sense that $d_X(u+s,u+t)=d_X(s,t)$ for all $u,s,t\in\mathbf{R}_+^k$. Let T be a compact subset of $\mathbf{R}_+^k$ with non-empty interior. Then X is almost surely bounded and (uniformly) continuous on T if and only if d_X is continuous on $T\times T$ and*

$$H(T,d_X):=\int_0^\infty(\log N(T,d_X;\varepsilon))^{1/2}d\varepsilon<\infty.\tag{2.1.13}$$

Moreover, there is a constant $K>0$ such that

$$\begin{aligned}&K^{-1}\left\{H(T,d_X)-(\log(|T''|/|T|))^{1/2}D(T)\right\}\\
&\leqslant E\sup_{t\in T}X_t\leqslant KH(T,d_X),\end{aligned}\tag{2.1.14}$$

where $|\cdot|$ is the Lebesgue measure on $\mathbf{R}_+^k$.

Proof By Corollary 2.1.1, it is enough to show the left hand side inequality. By Theorem 2.1.4, it suffices to show that for $T\subset\mathbf{R}_+^k$.

$$\frac{1}{2}\int_0^D\left(\log\left(\frac{|T|}{|T''|}N(T,d_X;\varepsilon)\right)d\varepsilon\right)^{1/2}\leqslant\sup_{t\in T}\int_0^D g(m(B(t,\varepsilon)))\,d\varepsilon,\tag{2.1.15}$$

where m is a probability measure on T and $D=D(T)$ is the d_X-diameter of T. Let us denote by M the right side of the inequality (2.1.15), and let $\lambda(\cdot)=|\cdot|$ be the Lebesgue measure on $\mathbf{R}_+^k$. Since $g(x)$ is convex, by Jensen's inequality

$$M \geqslant \int_0^D \frac{1}{|T|}\int_T g\left(m(B(t,\varepsilon))\right) \mathrm{d}\lambda(t)\mathrm{d}\varepsilon$$

$$\geqslant \int_0^D \left(\log \frac{|T|}{\int_T m(B(t,\varepsilon))\mathrm{d}\lambda(t)} \right)^{1/2} \mathrm{d}\varepsilon.$$

Now, by the Fubini theorem and the translation invariance,

$$\int_T m(B(t,\varepsilon))\mathrm{d}\lambda(t) = \int_{T'} |T \cap B(t,\varepsilon)|\mathrm{d}m(s) \leqslant |T' \cap B(0,\varepsilon)|.$$

Hence,

$$M \geqslant \int_0^D \left(\log \frac{|T|}{|T' \cap B(0,\varepsilon)|} \right)^{1/2} \mathrm{d}\varepsilon.$$

Let $\{t_1, \cdots, t_p\}$ be maximum in T under the conditions $(t_i + B(0,\varepsilon)) \cap (t_j + B(0,\varepsilon)) = \varnothing\ \forall\, i \neq j$. If $t \in T$, by maximality, $(t + B(0,\varepsilon)) \cap (t_i + B(0,\varepsilon)) \neq \varnothing$ for some $i = 1, \cdots, p$. Hence $t \in \{t_i + B(0,2\varepsilon)\}$. It follows that $T \subset \bigcup_{i=1}^p \{t_i + B(0,2\varepsilon)\}$, and then $N(T,d_X;2\varepsilon) \leqslant p$. Noting that $t_i + T' \cap B(0,\varepsilon) \subset T''$, we have

$$\begin{aligned} |T' \cap B(0,\varepsilon)|p &= \sum_{i=1}^p |t_i + T' \cap B(0,\varepsilon)| \\ &= \left| \bigcup_{i=1}^p \{t_i + T' \cap B(0,\varepsilon)\} \right| \\ &\leqslant |T''|. \end{aligned}$$

It follows that

$$N(t,d_X;2\varepsilon) \leqslant \frac{|T''|}{|T' \cap B(0,\varepsilon)|}$$

from which (2. 1. 15) follows and the proof is completed.

The converse to Theorem 2. 1. 5 cannot be true in general. Here is a counterexample. Let $T = \mathbf{Z}_+$ so that our process is actually a sequence $\{X_n; n \geqslant 1\}$. We assume that X_n are independent with

$$\sigma_n = \sigma(X_n) = (EX_n^2)^{1/2} \leqslant (1 + \log n)^{-1/2}.$$

The sequence $\{X_n\}$ is almost surely bounded, moreover

$$E \sup_n |X_n| \leqslant K \tag{2. 1. 16}$$

for some K. To see this, note for each $n \geqslant 1$ and $x \geqslant 2$

$$P\{|X_n| \geqslant x\} \leqslant e^{-x^2/(2\sigma_n^2)} \leqslant e^{-\frac{1}{2}x^2(1+\log n)} \leqslant Cn^{-x^2/2}.$$

Thus

$$P\{\sup_n |X_n| \geqslant x\} \leqslant \sum_{n=1}^\infty P\{|X_n| \geqslant x\} \leqslant C\sum_{n=1}^\infty n^{-\frac{1}{2}x^2} \leqslant Ce^{-\frac{1}{2}x^2}.$$

This and simple integration prove (2. 1. 16).

On the other hand, the sequence $\{X_n; n \geqslant 1\}$ don't give a finite entropy integral. Given $\varepsilon > 0$, for $n < n_\varepsilon = \exp(-1 + 1/(2\varepsilon^2))$ we have $\sigma(X_n) > \varepsilon\sqrt{2}$. Thus

$$d_X(n,m) > 2\varepsilon \quad \text{for all } n, m < n_\varepsilon.$$

Since this means that n and m cannot belong to the same ε ball if $n, m < n_\varepsilon$, it follows that $N(T,d_X;\varepsilon) \geqslant n_\varepsilon - 1$, so that

$$\inf_{\varepsilon > 0} \varepsilon(\log N(T,d_X;\varepsilon))^{1/2} > 0,$$

and so the metric entropy integral $\int (\log N(T,d_X;\varepsilon))^{1/2}\mathrm{d}\varepsilon$ cannot be finite.

The following result about the special case of $k = 1$ shows that the Fernique's condition is necessary for a Gaussian process with stationary increments to be bounded and continuous.

Theorem 2. 1. 6 *Let $X = \{X_t; t \in [0,1]\}$ be a centered Gaussian process with stationary increments. Suppose that $\sigma^2(t) = E(X_t - X_0)^2$ is increasing. Then $X(t)$ is almost surely bounded and (uniformly) continuous if and only if*

$$\int_0^{1/2} \frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\mathrm{d}u < \infty.$$

Furthermore, there exists a constant K such that

$$K^{-1}\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\mathrm{d}u \leqslant \sup_{t\in[0,1]} X_t$$

$$\leqslant K\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\mathrm{d}u. \qquad (2.1.17)$$

Proof Note that

$$\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\mathrm{d}u = \int_{(\log 2)^{1/2}}^{\infty}\sigma(\mathrm{e}^{-u^2})\mathrm{d}u.$$

By Theorem 2.1.3, we need only to show the left hand side inequality. Noting that $D=\sup_{s,t\in[0,1]}\sigma(|t-s|)=\sigma(1)$ and $N([0,1],d_X;\varepsilon)=1\wedge\frac{1}{2\mathrm{inv}\sigma(\varepsilon)}$, it follows from Theorem 2.1.5 that

$$\int_0^{\sigma(1)}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\mathrm{d}\varepsilon \leqslant CE\sup_{t\in[0,1]}X_t + C\sigma(1)$$

$$\leqslant CE\sup_{t\in[0,1]}X_t + C(E(X_1-X_0)^2)^{1/2}$$

$$\leqslant CE\sup_{t\in[0,1]}X_t + CE|X_1-X_0|$$

$$\leqslant CE\sup_{t\in[0,1]}X_t + CE\sup_{s,t\in[0,1]}|X_s-X_t|$$

$$\leqslant CE\sup_{t\in[0,1]}X_t.$$

We can assume that $E\sup_{t\in[0,1]}X_t<\infty$, for otherwise there is nothing to prove. Then the integral in above inequalities converges. Integration by parts yields

$$\int_0^{\sigma(1)}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\mathrm{d}\varepsilon$$

$$=\varepsilon\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\Bigg|_0^{\sigma(1)} - \int_0^{\sigma(1)}\varepsilon\,\mathrm{d}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}$$

$$\geqslant\varepsilon\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\Bigg|_0^{\sigma(1)} + \int_0^{\infty}\sigma(\mathrm{e}^{-u^2})\mathrm{d}u.$$

Since $\mathrm{inv}\sigma(\varepsilon)$ is non-decreasing, we have $\varepsilon\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\to 0$ as $\varepsilon\to 0$. It follows that

$$\int_0^{\sigma(1)}\left(\log\frac{1}{\mathrm{inv}\sigma(\varepsilon)}\right)^{1/2}\mathrm{d}\varepsilon \geqslant \int_0^{\infty}\sigma(\mathrm{e}^{-u^2})\mathrm{d}u$$

$$\geqslant\int_{(\log 2)^{1/2}}^{\infty}\sigma(\mathrm{e}^{-u^2})\mathrm{d}u - (\log 2)^{1/2}\sigma(1).$$

Hence, we get

$$\int_0^{1/2}\frac{\sigma(u)}{u(\log u^{-1})^{1/2}}\mathrm{d}u \leqslant KE\sup_{t\in[0,1]}X_t + \sigma(1)$$

$$\leqslant KE\sup_{t\in[0,1]}X_t,$$

as required.

2.1.4 Vector valued Gaussian processes

Let B be a separable Banach space with dual space B'. A process $X=\{X_t;t\in T\}$ with values in B is a family $\{X_t;t\in T\}$ indexed by T of Borel random variables X_t with values in B. X is Gaussian if each finite sample $(X_{t_1},\cdots,X_{t_N})$, $t_i\in T$ is Gaussian in B^N. The following result is due to Fernique (1990).

Theorem 2.1.7 *Let (T,d) be a metric space and $X=\{X_t;t\in T\}$ be a centered Gaussian process with values in B. Suppose that there is a centered Gaussian random variable ξ with values in B such that*

$$Ef^2(X_t)\leqslant Ef^2(\xi),\ \forall\ (f,t)\in B'\times T,$$

then there is a constant $C>0$ such that

$$E\sup_{t\in T}\|X_t\|$$

$$\leqslant C\left\{E\|\xi\| + \sup_{\|f\|\leqslant 1}\int_0^{\infty}(\log N(T,d_{f(X)};\varepsilon))^{1/2}\mathrm{d}\varepsilon\right\}. \qquad (2.1.18)$$

Moreover, assume that (T,d) is a compact metric space, if

$\| f(X_t)-f(X_s) \|_2$ *is continuous on* (T,d) *for each* $f\in B'$ *and*

$$\lim_{\eta\to 0}\sup_{\|f\|\leqslant 1}\int_0^{\eta}(\log N(T,d_{f(X)};\varepsilon))^{1/2}\mathrm{d}\varepsilon=0, \qquad (2.1.19)$$

then X *is almost surely continuous on* (T,d).

Proof We first show (2.1.18). Let B_1' be the unit ball of B'. We consider X as a process indexed by $T^*=B_1'\times T$. By Theorem 2.1.4, the real valued Gaussian process $\xi=\{f(\xi);f\in B_1'\}$ indexed by B_1' has a majorizing measure; that is, there exists a probability measure μ on (B_1',d_ξ) such that

$$\sup_{f\in B_1'}\int_0^{\infty}g(\mu(B(f,\varepsilon)))\mathrm{d}\varepsilon\leqslant CE\sup_{f\in B_1'}f(\xi)=CE\|\xi\|,$$

where $B(f,\varepsilon)$ is the ball of ε with respect to the metric on the space which contains its center f, that is, $d_\xi(f,g)=\|f(\xi)-g(\xi)\|_2$, $f,g\in B_1'$. (We use further this convention about balls in metric spaces below.) Let $D=D(T^*)$ be the d_X-diameter of $B_1'\times T$ and

$$d_X((f,t),(g,s))=\|f(X_t)-g(X_s)\|_2,\ f,g\in B',\ s,t\in T.$$

For each $f\in B_1'$, let $S(f,u)$ denote the set of the centers of minimal family such that the balls $B_{f(X)}(t,u), t\in S(f,u)$ cover T. By definition, Card $S(f,u)=N(T,d_{f(X)};u)$. Consider the probability measures π_f on T given by

$$\pi_f=\sum_{p=1}^{\infty}2^{-p}\sum_{s\in S(f,D/2^p)}\frac{\delta_s}{N(T,d_{f(X)};D/2^p)},$$

where δ_s is the Dirac measure at s. From μ and $\{\pi_f;f\in B_1'\}$, we construct a probability measure λ on $B_1'\times T$ by

$$\lambda=\int(\delta_f\otimes\pi_f)\mathrm{d}\mu(f).$$

From Theorem 2.1.4, it follows that

$$E\sup_{t\in T}\|X_t\|=E\sup_{(f,t)\in B_1'\times T}f(X_t)$$
$$\leqslant K\sup_{(f,t)\in B_1'\times T}\int_0^{\infty}g(\lambda(B((f,t),\varepsilon)))\mathrm{d}\varepsilon.$$

Given $(f,t)\in B_1'\times T$ and $p\geqslant 1$, for any $(g,s)\in B_1'\times T$, by the triangle inequality we have

$$d_X((f,t),(g,s))\leqslant\|f(X_t)-g(X_t)\|_2+\|g(X_t)-g(X_s)\|_2$$
$$\leqslant d_\xi(f,g)+d_{g(X)}(t,s)$$

and

$$d_{g(X)}(s,t)\leqslant\|f(X_s)-f(X_t)\|_2+\|f(X_s)-g(X_s)\|_2$$
$$+\|f(X_t)-g(X_t)\|_2$$
$$\leqslant d_{f(X)}(s,t)+2d_\xi(f,g).$$

Thus the following implication is true:

$$d_{g(X)}(t,s)\leqslant D/2^{p+1},\ d_\xi(g,f)\leqslant D/2^{p+3}\Rightarrow d_X((f,t),(g,s))\leqslant D/2^p.$$

It follows that

$$\lambda(B((f,t),D/2^p))$$
$$\geqslant\int\pi_g(B_{g(X)}(t,D/2^{p+1}))I\{g;g\in B(f,D/2^{p+3})\}\mathrm{d}\mu(g)$$
$$\geqslant\int\frac{1}{2^{p+1}N(T,d_{g(X)};D/2^{p+1})}I\{g;g\in B(f,D/2^{p+3})\}\mathrm{d}\mu(g).$$

Noting the following implication

$$d_{f(X)}(s,t)\leqslant D/2^{p+2},d_\xi(f,g)\leqslant D/2^{p+3}\Rightarrow d_{g(X)}(s,t)\leqslant D/2^{p+1},$$

and that $\{B_{f(X)}(s,D/2^{p+2});s\in S(f,D/2^{p+2})\}$ cover T, we conclude that $\{B_{g(X)}(s,D/2^{p+1});s\in S(f,D/2^{p+2})\}$cover T if $d_\xi(f,g)\leqslant D/2^{p+3}$, which implies the following implication

$$d_\xi(f,g)\leqslant D/2^{p+3}\Rightarrow N(T,d_{g(X)};D/2^{p+1})\leqslant N(T,d_{f(X)};D/2^{p+2}).$$

It follows that

$$\lambda(B((f,t),D/2^p))\geqslant\frac{1}{2^{p+1}N(T,d_{f(X)};D/2^{p+2})}\mu(B(f,D/2^{p+3})).$$

Thus, for each $(f,t)\in B_1'\times T$, we have

$$\int g(\lambda(B((f,t),\varepsilon)))\,d\varepsilon \leqslant \sum_{p=0}^{\infty}\frac{D}{2^p}g(\lambda(B((f,t),D/2^p)))$$

$$\leqslant \sum_{p=0}^{\infty}\frac{D}{2^p}g\left(\frac{\mu(B(f,D/2^{p+3}))}{2^{p+1}N(T,d_{f(X)};D/2^{p+2})}\right)$$

$$\leqslant \sum_{p=0}^{\infty}\frac{D}{2^p}\Big\{(\log 2^{p+1})^{1/2}+g(\mu(B(f,D/2^{p+3})))$$

$$+\left(\log N(T,d_{f(X)};D/2^{p+2})\right)^{1/2}\Big\}$$

$$\leqslant K\left\{D+\int_0^{\infty}g(\mu(B(f,u)))\,du+\int_0^{D}(\log N(T,d_{f(X)};u))^{1/2}du\right\}.$$

It follows that

$$E\sup_{t\in T}\|X_t\|\leqslant K\left\{D+E\|\xi\|+\sup_{f\in B_1'}\int_0^{D}(\log N(T,d_{f(X)};u))^{1/2}du\right\}.$$

Note that

$$D\leqslant \sup_{\substack{f,g\in B_1'\\ s,t\in T}}\|f(X_t)-g(X_t)\|_2\leqslant 2\sup_{(f,t)\in B_1'\times T}\|f(X_t)\|_2$$

$$\leqslant 2\sup_{f\in B_1'}\|f(\xi)\|_2=(2\pi)^{1/2}\sup_{f\in B_1'}E|f(\xi)|\leqslant(2\pi)^{1/2}E\|\xi\|.$$

Summarizing, we get for some constant C that

$$E\sup_{t\in T}\|X_t\|$$

$$\leqslant C\left\{E\|\xi\|+\sup_{f\in B_1'}\int_0^{(2\pi)^{1/2}E\|\xi\|}(\log N(T,d_{f(X)};u))^{1/2}du\right\}.$$

This inequality is stronger than (2.1.18).

Now, suppose that (2.1.19) holds. If F is a finite dimensional subspace of B, let T_F denote the quotient map $B\to B/F$, we get from the above inequality that

$$E\sup_{t\in T}\|T_F(X_t)\|$$

$$\leqslant K\left\{E\|T_F(\xi)\|+\sup_{f\in B_1'}\int_0^{\sqrt{2\pi}E\|T_F(\xi)\|}(\log N(T,d_{f(X)};u))^{\frac{1}{2}}du\right\}.$$

It follows that if F_n is a sequence of finite dimensional subspaces of B with $F_n\uparrow B$, then

$$\lim_{n\to\infty}E\|T_{F_n}(\xi)\|=0.$$

Thus

$$\lim_{n\to\infty}\sup_{t\in T}\|T_{F_n}(X_t)\|=0\quad \text{a.s.}\quad \text{as}\quad n\to\infty.$$

Also, for each $f\in B_1'$, by Corollary 2.1.1 and the assumption that $\|f(X_t)-f(X_s)\|_2$ is continuous, $f(X_t)$ is almost surely continuous on (T,d), which implies that the processes $X-T_{F_n}(X)$ that takes values in finite dimensional spaces F_n are continuous on (T,d). It follows that X is almost surely continuous on (T,d). This completes the proof.

The following result is a simple corollary of Theorem 2.1.7 together with Theorem 2.1.5.

Corollary 2.1.2 *Let $X=\{X_t;t\in T\}$ be a stationary centered Gaussian process with values in B, where T is a compact subset of $\mathbf{R}_+^k$ with non-empty interior. Then, for some constant $C>0$,*

$$\frac{1}{2}\left(E\|X_0\|+\sup_{\|f\|\leqslant 1}E\sup_{t\in T}f(X_t)\right)$$

$$\leqslant E\sup_{t\in T}\|X_t\|$$

$$\leqslant C\left(1+\log\frac{|T''|}{|T|}\right)\left(E\|X_0\|+\sup_{\|f\|\leqslant 1}E\sup_{t\in T}f(X_t)\right).\quad(2.1.20)$$

Moreover, X is almost surely continuous on T if and only if $\|f(X_t)-f(X_s)\|_2$ is continuous on $T\times T$ for each $f\in B'$ and

$$\lim_{\eta\to\infty}\sup_{\|f\|\leqslant 1}E\sup_{t\in T}f(X_t)=0.\qquad(2.1.21)$$

2.1.5 The infinite series of independent Ornstein-Uhlenbeck processes

Let $\{Y(t); -\infty<t<\infty\}=\{X_k(t); -\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck (OU) processes with coefficients $\gamma_k\geqslant 0$ and $\lambda_k>0$, i.e., $X_k(\cdot)$ is a stationary, mean zero Gaussian process with

$$EX_k(s)X_k(t)=(\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|), k=1,2,\cdots.$$

The process $Y(\cdot)$ was first introduced by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$\mathrm{d}X_i(t)=-\lambda_i X_i(t)\mathrm{d}t+(2\gamma_i)^{1/2}\mathrm{d}W_i(t) \quad (i=1,2,\cdots), \tag{2.1.22}$$

where $\{W_i(t); -\infty<t<\infty\}$ are independent Wiener processes (cf. also Dawson 1975 and Walsh 1981). Such processes have been extensively studied in the literature since the appearance of Dawson (1972). They have been used as a model for neuronal behavior (cf. Kallianpur and Wolpert 1984, Walsh 1981); they have been proved important in quantum field theory (cf. Carmona 1977 and Gross 1977); and they also arise as fluctuation limits in infinite particle system (cf. Holley and Stroock 1978). Owing to the variety of applications of these processes, they have been looked at from a number of different mathematical angles. For instance, they have been considered as examples of stochastic evolution equations (cf. DaPrato, Kwapien and Zabczyk 1987, Kotelencz 1987, Miyahara 1981) and of infinite dimensional diffusions (cf. Itô 1984, Kuo 1975, Piech 1975,

Stroock 1981 and Schmuland 1988a); or as reversible Markov processes which may be studied by using the associated theory of Dirichlet forms (cf. Schmuland 1988b); or as solutions to stochastic differential equations (cf. Dawson 1975, Ricciardi and Sacerdote 1979, Walsh 1981, Antoniadis and Carmona 1987 and Itô 1984).

Walsh (1981) presented a mathematical model for neural response and investigated many analytic properties of processes. One of the processes of interest in his study is the infinte series of independent O-U coordinate processes of $Y(\cdot)$, namely the process $X(\cdot)$ defined by

$$\{X(t); -\infty<t<\infty\}=\left\{\sum_{k=1}^{\infty}X_k(t); -\infty<t<\infty\right\}, \tag{2.1.23}$$

where the $X_k(\cdot)$ are the Ornstein-Uhlenbeck components of $Y(\cdot)$.

Csáki, Csörgő, Lin and Révész (1991) studied the properties of the process $X(\cdot)$. We present the results on the continuity of $X(\cdot)$ here. Also, $Y(\cdot)$ have been studied directily as l^p-valued Gaussian processes and will be discussed in the next chapter.

Theorem 2.1.8 *Let* $X(\cdot)$ *be defined as in* (2.1.23) *and*

$$\{X(t,n); -\infty<t<\infty, n=1,2,\cdots\}$$
$$=\left\{\sum_{k=1}^{n}X_k(t); -\infty<t<\infty, n=1,2\cdots\right\}. \tag{2.1.24}$$

Assume that for some $\delta>0$

$$\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(\log(\lambda_k\vee e))^{1+\delta}<\infty. \tag{2.1.25}$$

Then $X(t,n)\to X(t)$ uniformly in t over any finite interval with probability one, i.e. for any $\varepsilon>0$, $T>0$ and for almost all $\omega\in\Omega$ there exists an integer $n_0=n_0(\varepsilon,T,\omega)$ such that

$$\sup_{|t|\leqslant T}|X(t,n,\omega)-X(t,\omega)|\leqslant\varepsilon \tag{2.1.26}$$

whenever $n\geqslant n_0$. As a consequence, the Gaussian process $\{X(t);-\infty<t<\infty\}$ is continuous with probability one. where and throughout the remainder of this book, $\log x$ *denotes* $\log(x\vee \mathrm{e})$, *except that it is specially mentioned.*

Proof In order to verify (2.1.26), on account of Itô-Nisio's theorem (cf. Theorem 6.1 of Ledoux and Talagrand 1991), it suffices to prove that

$$\sup_{|t|\leqslant T}|X(t,n)-X(t)| = \sup_{|t|\leqslant T}\Big|\sum_{k=n+1}^{\infty}X_k(t)\Big|$$

converges to zero in probability as $n\to\infty$. Thus we want to show that for any $\varepsilon>0$

$$\lim_{n\to\infty}P\Big\{\sup_{|t|\leqslant T}\Big|\sum_{k=n+1}^{\infty}X_k(t)\Big|>\varepsilon\Big\}=0.$$

The latter, in turn, will be established by showing that, under the condition (2.1.25), for any $\varepsilon>0$ and $0<\eta<1$ there exists $n_0=n_0(\varepsilon,\eta)$ such that

$$P\Big\{\sup_{|t|\leqslant T}\Big|\sum_{k=n+1}^{m}X_k(t)\Big|>\varepsilon\Big\}\leqslant\eta,$$

whenever $m>n\geqslant n_0$. Letting

$$X_{m,n}(t)=X(t,m)-X(t,n)=\sum_{k=n+1}^{m}X_k(t),$$

we have that, for each $m>n$, the process $\{X_{m,n}(t);-\infty<t<\infty\}$ is a stationary, mean zero Gaussian process with

$$EX_{m,n}^2(t)=\sum_{k=n+1}^{m}\gamma_k/\lambda_k,$$

$$EX_{m,n}(t)X_{m,n}(s)=\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|)$$

and

$$\begin{aligned}\sigma_{m,n}^2(h):&=E(X_{m,n}(t+h)-X_{m,n}(t))^2\\ &=2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k h))\end{aligned}$$

for all $t,s\in\mathbf{R}$ and $h>0$. Set

$$K_1=\{k;\lambda_k<\mathrm{e}^{u^2/2}\},\quad K_2=\{k;\lambda_k\geqslant\mathrm{e}^{u^2/2}\}.$$

Then

$$\begin{aligned}&\int_1^\infty\Big(\sum_{\substack{k=n+1\\k\in K_1}}^{m}\frac{\gamma_k}{\lambda_k}\Big(1-\exp(-\lambda_k\mathrm{e}^{-u^2})\Big)\Big)^{1/2}\mathrm{d}u\\ \leqslant&\int_1^\infty\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}\cdot\lambda_k\mathrm{e}^{-u^2}I_{\{\lambda_k<\mathrm{e}^{u^2/2}\}}\Big)^{1/2}\mathrm{d}u\\ \leqslant&\int_1^\infty\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\mathrm{e}^{-u^2/4}\mathrm{d}u\leqslant4\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\int_1^\infty\Big(\sum_{\substack{k=n+1\\k\in K_2}}^{m}\frac{\gamma_k}{\lambda_k}\Big(1-\exp(-\lambda_k\mathrm{e}^{-u^2})\Big)\Big)^{1/2}\mathrm{d}u\\ \leqslant&\int_1^\infty\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}I_{\{\lambda_k\geqslant\mathrm{e}^{u^2/2}\}}\Big)^{1/2}\mathrm{d}u\\ \leqslant&\int_1^\infty\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(\log^+\lambda_k)^{1+\delta}\cdot\Big(\frac{2}{u^2}\Big)^{1+\delta}\Big)^{1/2}\mathrm{d}u\\ \leqslant&2^{(1+\delta)/2}\delta^{-1}\Big(\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(\log^+\lambda_k)^{1+\delta}\Big)^{1/2}.\end{aligned}$$

It follows that

$$\int_1^{\infty}\sigma_{m,n}(e^{-u^2})du \leqslant c\left\{\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(\log(\lambda_k \vee e))^{1+\delta}\right\}^{1/2}$$

which together with Theorem 2. 1. 3 implies that

$$E\sup_{0\leqslant t\leqslant 1}|X_{m,n}(t)| \leqslant K\left\{\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(\log(\lambda_k \vee e))^{1+\delta}\right\}^{1/2} =: K\eta_{m,n},$$

where K is a constant. Since the process $\{X_{m,n}(t);-\infty<t<\infty\}$ is stationary, we have

$$E\sup_{|t|\leqslant T}|X_{m,n}(t)|\leqslant 2(T+1)E\sup_{0\leqslant t\leqslant 1}|X_{m,n}(t)|\leqslant 2(T+1)K\eta_{m,n}.$$

It follows that

$$P\{\sup_{|t|\leqslant T}|X_{m,n}(t)|>\varepsilon\}\leqslant\frac{2K(T+1)}{\varepsilon}\eta_{m,n}<\eta$$

whenever $m>n\geqslant n_0$, on account of $\eta_{m,n}\to 0(n\to\infty)$ by condition (2. 1. 25). This completes the proof of Theorem 2. 1. 8.

2. 2 Fractional Wiener Processes

We start our main topic on the moduli of continuity and limit behavior of large increments for Gaussian processes with a simple but important Gaussian process, i. e. the fractional Wiener process or so called the fractional Brownian motion.

A fractional Wiener process $\{Z(t);t\in\mathbf{R}\}$ of order 2α with $0<\alpha<1$ is a real-valued Gaussian process with mean zero, stationary increments, $Z(0)=0$ and $\sigma^2(|t|)=EZ^2(t)=|t|^{2\alpha}(t\in\mathbf{R})$. Obviously, when $\alpha=1/2$, $\{Z(t);t\in\mathbf{R}\}$ is a standard Wiener process, and we denote it by $\{W(t);t\in\mathbf{R}\}$. It is easily seen that $\{Z(t);t\in\mathbf{R}\}$ and $\left\{\int_{\mathbf{R}}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}dW(x);t\in\mathbf{R}\right\}$

have the same distribution, where

$$K_\alpha^2=\int_{\mathbf{R}}(|x-1|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})^2dx$$

and $K_\alpha^{-1}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}$ is interpreted to be $I_{(0,t]}$ when $\alpha=1/2$. So such a fractional Wiener process $\{Z(t);t\in\mathbf{R}\}$ exists and can be rewritten as

$$Z(t)=\int_{\mathbf{R}}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}dW(x),\ t\in\mathbf{R}$$

and keep the path properties of $Z(\cdot)$ without a change.

The fractional Wiener process was introduced by Mandelbrot and Van Ness (1968). Such processes have been extensively playing an important role in econometrics, financial statistics and science of management. A fractional Wiener process is an immediate and simple extension of a Wiener process with many properties preserved. For example, it is continuous (by Theorem 2. 1. 3), its increments are stationary, etc. But, its increments are no longer independent except that it is the Wiener process itself. In this section, we pay attention to its moduli of continuity and behavior of large increments. Since $\{Z(t);t\leqslant 0\}$ has the same distribution as $\{Z(t);t\geqslant 0\}$, we consider $\{Z(t);t\geqslant 0\}$ only.

2. 2. 1 The moduli of continuity of $Z(\cdot)$

The following is the Lévy type moduli of continuity.

Theorem 2. 2. 1 *We have*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|Z(t+h)-Z(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}$$

$$=\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}\frac{|Z(t+s)-Z(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}$$
$$=1\quad \text{a. s.} \tag{2.2.1}$$

Before proving Theorem 2. 2. 1, we shall prove a lemma. First, we introduce the concepts of quasi-increasing, quasi-decreasing and regularly varying functions.

Definition 2. 2. 1 *A function $f(x)$ on (a,b) will be called quasi-increasing (resp. quasi-decreasing) on (a,b), if there exists a positive C_0 such that*

$$f(x)\leqslant C_0 f(y)\quad \textit{for all}\quad a<x<y<b$$
$$(\textit{resp.}\ f(x)\geqslant C_0 f(y)\quad \textit{for all}\quad a<x<y<b).$$

Definition 2. 2. 2 *A function $f(x)$ will be called regularly varying at zero (resp. at infinity) with a positive exponent α, if*

$$\lim_{s\to 0} f(\theta s)/f(s)=\theta^\alpha\quad \textit{for all}\ \theta>0$$
$$(\textit{resp.}\ \lim_{s\to\infty} f(\theta s)/f(s)=\theta^\alpha\quad \textit{for all}\ \theta>0).$$

It is easy to see that if a function $f(x)$ on $[0,1]$ (resp. on $[1,\infty)$) is regularly varying at zero (resp. at infinity) with exponent α, then $f(s)/s^{\alpha/2}$ is quasi-increasing on $[0,1]$ (resp. on $[1,\infty)$).

Lemma 2. 2. 1 *Assume that $\{X(t);t\geqslant 0\}$ is a separable Gaussian process with*

$$E(X(t)-X(s))^2\leqslant \Lambda^2(|t-s|),$$

where $\Lambda(x)$ is a non-decreasing continuous function such that for some $\alpha>0$ and $h_0>0$, $\Lambda(x)/x^\alpha$ is quasi-increasing on $(0,h_0)$. Then for any $\varepsilon>0$, there exist positive constants C_1, C_2 such that

$$P\Big\{\sup_{0\leqslant t\leqslant t'\leqslant T}\ \sup_{t'-t\leqslant h}|X(t')-X(t)|\geqslant x\Lambda(h)\Big\}\leqslant C_2\frac{T}{h}e^{-x^2/(2+\varepsilon)} \tag{2.2.2}$$

for all $x\geqslant C_1$, $0<h\leqslant h_0$ and $T>h$.

Proof By Corollary 1. 1. 1, it follows that for any $y\geqslant\sqrt{1+4\log p}$

$$P\Big\{\sup_{0\leqslant s\leqslant\Delta}|X(t+s)-X(t)|$$
$$\geqslant y\Big(\Lambda(\Delta)+(2+\sqrt{2})\int_1^\infty \Lambda\Big(\frac{\Delta}{2}p^{-u^2}\Big)du\Big)\Big\}$$
$$\leqslant \frac{5}{2}p^2\int_y^\infty e^{-u^2/2}du$$

for any $0<\Delta<h_0$, $t\geqslant 0$, where $p\geqslant 2$ is an integer.

Since $\Lambda(x)/x^\alpha$ is quasi-increasing on $(0,h_0)$, there exists $c_0\geqslant 1$ such that

$$\Lambda(ht)\leqslant c_0 t^\alpha\Lambda(h)$$

for all $0\leqslant t\leqslant 1$, $0<h<h_0$. It follows that

$$\int_1^\infty \Lambda\Big(\frac{\Delta}{2}p^{-u^2}\Big)du\leqslant c_0\Lambda(\Delta)\int_1^\infty p^{-\alpha u^2}du\leqslant c_0\Lambda(\Delta)\,\frac{2}{\alpha p^\alpha\log p}.$$

Hence for $y\geqslant\sqrt{1+4\log p}$,

$$P\Big\{\sup_{0\leqslant s\leqslant\Delta}|X(t+s)-X(t)|\geqslant y\Lambda(\Delta)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}$$
$$\leqslant\frac{5}{2}p^2\int_y^\infty e^{-u^2/2}du.$$

Given $0<\delta<h_0$, let $N=(2c_0/\delta)^{1/\alpha}/h$. Noting that $\Lambda\Big(\frac{1}{N}\Big)\leqslant c_0\Big(\frac{\delta}{2c_0}\Big)\Lambda(h)=\frac{\delta}{2}\Lambda(h)$, we have for $y\geqslant\sqrt{1+4\log p}$,

$$P\Big\{\sup_{0\leqslant t\leqslant t'\leqslant T}\ \sup_{t'-t\leqslant h}|X(t')-X(t)|\geqslant(1+\delta)y\Lambda(h)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}$$
$$\leqslant P\Big\{\sup_{0\leqslant i\leqslant[NT]}\ \sup_{0\leqslant s\leqslant h}\Big|X\Big(\frac{i}{N}+s\Big)-X\Big(\frac{i}{N}\Big)\Big|\geqslant y\Lambda(h)\Big(1+\frac{8c_0}{\alpha p^\alpha\log p}\Big)\Big\}$$
$$+P\Big\{\sup_{0\leqslant i\leqslant[NT]}\ \sup_{0\leqslant s\leqslant 1/N}\Big|X\Big(\frac{i}{N}+s\Big)-X\Big(\frac{i}{N}\Big)\Big|$$

$$\geqslant \frac{\delta}{2} y A(h)\left(1+\frac{8c_0}{\alpha p^\alpha \log p}\right)\Bigg\}$$

$$\leqslant 2(NT+1)\frac{5}{2}p^2\int_y^\infty e^{-u^2/2}du$$

$$\leqslant 5\left(\left(\frac{2c_0}{\delta}\right)^{1/\alpha}+1\right)\frac{T}{h}p^2\int_y^\infty e^{-u^2/2}du.$$

Now, for any given $\varepsilon>0$, choose $\delta>0$ small enough and p large enough such that $(1+\delta)\left(1+\frac{8c_0}{\alpha p^\alpha \log p}\right)\leqslant\sqrt{1+\varepsilon/2}$. Let $x=y(1+\delta)\left(1+\frac{8c_0}{\alpha p^\alpha \log p}\right)$, it follows that

$$P\{\sup_{0\leqslant t\leqslant t'\leqslant T}\sup_{t'-t\leqslant h}|X(t')-X(t)|\geqslant xA(h)\}$$

$$\leqslant C_2\frac{T}{h}\int_{\frac{x}{\sqrt{1+\varepsilon/2}}}^\infty e^{-u^2/2}du\leqslant C_2\frac{T}{h}e^{-x^2(2+\varepsilon)}$$

for $x\geqslant c_1:=2\sqrt{1+4\log p}$. This is what we have to prove.

Now we prove Theorem 2.2.1. Let

$$A_h=\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|Z(t+s)-Z(t)|.$$

By Lemma 2.2.1, we have for h small enough

$$P\left\{\frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\geqslant 1+\varepsilon\right\}$$

$$\leqslant\frac{c}{h}\exp\left(-\frac{2(\log h^{-1})(1+\varepsilon)^2}{2+\varepsilon}\right)\leqslant ch^\varepsilon.$$

Take $A>1/\varepsilon$ and let $h=h_n=n^{-A}$. Then

$$\sum_{n=1}^\infty P\left\{\frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}}\geqslant 1+\varepsilon\right\}\leqslant\sum_{n=1}^\infty cn^{-A\varepsilon}<\infty,$$

and the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty}\frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}}\leqslant 1+\varepsilon \quad \text{a.s.}$$

When $h_{n+1}<h\leqslant h_n$, we have

$$\limsup_{h\to0}\frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\leqslant\limsup_{n\to\infty}\frac{A_{h_n}}{\{2\sigma^2(h_{n+1})\log h_n^{-1}\}^{1/2}}$$

$$=\limsup_{n\to\infty}\frac{A_{h_n}}{\{2\sigma^2(h_n)\log h_n^{-1}\}^{1/2}}\cdot\frac{\sigma(h_n)}{\sigma(h_{n+1})}\leqslant 1+\varepsilon \quad \text{a.s.}$$

It follows that

$$\limsup_{h\to0}\frac{A_h}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\leqslant 1 \quad \text{a.s.} \qquad (2.2.3)$$

Next we show that

$$\liminf_{h\to0}\sup_{0\leqslant t\leqslant 1-h}\frac{Z(t+h)-Z(t)}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}\geqslant 1 \quad \text{a.s.} \qquad (2.2.4)$$

Given $\varepsilon>0$ and $\delta>0$, let m be an integer such that $\frac{1}{2}((l+1)^{2\alpha}+(l-1)^{2\alpha}-2l^{2\alpha})<\delta^2$ for all $l\geqslant m$. It is easily seen that for all $l\neq k$

$$E\left(Z\left(\frac{km+1}{n}\right)-Z\left(\frac{km}{n}\right)\right)\left(Z\left(\frac{lm+1}{n}\right)-Z\left(\frac{lm}{n}\right)\right)$$

$$=\frac{1}{2}\left(\frac{1}{n}\right)^{2\alpha}((|l-k|m+1)^{2\alpha}+(|l-k|m-1)^{2\alpha}-2(|l-k|m)^{2\alpha})$$

$$\leqslant\sigma^2\left(\frac{1}{n}\right)\delta.$$

Define $\xi_i=\left(Z\left(\frac{im+1}{n}\right)-Z\left(\frac{im}{n}\right)\right)\Big/\sigma\left(\frac{1}{n}\right)$, $i=0,\cdots,\left[\frac{n}{m}\right]-1$ and let τ, $\eta_i\left(i=0,\cdots,\left[\frac{n}{m}\right]-1\right)$ be independent mean zero normal variables with $E\eta_i^2=1-\delta^2$ and $E\tau^2=\delta^2$. Then $E\xi_i^2=E(\eta_i+\tau)^2=1$ and $E\xi_i\xi_j\leqslant\delta^2=E(\eta_i+\tau)(\eta_j+\tau)$, $i\neq j$. By Slepian's inequality (Corollary 1.2.1) we have

$$P\left\{\max_{0\leqslant i\leqslant[\frac{n}{m}]-1}\frac{Z\left(\frac{im+1}{n}\right)-Z\left(\frac{im}{n}\right)}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}\leqslant 1-3\varepsilon\right\}$$

$$\leq P\left\{\max_{0\leq i\leq\left[\frac{n}{m}\right]-1}\frac{\eta_i+\tau}{(2\log n)^{1/2}}\leq 1-3\varepsilon\right\}$$

$$\leq P\left\{\max_{0\leq i\leq\left[\frac{n}{m}\right]-1}\eta_i\leq(1-2\varepsilon)(2\log n)^{1/2}\right\}+P\{\tau>\varepsilon(2\log n)^{1/2}\}$$

$$=\prod_{i=0}^{[n/m]-1}P\left\{N(0,1)\leq(1-2\varepsilon)\left(\frac{2\log n}{1-\delta^2}\right)^{1/2}\right\}$$

$$+P\left\{N(0,1)\geq\frac{\varepsilon}{\delta}(2\log n)^{1/2}\right\}.$$

Choose $\delta>0$ small enough such that $\frac{1-2\varepsilon}{\sqrt{1-\delta^2}}<1-\frac{3}{2}\varepsilon$ and $\frac{\varepsilon}{\delta}>2$. By (1.1.1), it follows that for n large enough

$$P\left\{N(0,1)\leq(1-2\varepsilon)\left(\frac{2\log n}{1-\delta^2}\right)^{1/2}\right\}$$

$$\leq P\left\{N(0,1)\leq\left(1-\frac{3}{2}\varepsilon\right)(2\log n)^{1/2}\right\}$$

$$\leq 1-\frac{1}{n^{1-\varepsilon}}\leq\exp\left(-\frac{1}{n^{1-\varepsilon}}\right)$$

and $P\{N(0,1)\geq\varepsilon(2\log n)^{1/2}/\delta\}\leq n^{-2}$, we obtain

$$P\left\{\max_{0\leq i\leq[\frac{n}{m}]-1}\frac{\left|Z\left(\frac{im+1}{n}\right)-Z\left(\frac{im}{n}\right)\right|}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}\leq 1-3\varepsilon\right\}$$

$$\leq\exp\left(-\left[\frac{n}{m}\right]\frac{1}{n^{1-\varepsilon}}\right)+\frac{1}{n^2}$$

$$\leq\exp\left(-\frac{1}{2m}n^{\varepsilon}\right)+\frac{1}{n^2}.$$

Then

$$\sum_{n=1}^{\infty}P\left\{\max_{0\leq i\leq\left[\frac{n}{m}\right]-1}\frac{Z\left(\frac{im+1}{n}\right)-Z\left(\frac{im}{n}\right)}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}\leq 1-3\varepsilon\right\}<\infty,$$

and the Borel-Cantelli lemma impiles

$$\liminf_{n\to\infty}\sup_{0\leq t\leq 1-\frac{1}{n}}\frac{Z\left(t+\frac{1}{n}\right)-Z(t)}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}$$

$$\geq\liminf_{n\to\infty}\max_{0\leq i\leq\left[\frac{n}{m}\right]-1}\frac{Z\left(\frac{im+1}{n}\right)-Z\left(\frac{im}{n}\right)}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}\geq 1-3\varepsilon\quad\text{a.s.}\quad(2.2.5)$$

Considering now $h_{n+1}<h\leq h_n$ with $h_n=1/n$, we get

$$\liminf_{h\to 0}\sup_{0\leq t\leq 1-h}\frac{Z(t+h)-Z(t)}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}$$

$$\geq\liminf_{n\to\infty}\sup_{0\leq t\leq 1-\frac{1}{n+1}}\frac{Z\left(t+\frac{1}{n+1}\right)-Z(t)}{\left\{2\sigma^2\left(\frac{1}{n+1}\right)\log(n+1)\right\}^{1/2}}$$

$$\times\frac{\left\{2\sigma^2\left(\frac{1}{n+1}\right)\log(n+1)\right\}^{1/2}}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}}$$

$$-\limsup_{n\to\infty}\sup_{0\leq t\leq 1-\frac{1}{n+1}}\sup_{0\leq s\leq\frac{1}{n(n+1)}}\frac{|Z(t+s)-Z(t)|}{\left\{2\sigma^2\left(\frac{1}{n}\right)\log n\right\}^{1/2}},$$

where the latter r. v. is $\overset{\text{a.s.}}{=}o(1)$ by (2.2.3) and the first one is not less than (a. s.) $1-3\varepsilon$ by (2.2.5). Hence we get (2.2.4).

2.2.2 How big are the increments of $Z(\cdot)$

In Theorem 2.2.1 we saw how big the increments of a fractional Wiener process over subintervals of length h of the unit interval can be when h is small. The following Theorem 2.2.2 deals with the similar problem of how big the increments of a fractional Wiener process over subinterval of length a_T of the in-

terval $[0,T]$ can be when $T\to\infty$ and a_T is a non-decreasing function of T.

Theorem 2.2.2 (*Ortega* 1984) *Let $a_T(0<a_T\leqslant T)$ be a function of T for which*

(i) *a_T is non-decreasing,*

(ii) *T/a_T is non-decreasing.*

Then

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|Z(t+s)-Z(t)|$$
$$=\limsup_{T\to\infty}\beta_T|Z(T+a_T)-Z(T)|=1\quad \text{a. s.}, \tag{2.2.6}$$

where

$$\beta_T=\{2\sigma^2(a_T)(\log(T/a_T)+\log\log T)\}^{-1/2}.$$

If we also have

(iii) $\lim_{T\to\infty}(\log(T/a_T))(\log\log T)^{-1}=\infty$,

then

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|Z(t+s)-Z(t)|$$
$$=\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|Z(t+a_T)-Z(T)|=1\quad \text{a. s.} \tag{2.2.7}$$

Proof The proof is formulated in three steps.

Step 1 Let

$$A(T)=\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|Z(t+s)-Z(t)|,$$

suppose that conditions (i) and (ii) are fulfilled. Then

$$\limsup_{T\to\infty}A(T)\leqslant 1\quad \text{a. s.} \tag{2.2.8}$$

Proof For any given $\varepsilon>0$, by Lemma 2.2.1 we have for T large enough

$$P\{A(T)\geqslant 1+\varepsilon\}\leqslant C\frac{T}{a_T}\exp\left\{-(1+\varepsilon)\left(\log\frac{T}{a_T}+\log\log T\right)\right\}$$
$$=C\left(\frac{a_T}{T}\right)^{\varepsilon}\frac{1}{(\log T)^{1+\varepsilon}}.$$

Let $T_k=\theta^k(\theta>1)$. Then

$$\sum_{k=1}^{\infty}P\{A_{T_k}\geqslant 1+\varepsilon\}<\infty$$

for every $\varepsilon>0,\theta>1$. Hence by the Borel-Cantelli lemma

$$\limsup_{k\to\infty}A(T_k)\leqslant 1. \tag{2.2.9}$$

We also have

$$1\leqslant\limsup_{k\to\infty}\frac{\beta_{T_k}}{\beta_{T_{k+1}}}\leqslant\theta^{\varepsilon}. \tag{2.2.10}$$

Now choosing θ near enough to one, (2.2.8) follows from (2.2.9) and (2.2.10), because $\beta_T^{-1}A(T)$ is non-decreasing and β_T is non-increasing in T.

Step 2 Let

$$B(T)=\beta_T(Z(T)-Z(T-a_T)),$$

suppose that the conditions (i) and (ii) are fulfilled. Then

$$\limsup_{T\to\infty}B(T)\geqslant 1\quad \text{a. s.} \tag{2.2.11}$$

Proof For any $\varepsilon>0$, by (1.1.1), we have

$$P\{B(T)\geqslant 1-\varepsilon\}\geqslant\frac{\exp\left(-(1-\varepsilon)^2\left(\log\frac{T}{a_T}+\log\log T\right)\right)}{\sqrt{2\pi}\left\{2\left(\log\frac{T}{a_T}+\log\log T\right)\right\}^{1/2}}$$
$$\geqslant\left(\frac{a_T}{T\log\log T}\right)^{1-\varepsilon}=:b_T^{1-\varepsilon} \tag{2.2.12}$$

if T is large enough. Let $\rho=\lim_{T\to\infty}a_T/T$. If $\rho<1$, let $T_1=1$ and define T_{k+1} by $T_{k+1}-a_{T_{k+1}}=T_k$. Then

$$\lim_{k\to\infty}\frac{a_{T_k}}{a_{T_{k+1}}}=\lim_{k\to\infty}\frac{T_k}{T_{k+1}}=1-\rho>0. \tag{2.2.13}$$

It follows that for any $\varepsilon>0$

$$\sum_{k=2}^{\infty}b_{T_k}^{1-\varepsilon}\geqslant c\sum_{k=2}^{\infty}\left(\frac{a_{T_{k+1}}}{T_k\log T_k}\right)^{1-\varepsilon}\geqslant c\sum_{k=2}^{\infty}\frac{a_{T_{k+1}}}{T_k(\log T_k)^{1-\varepsilon}}$$

$$= c\sum_{k=2}^{\infty}\frac{T_{k+1}-T_k}{T_k(\log T_k)^{1-\epsilon}} \geqslant c\sum_{k=2}^{\infty}\int_{T_k}^{T_{k+1}}\frac{\mathrm{d}x}{x(\log x)^{1-\epsilon}}$$
$$= c\int_{T_2}^{\infty}\frac{\mathrm{d}x}{x(\log x)^{1-\epsilon}} = \infty. \tag{2.2.14}$$

Let $Y_k := (Z(T_k)-Z(T_k-a_{T_k}))/\sigma(a_{T_k})$, then $EY_k=0$, $EY_k^2=1$. For given $\epsilon>0$, define

$$A_n=\{\beta_{T_n}Y_n>(1-3\epsilon)/\sigma(a_{T_n})\}.$$

If $0<\alpha\leqslant 1/2$, then $EY_jY_k\leqslant 0$. By Slepian's inequality, it follows that $P(A_jA_k)\leqslant P(A_j)P(A_k)$ for $j\neq k$. Then by the Borel-Cantelli lemma (Lemma 2.1.1) it follows that $P(A_n, i.o.)=1$, which implies (2.2.11) immediately.

In the case of $1/2<\alpha<1$. Suppose $k\geqslant j+3$, we have

$$EY_jY_k=(P^{2\alpha}+G(Q,R))/(2Q^\alpha R^\alpha),$$

where $P = \sum_{i=j+1}^{k-1}a_{T_i}$, $Q = a_{T_j}$, $R = a_{T_k}$ and

$$G(U,V)=(P+U+V)^{2\alpha}-(P+U)^{2\alpha}-(P+V)^{2\alpha}.$$

Taylor's theorem implies

$$G(Q,R)=-P^{2\alpha}+2\alpha(2\alpha-1)P^{2\alpha-2}QR+S,$$

where for some $0<\tau<1$,

$$S=\frac{1}{3!}2\alpha(2\alpha-1)(2\alpha-2)\{(Q+R)^3(P+\tau Q+\tau R)^{2\alpha-3}$$
$$-Q^3(P+\tau Q)^{2\alpha-3}-R^3(P+\tau R)^{2\alpha-3}\}.$$

It is easy to see that

$$S\leqslant\frac{2\alpha(2\alpha-1)(2-2\alpha)}{3!}Q^3(P+\tau Q)^{2\alpha-3},$$
$$G(Q,R)\leqslant -P^{2\alpha}+3\alpha(2\alpha-1)QRP^{2\alpha-2}.$$

It follows that

$$EY_jY_k\leqslant C(a_{T_j}a_{T_k})^{1-\alpha}\Big(\sum_{i=j+1}^{k-1}a_{T_i}\Big)^{2(\alpha-1)}.$$

Recalling that (2.2.13) implies $a_{T_k}\leqslant Ca_{T_{k-1}}$, it follows that for $k\geqslant j+3$

$$EY_jY_k\leqslant C\Big(a_{T_j}/\sum_{i=j+1}^{k-1}a_{T_i}\Big)^{1-\alpha}\leqslant C(k-j-1)^{\alpha-1}.$$

It follows that for all $j\neq i$ and $k\geqslant 3$

$$EY_{ik}Y_{jk}\leqslant C(|j-i|k)^{\alpha-1}\leqslant Ck^{\alpha-1}=:\rho_k^2\to 0 \text{ as } k\to\infty.$$

Let k be large enough such that $1-2\epsilon<(1-\epsilon)\sqrt{1-\rho_k^2}$. Let τ and ξ_i, $i=1,2,\cdots$ be independent mean zero normal random variables with $E\tau^2=\rho_k^2$ and $E\xi_i^2=1-\rho_k^2$. Then

$$EY_{ik}Y_{jk}\leqslant\rho_k^2=E(\tau+\xi_i)(\tau+\xi_j).$$

By Slepian's inequality, we have for any $n>l\geqslant 1$

$$P\{\bigcap_{j=l}^{n}A_{jk}^c\}\leqslant P\{\bigcap_{j=l}^{n}\{\tau+\xi_j<(1-3\epsilon)\beta_{T_{jk}}^{-1}/\sigma(T_{jk})\}\}$$
$$\leqslant P\{\bigcap_{j=l}^{n}\{\xi_j<(1-2\epsilon)\beta_{T_{jk}}^{-1}/\sigma(T_{jk})\}\}+P\{\tau\geqslant\epsilon\beta_{T_{lk}}^{-1}/\sigma(T_{lk})\}$$
$$\leqslant\prod_{j=l}^{n}P\{N(0,1)<(1-\epsilon)\beta_{T_{jk}}^{-1}/\sigma(T_{jk})\}$$
$$+P\Big\{N(0,1)\geqslant\frac{\epsilon}{\rho_k}\beta_{T_{lk}}^{-1}/\sigma(T_{lk})\Big\}$$
$$\leqslant\exp\Big(-\sum_{j=1}^{n}b_{T_{jk}}^{1-\epsilon}\Big)+\exp\Big(-\frac{\epsilon^2}{\rho_k^2}\Big(\log\frac{T_{lk}}{a_{T_{lk}}}+\log\log T_{lk}\Big)\Big).$$

Noting that (2.2.14) implies

$$k\sum_{j=2}^{\infty}b_{T_{jk}}^{1-\epsilon}\geqslant\sum_{j=2}^{\infty}\sum_{i=1}^{k}b_{T_{jk+i}}^{1-\epsilon}=\sum_{i=2k+1}^{\infty}b_{T_i}^{1-\epsilon}=\infty,$$

it follows that for $lk\geqslant N$

$$P\{\bigcap_{i=N}^{\infty}A_i^c\}\leqslant P\{\bigcap_{j=l}^{\infty}A_{jk}^c\}\leqslant\exp\Big(-\frac{\epsilon^2}{\rho_k^2}\log\log T_N\Big).$$

Letting $k\to\infty$ implies $P(\bigcap_{i=N}^{\infty}A_i^c)=0$. It follows that $P\{\bigcup_{N=1}^{\infty}$

$\bigcup_{i=N}^{\infty} A_i^c\}=0$, which implies (2.2.11) also.

In the case of $\rho=1$, we have $a_T=T$ and $Z(T)-Z(T-a_T)=Z(T)$. We define $T_k=e^{k^p}$, $1<p<1+\varepsilon$. Let $Y_k=Z(T_k)/\sigma(T_k)$. Then

$$\sum_{k=2}^{\infty} P\{Y_k>(1-3\varepsilon)\beta_{T_k}^{-1}\sigma(T_k)\}\geqslant\sum_{k=2}^{\infty}\left(\frac{1}{\log T_k}\right)^{1-3\varepsilon}$$
$$\geqslant C\sum_{k=2}^{\infty} k^{-p(1-3\varepsilon)}=\infty.$$

Also, for $k>j$ we have

$$EY_jY_k=\frac{1}{2}\left\{\left(\frac{T_j}{T_k}\right)^{\alpha}+\left(\frac{T_k}{T_j}\right)^{\alpha}\left(1-\left(1-\frac{T_j}{T_k}\right)^{2\alpha}\right)\right\}$$
$$\leqslant C\left(\left(\frac{T_j}{T_k}\right)^{\alpha}+\left(\frac{T_j}{T_k}\right)^{1-\alpha}\right).$$

It follows that

$$\sup_{j\neq i} EY_{jk}Y_{ik}\to 0 \text{ as } k\to\infty$$

and then the proof is the same as the case of $\rho<1$ and $1/2<\alpha<1$.

Step 3 Let

$$C(T)=\sup_{0\leqslant t\leqslant T-a_T}\beta_T(Z(t+a_T)-Z(t)),$$

suppose that the conditions (i)—(iii) are fulfilled. Then

$$\liminf_{T\to\infty} C(T)\geqslant 1 \quad \text{a.s.} \qquad (2.2.15)$$

Proof For any given $\delta>0$, let $T_n=(1+\delta)^n$, $\lambda_n=\beta_{T_n}^{-1}/\sigma(a_{T_n})$, and define $\xi(T)=[Ta_T^{-1}]-1$,

$$V(k,n)=(Z((k+1)a_{T_n})-Z(ka_{T_n}))/\sigma(a_{T_n}),$$

where $0\leqslant k\leqslant\xi(T_n)$, $n\geqslant 1$. Then for all k,n, $EV(k,n)=0$, $EV^2(k,n)=1$ and if $k\geqslant j+1$,

$$r_n(k,j)=EV(k,n)V(j,n)$$

$$=\frac{1}{2}((k-j+1)^{2\alpha}+(k-j-1)^{2\alpha}-2(k-j)^{2\alpha})$$

is less than or equal to 0 if $0<\alpha\leqslant 1/2$ and less than $C(k-j)^{2\alpha-2}$ if $1/2<\alpha<1$.

If $\alpha\leqslant 1/2$, Slepian's inequality and (2.2.12) imply

$$P\left\{\max_{0\leqslant k\leqslant\xi(T_n)} V(k,n)\leqslant(1-3\varepsilon)\lambda_n\right\}$$
$$\leqslant\prod_{k=0}^{\xi(T_n)} P\{V(k,n)\leqslant(1-3\varepsilon)\lambda_n\}$$
$$\leqslant\left(1-\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{1-3\varepsilon}\right)^{T_n/a_{T_n}}$$
$$\leqslant 2\exp\left(-\left(\frac{T_n}{a_{T_n}}\right)^{3\varepsilon}\left(\frac{1}{\log T_n}\right)^{1-3\varepsilon}\right).$$

By condition (iii), we have

$$\sum_{n=1}^{\infty}\exp\left(-\left(\frac{T_n}{a_{T_n}}\right)^{3\varepsilon}\left(\frac{1}{\log T_n}\right)^{1-3\varepsilon}\right)<\infty$$

and whence, we have proved

$$\liminf_{n\to\infty} C(T_n)\geqslant\liminf_{n\to\infty}\beta_{T_n}\max_{0\leqslant k\leqslant\xi(T_n)} V(k,n)\sigma(a_{T_n})\geqslant 1-3\varepsilon \quad \text{a.s.}$$

In the case of $\alpha>1/2$, let $\delta>0$ be small enough such that $\varepsilon/\delta>2$ and $1-2\varepsilon<(1-\varepsilon)\sqrt{1-\delta^2}$, and then let m be large enough such that $r_n(km,jm)\leqslant Cm^{2\alpha-2}<\delta^2$. Define $\xi_1(T)=\left[\frac{T}{ma_T}\right]-1$ and let $\tau,\eta_k,k=0,1,\cdots,\xi_1(T_n)$ be independent mean zero normal variables with $E\tau^2=\delta^2$ and $E\eta_k^2=1-\delta^2$. Then $E(\tau+\eta_k)^2=1$ and

$$EV(km,n)V(jm,n)\leqslant E(\tau+\eta_k)(\tau+\eta_j), \quad k\neq j.$$

By Slepian's inequality, we have

$$P\{\max_{0\leqslant k\leqslant\xi_1(T_n)} V(km,n)\leqslant(1-3\varepsilon)\lambda_n\}$$

$$\leqslant P\{\max_{0\leqslant k\leqslant \xi_1(T_n)}(\tau+\eta_k)\leqslant(1-3\varepsilon)\lambda_n\}$$
$$\leqslant P\{\max_{0\leqslant k\leqslant \xi_1(T_n)}\eta_k\leqslant(1-2\varepsilon)\lambda_n\}+P\{\tau\geqslant\varepsilon\lambda_n\}$$
$$\leqslant \prod_{k=0}^{\xi_1(T_n)} P\{N(0,1)\leqslant(1-\varepsilon)\lambda_n\}+P\{N(0,1)\geqslant 2\lambda_n\}$$
$$\leqslant\left(1-\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{1-\varepsilon}\right)^{\left[\frac{T}{ma_T}\right]}+\left(\frac{a_{T_n}}{T_n\log T_n}\right)^2$$
$$\leqslant 2\exp\left(-\frac{1}{m}\left(\frac{T_n}{a_{T_n}}\right)^{\varepsilon}\left(\frac{1}{\log T_n}\right)^{1-\varepsilon}\right)+c\frac{1}{n^2}.$$
It follows that
$$\sum_{n=1}^{\infty}P\{\max_{0\leqslant k\leqslant \xi_1(T_n)}V(km,n)\leqslant(1-3\varepsilon)\lambda_n\}<\infty$$
and whence, we have proved
$$\liminf_{n\to\infty}C(T_n)\geqslant\liminf_{n\to\infty}\beta_{T_n}\max_{0\leqslant k\leqslant \xi_1(T_n)}V(km,n)\sigma(a_{T_n})\geqslant 1-3\varepsilon \text{ a.s.}$$

Finally, when $T_n\leqslant T<T_{n+1}$, we first observe that $0\leqslant a_T-a_{T_n}$ and that, by condition (ii), $a_T/a_{T_n}\leqslant T/T_n\leqslant 1+\delta$, which implies $0\leqslant a_T-a_{T_n}\leqslant\delta a_T$. Whence, we have
$$C(T)\geqslant\frac{\beta_{T_{n+1}}}{\beta_{T_n}}\cdot C(T_n)-\beta_T\sup_{0\leqslant t\leqslant T-\delta a_T}\ \sup_{0\leqslant s\leqslant\delta a_T}|Z(t+s)-Z(t)|.$$
By Step 1 and the fact that $\beta_{T_{n+1}}/\beta_{T_n}$ can be made to near 1 arbitrarily if δ is small enough, we have proved (2.2.15).

There is another form of increments, i.e. the lag increments of a process, which was first studied by Hanson and Russo (1983a,b) for a Wiener process $W(\cdot)$ and was generalized by Chen, Kong and Lin (1986). The following is the results about the lag increments of a fractional Wiener process $Z(\cdot)$.

Theorem 2.2.3 *We have*

$$\limsup_{T\to\infty}\sup_{0<t\leqslant T}|Z(T)-Z(T-t)|/d(T,t)$$
$$=\limsup_{T\to\infty}\sup_{0<t\leqslant T}\sup_{t\leqslant s\leqslant T}|Z(s)-Z(s-t)|/d(T,t)$$
$$=\limsup_{T\to\infty}\sup_{0<t\leqslant T}\sup_{0\leqslant s\leqslant t}|Z(s)-Z(T-s)|/d(T,t)=1 \text{ a.s.},\tag{2.2.16}$$
where $d(T,t)=\{2\sigma^2(t)(\log(T/t)+\log\log t)\}^{1/2}$.

Theorem 2.2.3 was due to Lu (1986) and Hong (1990). The proof was similar to that for Wiener process, and we will not present it here. One can refer to Lin and Lu (1992, page 13).

Remark 2.2.1 For a fractional Wiener process $\{Z(t);t\geqslant 0\}$, there also is a general form of the increments similar to that for a Wiener process. Wang (1997) shows that if $\{Z(t);t\geqslant 0\}$ is a fractional Wiener process of order $\alpha(0<\alpha<1)$, and a_T, b_T, c_T be non-negative function of T satisfying that

(i) $a_T+b_T\geqslant T$ and $c_T\geqslant T$ for T large enough,

(ii) there exists a constant $A>0$ such that for any $T\geqslant 1$,
$$b_T-b_{T-1}\leqslant Aa_T,\quad a_T+b_T\leqslant A(a_{T-1}+b_{T-1}),$$
then
$$\limsup_{T\to\infty}\sup_{0\leqslant t}\sup_{0<s}\sup_{0\leqslant r\leqslant s}\frac{|Z(t+r)-Z(t)|}{d(t+s\vee c_T,s)}=1\quad\text{a.s.},$$
$$\limsup_{T\to\infty}\beta(a_T+b_T,a_T)|Z(a_T+b_T)-Z(b_T)|=1\quad\text{a.s.},$$
where $\beta(M,m)=\{2\sigma^2(m)(\log(M/m)+\log\log M)\}^{-1/2}$; furthermore, if
$$\sum_{N=1}^{\infty}\exp\{-b_N/(a_N^{\varepsilon}((a_N+b_N)\log(a_N+b_N))^{1-\varepsilon})\}<\infty$$
for any $0\leqslant\varepsilon<1$, and
$$\lim_{T\to\infty}b_T/b_{[T]}=\lim_{T\to\infty}a_T/a_{[T]}=1,$$

then

$$\limsup_{T\to\infty}\sup_{0\le t}\sup_{0<s}\sup_{0\le r\le s}\frac{|Z(t+r)-Z(t)|}{d(t+s\vee c_T,s)}=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\beta(t+b_T,a_T)|Z(t+a_T)-Z(t)|=1\quad \text{a.s.}$$

2.2.3 Liminfs on the large increments of $Z(\cdot)$

In Theorem 2.2.2, we saw that the limsups of the increments of $Z(\cdot)$ over subintervals of length a_T of the interval $[0, T]$ equals to the respective liminfs whenever (iii) is fulfilled. When condition (iii) is not satisfied, the limsups may not equal to the respective liminfs. What about the liminfs? The following is a general result which was due to Zhang (1996a).

Theorem 2.2.4 *Let $a_T(0<a_T\le T)$ be a function of T satisfying (i), (ii) of Theorem 2.2.2 and*

(iv) $\lim_{T\to\infty}(T/a_T)/\log\log T=\infty$.

Then

$$\liminf_{T\to\infty}\gamma(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$

$$=\liminf_{T\to\infty}\gamma(T)\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=1\quad \text{a.s.},\quad (2.2.17)$$

where $\gamma(T)=\{2\sigma^2(a_T)(\log T/a_T-\log\log\log T)\}^{-1/2}$.

If condition (iv) *is replaced by*

(iv)′ $\lim_{T\to\infty}(T/a_T)/\log\log T=0$,

then there exist two positive constants $c_1=c_1(\alpha), c_2=c_2(\alpha)$ depending only on α, such that

$$c_1\le\liminf_{T\to\infty}\gamma'(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le c_2\quad \text{a.s.},$$

(2.2.18)

where

$$\gamma'(T)=\left\{a_T\log\left(1+\frac{T}{a_T\log\log T}\right)\right\}^{-\alpha}.$$

From Theorem 2.2.4, there are some simple consequences:

Corollary 2.2.1 *Let $a_T(0<a_T\le T)$ be a function of T satisfying (i), (ii) of Theorem 2.2.2 and*

(v) $\lim_{T\to\infty}(\log T/a_T)/\log\log\log T=\infty$.

Then

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$

$$=\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=1\quad \text{a.s.},\quad (2.2.19)$$

where $\gamma_1(T)=\{2\sigma^2(a_T)\log T/a_T\}^{-1/2}$.

Corollary 2.2.2 *Let $a_T(0<a_T\le T)$ be a function of T satisfying (i), (ii) of Theorem 2.2.2 and*

(iii)′ $\lim_{T\to\infty}(\log T/a_T)/\log\log T=r$.

Then

$$\liminf_{T\to\infty}\beta_T\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$

$$=\liminf_{T\to\infty}\beta_T\sup_{0\le t\le T-a_T}|Z(t+a_T)-Z(t)|=\left(\frac{r}{r+1}\right)^{1/2}\quad \text{a.s.},$$

(2.2.20)

where $\frac{r}{r+1}=1$ *if* $r=\infty$.

Remark 2.2.2 Corollary 2.2.1 was first obtained by Csáki and Révész (1979) for a Wiener process. Corollary 2.2.2 was established by Hong (1990). From Theorem 2.2.4, it is easily seen that the liminf behavior of increments of $\{Z(t)\}$ vary with limit behavior of $(T/a_T)/\log\log T$. But Theorem 2.2.2 tells us that the limsup behavior is the same.

To prove Theorem 2.2.4, we shall first state some lemmas.

In the sequel of this section, c_α will denote a positive constant depending only on α whose value is uninteresting and may vary from line to line.

First of all, suppose that $\{X(t);t\in T\}$ is a mean zero Gaussian process and $x(t)>0$ is a real function on T. If there are a Gaussian process $\{U(t);t\in T_c\}$ and a function $u(t)>0$ on T_c with T_c being a countable set such that

$$\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}\supset\left\{\sup_{t\in T_c}\frac{|U(t)|}{u(t)}\leqslant 1\right\}\text{ a. s. },$$

then the Khatri-Šidák's inequality (Theorem 1. 2. 4′) implies

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}\geqslant\prod_{t\in T_c}P\left\{\frac{|U(t)|}{u(t)}\leqslant 1\right\}=:p_X.$$

If such $\{U(t);t\in T_c\}$ and $u(t)$ exist and furthermore, for each $t\in T_c$, $U(t)$ is a linear combination of $X(s_1),\cdots,X(s_m)$ for some $s_1,\cdots,s_m\in T$, the lower bound p_X will be called a KS lower bound (KSLB) of $P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}$ and write

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}\overset{\text{KS}}{\geqslant}p_X.$$

Also, if $\{Y(s);s\in S\}$ is another mean zero Gaussian process with a real function $y(s)$ on S, the inequality

$$P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}\overset{\text{KS}}{\geqslant}P\left\{\sup_{s\in S}\frac{|Y(s)|}{y(s)}\leqslant 1\right\}$$

means that a KS lower bound p_Y of $P\left\{\sup_{s\in S}\frac{|Y(s)|}{y(s)}\leqslant 1\right\}$ is also a KS lower bound of $P\left\{\sup_{t\in T}\frac{|X(t)|}{x(t)}\leqslant 1\right\}$.

The following lemma is a direct consequece of the Khatri-Šidák's inequality (Theorem 1. 2. 4′).

Lemma 2. 2. 2 *Let T_i, $i=1,2,\cdots$ be the parameter sets, $\{Y_i(t);t\in T_i,i=1,2,\cdots\}$ be combined Gaussian processes with mean zero. Assume that*

$$P\left\{\sup_{t\in T_i}\frac{|Y_i(t)|}{x_i(t)}\leqslant 1\right\}\overset{\text{KS}}{\geqslant}p_i,\quad i=1,2,\cdots.$$

Then

$$P\left\{\sup_i\sup_{t\in T_i}\frac{|Y_i(t)|}{x_i(t)}\leqslant 1\right\}\overset{\text{KS}}{\geqslant}\prod_{i=1}^{\infty}p_i,$$

where $x_i(t)>0$, $t\in T_i$, $i=1,2,\cdots$.

The following lemma is an analogue of Lemma 2. 3 of Révész (1982). But the proof of Révész seems not true, since the lower bound of their Lemma 2. 3 cannot be obtained by using Slepian's inequality as proving their Lemma 2. 2.

Lemma 2. 2. 3 *Let $\{\Gamma(t);-\infty<t<\infty\}$ be an almost sure continuous Gaussian process with mean zero. Assume that there exists a non-decreasing function $u(h)$ on $[0,\infty)$ such that*

$$E(\Gamma(t+h)-\Gamma(t))^2\leqslant u^2(h)\quad\text{for all } t\geqslant 0,\ h\geqslant 0.$$

Then

$$P\left\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|\Gamma(t+s)-\Gamma(t)|\leqslant x(u(a)+u(a,k)+u^*(a,k))\right\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left(-\frac{76}{\sqrt{2\pi}}\left(\frac{T}{a}+1\right)2^{2k}x^{-1}\exp\left(-\frac{x^2}{2}\right)\right)$$

for all $x\geqslant 0.68$, $T>0$, $a>0$ and $k\geqslant 1$, where

$$u(a,k)=u\left(\frac{2a}{2^k}\right)+2\sum_{j=0}^{\infty}u(a2^{-k-j-1}),$$

$$u^*(a,k)=2\sum_{j=0}^{\infty}\sqrt{j}\,u(a2^{-k-j-1}).$$

Proof For any positive number t and integer k, put $R=2^k$, $t_j=a\left[t\frac{2^j}{a}\right]/2^j$. Then we have

$$|\Gamma(t+s)-\Gamma(t)|$$
$$\leqslant|\Gamma((t+s)_k)-\Gamma(t_k)|+|\Gamma(t+s)-\Gamma((t+s)_k)|$$
$$+|\Gamma(t)-\Gamma(t_k)|$$
$$\leqslant|\Gamma((t+s)_k)-\Gamma(t_k)|+\sum_{j=0}^{\infty}|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})|$$
$$+\sum_{j=0}^{\infty}|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})|.$$

Obviously,

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-1/R)}|(t+s)_k-t_k|\leqslant a,$$
$$\sup_{0\leqslant t\leqslant T}\sup_{a(1-1/R)\leqslant s\leqslant a}|(t+s)_k-(t+a(1-1/R))_k|\leqslant 2a2^{-k},$$
$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|(t+s)_{k+j+1}-(t+s)_{k+j}|\leqslant 2a2^{-k-j-1},$$
$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|\Gamma((t+s)_k)-\Gamma(t_k)|$$
$$\leqslant\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-1/R)}|\Gamma((t+s)_k)-\Gamma(t_k)|$$
$$+\sup_{0\leqslant t\leqslant T}\sup_{a(1-1/R)\leqslant s\leqslant a}|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)|$$

and

$$\operatorname{Card}\{\Gamma((t+s)_k)-\Gamma(t_k);0\leqslant t\leqslant T,0\leqslant s\leqslant a(1-1/R)\}$$
$$\leqslant 2R^2\left(\frac{T}{a}+1\right),$$
$$\operatorname{Card}\{\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k);a(1-1/R)\leqslant s\leqslant a\}$$
$$\leqslant 2R\left(\frac{T}{a}+1\right),0\leqslant t\leqslant T,$$
$$\operatorname{Card}\{\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j});0\leqslant t\leqslant T,0\leqslant s\leqslant a\}$$
$$\leqslant 2^{2+j+1}\left(\frac{T}{a}+1\right),$$
$$\operatorname{Card}\{\Gamma(t_{k+j+1})-\Gamma(t_{k+j});0\leqslant t\leqslant T\}\leqslant 2^{k+j+1}\left(\frac{T}{a}+1\right).$$

By Lemma 2.2.2, we have

$$J:=P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|\Gamma(t+s)-\Gamma(t)|$$

$$\leqslant xu(a)+xu(2a/R)+\sum_{j=0}^{\infty}2x_ju(a2^{-k-j-1})\Big\}$$
$$\overset{\text{KS}}{\geqslant}P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-1/R)}|\Gamma((t+s)_k)-\Gamma(t_k)|\leqslant xu(a)\Big\}$$
$$\times P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{a(1-1/R)\leqslant s\leqslant a}|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)|$$
$$\leqslant xu(2a/R)\Big\}$$
$$\times\prod_{j=0}^{\infty}P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})|$$
$$\leqslant x_ju(a2^{-k-j-1})\Big\}$$
$$\times\prod_{j=0}^{\infty}P\Big\{\sup_{0\leqslant t\leqslant T}|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})|\leqslant x_ju(a2^{-k-j-1})\Big\}$$
$$\overset{\text{KS}}{\geqslant}(\psi(x))^{2R^2\left(\frac{T}{a}+1\right)}(\psi(x))^{2R\left(\frac{T}{a}+1\right)}\prod_{j=0}^{\infty}(\psi(x_j))^{2^{k+j+1}\left(\frac{T}{a}+1\right)}$$
$$\times\prod_{j=0}^{\infty}(\psi(x_j))^{2^{k+j+1}\left(\frac{T}{a}+1\right)}$$
$$=(\psi(x))^{2(R^2+R)\left(\frac{T}{a}+1\right)}\prod_{j=0}^{\infty}(\psi(x_j))^{4R2^j\left(\frac{T}{a}+1\right)},$$

where $\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t^2}dt\geqslant 1/2(x\geqslant 0.68)$. Since $(1-y)e^{2y}\geqslant 1$ for $0\leqslant y\leqslant 1/2$, we have for $x\geqslant 0.68$

$$\psi(x)\geqslant\exp(-2(1-\psi(x)))\geqslant\exp\left(-\frac{4}{\sqrt{2\pi}}x^{-1}e^{-x^2/2}\right).$$

Now we choose $\frac{1}{2}x_j^2=\frac{1}{2}x^2+j$, then

$$J\geqslant\exp\Bigg(-\Bigg(\frac{4}{\sqrt{2\pi}}2(R^2+R)\left(\frac{T}{a}+1\right)x^{-1}e^{-\frac{x^2}{2}}$$
$$+\frac{4}{\sqrt{2\pi}}4R\left(\frac{T}{a}+1\right)\sum_{j=0}^{\infty}2^jx_j^{-1}e^{-\frac{x^2}{2}-j}\Bigg)\Bigg)$$
$$\geqslant\exp\left(-\frac{4}{\sqrt{2\pi}}\left(\frac{T}{a}+1\right)x^{-1}e^{-x^2/2}\left(2R^2+2R+4R\sum_{j=0}^{\infty}2^je^{-j}\right)\right)$$

$$=\exp\left(-\frac{4}{\sqrt{2\pi}}\left(\frac{T}{a}+1\right)x^{-1}e^{-x^2/2}\left(2R^2+2R+R\frac{4e}{e-2}\right)\right)$$

$$\geqslant\exp\left(-\frac{76}{\sqrt{2\pi}}x^{-1}e^{-x^2/2}R^2\left(\frac{T}{a}+1\right)\right).$$

Noting that

$$xu(2a/R)+2\sum_{j=0}^{\infty}x_j u(a2^{-k-j-1})$$

$$\leqslant x\left(u(2a/R)+2\sum_{j=0}^{\infty}u(a2^{-k-j-1})\right)+2\sum_{j=0}^{\infty}\sqrt{j}\,u(a2^{-k-j-1})$$

$$=xu(a,k)+u^*(a,k),$$

we have proved the lemma.

Lemma 2.2.4 *Let $\{\Gamma(t);-\infty<t<\infty\}$, $u(h)$ be as in Lemma 2.2.3 and assume that $u(x)/x^{\alpha}$ is quasi-increasing for some $\alpha>0$. Then there exist constants $c_{\alpha}>0$, $c_{\alpha}'>1$ depending only on α such that*

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|\Gamma(t+s)-\Gamma(t)|\leqslant xu(h)\}$$

$$\overset{\mathrm{KS}}{\geqslant}\exp\left(-c_{\alpha}\left(\frac{T}{h}+1\right)x^{4/\alpha-1}e^{-x^2/2}\right)$$

holds for all $x\geqslant c_{\alpha}'$.

Proof Since $u(x)/x^{\alpha}$ is quasi-increasing, there exists $c_0>0$ such that

$$u(ht)\leqslant c_0t^{\alpha}u(h)$$

for $0\leqslant t\leqslant1, h\geqslant0$. And then

$$u(h,k)\leqslant c_0u(h)\left(\left(\frac{2}{2^k}\right)^{\alpha}+2\sum_{j=0}^{\infty}2^{-\alpha(k+j+1)}\right)$$

$$=c_0u(h)2^{-\alpha k}\left(2^{\alpha}+\frac{2}{2^{\alpha}-1}\right),$$

$$u^*(h,k)\leqslant2\left(\sum_{j=0}^{\infty}\sqrt{j}\,2^{-\alpha(k+j+1)}\right)c_0u(h)$$

$$\leqslant2c_0u(h)2^{-\alpha k}(2\alpha\log2)^{-3/2}.$$

Let $K_{\alpha}=c_0\left(2^{\alpha}+\frac{2}{2^{\alpha}-1}\right)+2c_0(2\alpha\log2)^{-3/2}$, then

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|\Gamma(t+s)-\Gamma(t)|\leqslant xu(h)(1+2^{-\alpha k}K_{\alpha})\}$$

$$\overset{\mathrm{KS}}{\geqslant}\exp\left(-\frac{76}{\sqrt{2\pi}}\left(\frac{T}{h}+1\right)2^{2k}x^{-1}e^{-x^2/2}\right)$$

for $x\geqslant1$. Denote $y=x(1+2^{-\alpha k}K_{\alpha})$, then

$$J:=P\left\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|\Gamma(t+s)-\Gamma(t)|\leqslant yu(h)\right\}$$

$$\overset{\mathrm{KS}}{\geqslant}\exp\left(-\frac{76}{\sqrt{2\pi}}\left(\frac{T}{h}+1\right)y^{-1}e^{-y^2/2}2^{2k}(1+2^{-\alpha k}K_{\alpha})\right.$$

$$\left.\times\exp\left(y^2\frac{2\cdot2^{-\alpha k}K_{\alpha}+2^{-2\alpha K}K_{\alpha}^2}{(1+2^{-\alpha k}K_{\alpha})^2}\right)\right)$$

$$\geqslant\exp\left(-\frac{76}{\sqrt{2\pi}}\left(\frac{T}{h}+1\right)y^{-1}e^{-y^2/2}2^{2k}(1+K_{\alpha})\right.$$

$$\left.\times\exp(3K_{\alpha}^22^{-\alpha k}y^2)\right).$$

Now for $y\geqslant1+K_{\alpha}$, we can choose k such that

$$2^{k-1}\leqslant y^{2/\alpha}\leqslant2^k,$$

then

$$2^{2k}\exp(3K_{\alpha}^22^{-\alpha k}y^2)\leqslant4y^{4/\alpha}\exp(3K_{\alpha}^2).$$

Hence,

$$J\geqslant\exp\left(-\frac{4\times76}{\sqrt{2\pi}}\left(\frac{T}{h}+1\right)y^{4/\alpha-1}e^{-y^2/2}(1+K_{\alpha})\exp(3K_{\alpha}^2)\right).$$

The proof of Lemma 2.2.4 is over.

Lemma 2.2.5 *Let $\{Z(t);t\geqslant0\}$ be a fractional Wiener process of order α with $0<\alpha\leqslant1/2$. Then for any $\varepsilon>0$, there exist $u_0=u_0(\varepsilon)>0$, $T_0=T_0(\varepsilon)>0$ such that*

$$P\{\sup_{0\leqslant t\leqslant T}(Z(t+1)-Z(t))\leqslant u\}$$

$$\leqslant \exp\left(-(1-\varepsilon)H_{2\alpha}\frac{T}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\right)$$

for all $u \geqslant u_0$ *and* $T \geqslant T_0$. *Where*

$$H_{2\alpha}=\lim_{T\to\infty}T^{-1}\int_0^\infty e^s P\{\sup_{0<t<T}Y(t)>s\}ds,$$

and $\{Y(t);0\leqslant t\leqslant\infty\}$ *is a non-stationary Gaussian process with* $Y(0)=0$ a. s., $EY(t)=-|t|^{2\alpha}$, Cov $(Y(t_1),Y(t_2))=|t_1|^{2\alpha}+|t_2|^{2\alpha}-|t_1-t_2|^{2\alpha}$.

Proof For any integer $k<T$ we have

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}(Z(t+1)-Z(t))\\ &\geqslant \max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(Z(t+1)-Z(t))\\ &\geqslant \max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<i(k+1)+k}(Z(t+1)-Z(t)),\end{aligned}$$

where $l=\max\{i,(i+1)(k+1)-1\leqslant T\}$. Noting that $\alpha\leqslant 1/2$, by Slepian's inequality we have

$$\begin{aligned}&P\{\sup_{0\leqslant t\leqslant T}(Z(t+1)-Z(t))\leqslant u\}\\ &\leqslant P\{\max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<i(k+1)+k}(Z(t+1)-Z(t))\leqslant u\}\\ &\leqslant (P\{\sup_{0\leqslant t\leqslant k}(Z(t+1)-Z(t))\leqslant u\})^{l+1}.\end{aligned}$$

By Theorem 1. 1. 2, we have

$$\lim_{x\to\infty}\frac{P\{\sup_{0\leqslant t\leqslant k}(Z(t+1)-Z(t))>x\}}{\frac{k}{\sqrt{2\pi}}x^{1/\alpha-1}e^{-x^2/2}}=H_{2\alpha}.$$

Hence

$$\begin{aligned}&P\{\sup_{0\leqslant t\leqslant T}(Z(t+1)-Z(t))\leqslant u\}\\ &\leqslant\left(1-\frac{(1+o(1))H_{2\alpha}}{\sqrt{2\pi}}ku^{1/\alpha-1}e^{-u^2/2}\right)^{l+1}\\ &\leqslant\exp\left(-(1+o(1))\frac{k(l+1)H_{2\alpha}}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\right)\end{aligned}$$

$$\leqslant \exp\left(-(1-\varepsilon)H_{2\alpha}\frac{T}{\sqrt{2\pi}}u^{1/\alpha-1}e^{-u^2/2}\right)$$

for $T\geqslant T_0(\varepsilon)$, $u\geqslant u_0(\varepsilon)$, the proof of Lemma 2. 2. 5 is completed.

Lemma 2. 2. 6 *Let* $\{Z(t);t\geqslant 0\}$ *be a fractional Wiener process of order* α *with* $1/2<\alpha<1$. *Then for any* $\delta>0$, *there exists* $c_\alpha=c(\alpha,\delta)>0$ *such that*

$$\begin{aligned}&P\{\sup_{0\leqslant t\leqslant T}|Z(t+1)-Z(t)|\leqslant u\}\\ &\leqslant\exp\left(-c_\alpha\frac{T}{k}\frac{1}{u}\exp\left(-\frac{(1+\delta)u^2}{2}\right)\right)\end{aligned}$$

for T, $k(k\leqslant T)$ *large enough and all* $u>0$.

Proof Let $Y(t)=Z(t+1)-Z(t)$, then

$$\begin{aligned}&EY(t+h)Y(t)=E(Z(t+h+1)-Z(t+h))(Z(t+1)-Z(t))\\ &=\frac{1}{2}(|h+1|^{2\alpha}+|h-1|^{2\alpha}-2h^{2\alpha})=:\rho(h).\end{aligned}$$

It is easy to show that $\rho(h)$ is strictly decreasing on $[0,\infty)$ and convex on $[1,\infty)$ with $\rho(0)=1,\rho(h)\approx h^{2\alpha-2}$ as $h\to\infty$.

For $k>1$ (large enough), let $t_i=ik$, $i=0,1,2,\cdots,Y(t_{-1})=0$ and

$$a_{ij}=E(Y(t_i)-Y(t_{i-1}))(Y(t_j)-Y(t_{j-1})),\ i,j\geqslant 0.$$

Then

$$\begin{aligned}&a_{ij}=2\rho(|i-j|k)-\rho(|i-j+1|k)-\rho(|i-j-1|k),\ i,j\geqslant 1;\\ &a_{ii}=2(1-\rho(k)),\ i\geqslant 1,\ a_{00}=1;\\ &a_{i0}=a_{0i}=\rho(ik)-\rho((i-1)k),\ i\geqslant 1.\end{aligned}$$

So, for $i,j\geqslant 1$ and $|i-j|\geqslant 2$, by the convexity of $\rho(h)$ on $[1,\infty)$ we have $a_{ij}<0$; for $i,j\geqslant 1$ and $|i-j|=1$, we have $a_{ij}=2\rho(k)-\rho(2k)-1<0$ for k large enough since $\rho(h)\to 0$ as $h\to\infty$; for $i\geqslant 1$, by the monotoneity of $\rho(h)$ we have $a_{i0}<0$.

Now for any $n\geqslant 2$ and $1\leqslant i\leqslant n$, let

$$S_i^n = \sum_{\substack{j=0\\j\neq i}}^{n} |a_{ij}| = a_{ii} - \sum_{j=0}^{n} a_{ij}.$$

Then, for $i \geqslant 1$,

$$S_i^n = 2(1-\rho(k)) - E((Y(t_i)-Y(t_{i-1}))Y(t_n))$$
$$= 2(1-\rho(k)) - \rho((n-i)k) + \rho((n-i+1)k)$$
$$< 2(1-\rho(k)) = a_{ii};$$

$$S_i^i = 1-\rho(k) = \frac{1}{2}a_{ii};$$

for $i=0$,

$$S_0^n = \sum_{\substack{j=0\\j\neq 0}}^{n} |a_{0j}| = \sum_{j=1}^{n} (\rho((j-1)k) - \rho(jk))$$
$$= 1 - \rho(nk) < 1 = a_{00};$$

$$S_0^0 = 0 < \frac{1}{2}a_{00}.$$

Noting that $\boldsymbol{A} = (a_{ij})$ is the covariance matrix of $(Y(t_0)-Y(t_{-1}), \cdots, Y(t_n)-Y(t_{n-1}))$, similar to the proof of Corollary 1.2.6, we have

$$P\{\sup_{0\leqslant i\leqslant n} |Y(t_i)| \leqslant u\} = P\{\sup_{0\leqslant i\leqslant n} |Y(t_i)-Y(t_{-1})| \leqslant u\}$$

$$\leqslant \prod_{i=0}^{n} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}u/a_{ii}^{1/2}} e^{-x^2/2} dx \leqslant \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}u/a_{ii}^{1/2}} e^{-x^2/2} dx$$

$$= \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{u/\sqrt{1-\rho(k)}} e^{-x^2/2} dx$$

$$\leqslant \prod_{i=1}^{n} \exp\left(-\sqrt{\frac{2}{\pi}} \int_{u/\sqrt{1-\rho(k)}}^{\infty} e^{-x^2/2} dx\right)$$

$$\leqslant \exp\left(-\frac{\sqrt{1-\rho(k)}}{\sqrt{2\pi}} \cdot n \cdot \frac{1}{u} \exp\left(-\frac{u^2}{2(1-\rho(k))}\right)\right).$$

It follows that

$$P\{\sup_{0\leqslant t\leqslant T} |Z(t+1)-Z(t)| \leqslant u\} \leqslant P\{\sup_{0\leqslant i\leqslant T/k} |Y(t_i)| \leqslant u\}$$

$$\leqslant \exp\left(-\frac{\sqrt{1-\rho(k)}}{\sqrt{2\pi}} \cdot \frac{T}{k} \cdot \frac{1}{u} \exp\left(-\frac{u^2}{2(1-\rho(k))}\right)\right).$$

Hence Lemma 2.2.6 is proved.

Lemma 2.2.7 *There exists $c_\alpha > 0$ such that*

$$P\{\sup_{0\leqslant t\leqslant T} \sup_{0\leqslant s\leqslant 1} |Z(t+s)-Z(t)| \leqslant x^\alpha\} \leqslant \exp\left(-c_\alpha \frac{T}{x}\right)$$

for $0<x<1$, $T>0$. When $0<\alpha\leqslant 1/2$, we can choose

$$c_\alpha = -\log\varphi(1) < 0.18;$$

when $1/2<\alpha<1$, we can choose

$$c_\alpha = -\frac{1}{2}\log\varphi(1/\sqrt{1-4^{\alpha-1}}).$$

Proof Let $\xi_i = Z((i+1)x) - Z(ix)$, $\eta_i = \xi_{2i} - \xi_{2i-1}$. Then

$$P\left\{\sup_{0\leqslant t\leqslant T} \sup_{0\leqslant s\leqslant 1} |Z(t+s)-Z(t)| \leqslant x^\alpha\right\}$$
$$\leqslant P\left\{\sup_{0\leqslant i<T/x} \sup_{0\leqslant s\leqslant 1} |Z(ix+s)-Z(ix)| \leqslant x^\alpha\right\}$$
$$\leqslant P\left\{\sup_{0\leqslant i<T/x} |Z((i+1)x)-Z(ix)| \leqslant x^\alpha\right\}$$
$$\leqslant P\left\{\sup_{0\leqslant i<T/x} \xi_i \leqslant x^\alpha\right\}.$$

For $0<\alpha\leqslant 1/2$, we have $E\xi_i\xi_j \leqslant 0$, $i\neq j$. Slepian's inequality (Corollary 1.2.1) implies

$$P\left\{\sup_{0\leqslant t\leqslant T} \sup_{0\leqslant s\leqslant 1} |X(t+s)-X(t)| \leqslant x^\alpha\right\} \leqslant (\varphi(1))^{T/x}$$
$$= \exp\left(\frac{T}{x}\log\varphi(1)\right).$$

For $1/2<\alpha<1$, we have $E\eta_i^2 = (4-4^\alpha)x^{2\alpha}$ and

$$E\eta_i\eta_j = \frac{1}{2}\{4(2|j-i|-1)^{2\alpha} + 4(2|j-i|+1)^{2\alpha} - (2|j-i|-2)^{2\alpha}$$
$$-(2|j-i|+2)^{2\alpha} - 6(2|j-i|)^{2\alpha}\}x^{2\alpha} \leqslant 0$$

for $j\neq i$. Hence Slepian's inequality implies

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}|Z(t+s)-Z(t)|\leqslant x^{\alpha}\}$$
$$\leqslant P\{\sup_{0\leqslant i<\frac{1}{2}\frac{T}{x}}|\eta_i|\leqslant 2x^{\alpha}\}\leqslant P\{\sup_{0\leqslant i<\frac{1}{2}\frac{T}{x}}\eta_i\leqslant 2x^{\alpha}\}$$
$$\leqslant\left(\varphi\left(\frac{2}{\sqrt{4-4^{\alpha}}}\right)\right)^{T/(2x)}=\exp\left(\frac{1}{2}\log\varphi\left(\frac{1}{\sqrt{1-4^{\alpha-1}}}\right)\frac{T}{x}\right).$$

Lemma 2.2.7 is proved.

Let $T_n=e^{n^p}(p>1), d_n=ne^{n^p}, A_n=\left(\frac{d_{n-1}}{T_n},\frac{d_n}{T_n}\right)$,

$$Y_n(t)=\int_{|x|\notin A_n}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\mathrm{d}W(x). \tag{2.2.21}$$

The next lemma is about the estimate of the variance of $Y_n(\cdot)$.

Lemma 2.2.8 *Let $0<\alpha<1$, $\gamma>0$ and $p>1$. Then there exists a constant $c\in(0,\infty)$ depending only on α,γ,p such that uniformly in $n>2$, $0\leqslant t\leqslant n/2$ and $0<h\leqslant 1$,*

$$\sigma_n^2(t,h)=E(Y_n(t+h)-Y_n(t))^2\leqslant ch^{2\alpha}(\log h^{-1})^{\gamma}n^{-\delta},$$

where $\delta=\min\{(2-2\alpha), (p-1)/2, \gamma(p-1)/2\}$.

Proof If $0<\alpha<1$, $\alpha\neq 1/2$, let

$$f^2(y)=\frac{1}{K_\alpha^2}(|y-1/2|^{(2\alpha-1)/2}-|y+1/2|^{(2\alpha-1)/2})^2,\ -\infty<y<\infty.$$

Then $\int_{\mathbf{R}}f^2(y)\mathrm{d}y=1$ and $f^2(y)\leqslant 1/K_\alpha^2$ on $[0,\infty)$. Changing variables implies

$$\sigma_n^2(t,h)=h^{2\alpha}\int_{|y+t/h+1/2|\notin A_n/h}f^2(y)\mathrm{d}y.$$

$|y+t/h+1/2|\notin A_n/h$ implies

$$|y+t/h+1/2|\geqslant n/h \text{ or } |y+t/h+1/2|\leqslant\frac{d_{n-1}}{T_n}/h.$$

For $|y+t/h+1/2|\geqslant n/h$ we have $\left(0\leqslant t\leqslant\frac{1}{2}n\right)$

$$y+1/2\geqslant n/(2h) \text{ or } y+1/2\leqslant -n/h.$$

Thus for $n>2$,

$$\int_{|y+t/h+1/2|\geqslant n/h}f^2(y)\mathrm{d}y\leqslant\int_{n/(2h)}^{\infty}f^2(y)\mathrm{d}y+\int_{n/(4h)}^{\infty}f^2(y)\mathrm{d}y$$
$$\leqslant 2\int_{n/(4h)}^{\infty}f^2(y)\mathrm{d}y\leqslant c\int_{n/(4h)}^{\infty}y^{2\alpha-3}\mathrm{d}y\leqslant ch^{2-2\alpha}n^{-(2-2\alpha)}.$$

Now suppose

$$|y+t/h+1/2|\leqslant\frac{d_{n-1}}{T_n}/h.$$

Since $f^2(y)\leqslant 1/K_\alpha^2$, we have

$$\int_{-\frac{1}{2}-\left(\frac{d_{n-1}}{T_n}+t\right)/h}^{-\frac{1}{2}+\left(\frac{d_{n-1}}{T_n}-t\right)/h}f^2(y)\mathrm{d}y\leqslant 2\frac{d_{n-1}}{T_n}/(hK_\alpha^2).$$

Thus for $n>2$, we have

$$\sigma_n^2(t,h)\leqslant h^{2\alpha}\left(ch^{2-2\alpha}n^{-(2-2\alpha)}+2\frac{d_{n-1}}{T_nhK_\alpha^2}\right).$$

If $1\geqslant h\geqslant e^{-n^{(p-1)/2}}$, then

$$\sigma_n^2(t,h)\leqslant ch^{2\alpha}(\log h^{-1})^{\gamma}\Bigg((\log h^{-1})^{-\gamma}h^{2-2\alpha}n^{-(2-2\alpha)}$$
$$+e^{n^{(p-1)/2}}\frac{(n-1)e^{(n-1)^p}}{e^{n^p}}\Bigg)$$
$$\leqslant ch^{2\alpha}(\log h^{-1})^{\gamma}(n^{-(2-2\alpha)}+n^{-(2-2\alpha)})$$
$$\leqslant ch^{2\alpha}(\log h^{-1})^{\gamma}n^{-(2-2\alpha)}.$$

If $0<h\leqslant e^{-n^{(p-1)/2}}$, then

$$\sigma_n^2(t,h)\leqslant h^{2\alpha}\leqslant ch^{2\alpha}(\log h^{-1})^{\gamma}(\log e^{n^{(p-1)/2}})^{-\gamma}$$
$$=ch^{2\alpha}(\log h^{-1})^{\gamma}n^{-\gamma(p-1)/2}.$$

If $\alpha=1/2$, recall the kernel is interpreted to be $I_{(0,t]}(x)$. Then, for $h\geqslant 0$ and $n>2$

$$\sigma_n^2(t,h)=\begin{cases}0 & \text{if } t\geqslant \dfrac{d_{n-1}}{T_n},\\ \dfrac{d_{n-1}}{T_n}-t & \text{if } 0\leqslant t\leqslant \dfrac{d_{n-1}}{T_n}<t+h,\\ h & \text{if } 0\leqslant t\leqslant t+h\leqslant \dfrac{d_{n-1}}{T_n}\end{cases}$$

$$\leqslant h\wedge \frac{d_{n-1}}{T_n}.$$

If $h<d_{n-1}/T_n$, then

$$\sigma_n^2(t,h)\leqslant h\leqslant h(\log h^{-1})^\gamma\left(\log\frac{T_n}{d_{n-1}}\right)^{-\gamma}\leqslant ch(\log h^{-1})^\gamma n^{-\gamma(p-1)/2}.$$

If $h\geqslant d_{n-1}/T_n$, then for $0<\gamma\leqslant 1$, we have

$$h(\log h^{-1})^\gamma\geqslant\frac{d_{n-1}}{T_n}\left(\log\frac{T_n}{d_{n-1}}\right)^\gamma,$$

and so

$$\sigma_n^2(t,h)\leqslant\frac{d_{n-1}}{T_n}\leqslant h(\log h^{-1})^\gamma\left(\log\frac{T_n}{d_{n-1}}\right)^{-\gamma}\leqslant ch(\log h^{-1})^\gamma n^{-\gamma(p-1)/2};$$

for $\gamma>1$, we have

$$h(\log h^{-1})^\gamma\geqslant h\log h^{-1}\geqslant\frac{d_{n-1}}{T_n}\log\frac{T_n}{d_{n-1}},$$

and so

$$\sigma_n^2(t,h)\leqslant\frac{d_{n-1}}{T_n}\leqslant h(\log h^{-1})^\gamma\left(\log\frac{T_n}{d_{n-1}}\right)^{-1}$$
$$\leqslant ch(\log h^{-1})^\gamma n^{-(p-1)/2}.$$

Hence, for any case, there exists a constant $C\in(0,\infty)$ depending only on α,γ,p such that uniformly in $n>2$, $0\leqslant t\leqslant n/2$ and $0<h\leqslant 1$,

$$\sigma_n^2(t,h)\leqslant Ch^{2\alpha}(\log h^{-1})^\gamma n^{-\delta}$$

for δ as in hypothesis.

Proof of Theorem 2.2.4

The proof is formulated in four steps, which together will imply our statements.

Step 1 Suppose that condition (iv) is satisfied. Then

$$\liminf_{T\to\infty}\gamma(T)\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|Z(t+s)-Z(t)|\leqslant 1\quad\text{a.s.}\quad(2.2.22)$$

Proof If

$$\limsup_{T\to\infty}(\log(T/a_T))/(\log\log\log T)=\infty,\quad(2.2.23)$$

then there exists $\{T_N\}$ such that

$$\lim_{N\to\infty}(\log(T_N/a_{T_N}))/(\log\log\log T_N)=\infty.\quad(2.2.24)$$

By Lemma 2.2.4,

$$P\left\{\left(2\sigma^2(a_{T_N})\log\frac{T_N}{a_{T_N}}\right)^{-1/2}\sup_{0\leqslant t\leqslant T_N-a_{T_N}}\sup_{0\leqslant s\leqslant a_{T_N}}|Z(t+s)-Z(t)|\right.$$
$$\left.\geqslant(1+\varepsilon)^{1/2}\right\}$$
$$\leqslant 1-\exp\left(-c_\alpha\frac{T_N}{a_{T_N}}\left(2\log\frac{T_N}{a_{T_N}}\right)^{\frac{1}{2}\left(\frac{4}{\alpha}-1\right)}\exp\left(-(1+\varepsilon)\log\frac{T_N}{a_{T_N}}\right)\right)$$
$$\leqslant c_\alpha\left(\log\frac{T_N}{a_{T_N}}\right)^{\frac{2}{\alpha}-\frac{1}{2}}\left(\frac{a_{T_N}}{T_N}\right)^\varepsilon\to 0\ (N\to\infty).$$

Hence,

$$\liminf_{N\to\infty}\left(2\sigma^2(a_{T_N})\log\frac{T_N}{a_{T_N}}\right)^{-1/2}\sup_{0\leqslant t\leqslant T_N-a_{T_N}}\sup_{0\leqslant s\leqslant a_{T_N}}|Z(t+s)-Z(t)|$$
$$\leqslant 1\quad\text{a.s.},$$

which together with (2.2.24) implies (2.2.22).

Now, assume

$$\limsup_{T\to\infty}(\log(T/a_T))/(\log\log\log T)<\infty.$$

Then there exists a constant $r_0>0$ such that

$$T/a_T\leqslant(\log\log T)^{r_0}.\quad(2.2.25)$$

Let $T_n,d_n,\{Y_n(t)\}(n=1,2,\cdots)$ be as in Lemma 2.2.8. For any

$\varepsilon>0$, by Lemma 2.2.4 and (iv) we have

$$P\Big\{\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|Z(t+s)-Z(t)|\leqslant(1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\Big\}$$

$$\geqslant\exp\left\{-c_\alpha\frac{T_n}{a_{T_n}}\frac{\left((1+\varepsilon)^{1/2}\left(2\log\dfrac{T_n}{a_{T_n}\log\log T_n}\right)^{1/2}\right)^{4/\alpha-1}}{(T_n/a_{T_n})^{1+\varepsilon}}(\log\log T_n)^{1+\varepsilon}\right\}$$

$$\geqslant n^{-2/3} \tag{2.2.26}$$

for n large enough. Let

$$X_n(t)=\int_{|x|\in(d_{n-1},d_n)}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\mathrm{d}W(x),$$

$$\widetilde{X}_n(t)=\int_{|x|\notin(d_{n-1},d_n)}\frac{1}{K_\alpha}\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\}\mathrm{d}W(x). \tag{2.2.27}$$

Then $\{X_n(t)\}, n=1,2,\cdots$ are independent, $Z(t)=X_n(t)+\widetilde{X}_n(t)$, and

$$\{\widetilde{X}_n(t+s)-\widetilde{X}_n(t);0\leqslant t\leqslant T_n-a_{T_n},0\leqslant s\leqslant a_{T_n}\}$$

$$\overset{D}{=}\left\{T_n^\alpha(Y_n(t+s)-Y_n(t));0\leqslant t\leqslant 1-\frac{a_{T_n}}{T_n},0\leqslant s\leqslant\frac{a_{T_n}}{T_n}\right\}.$$

Hence, by Lemma 2.2.8, Lemma 2.2.4 and (2.2.25) we have

$$J_n:=P\Big\{\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|\widetilde{X}_n(t+s)-\widetilde{X}_n(t)|\geqslant\varepsilon\gamma^{-1}(T_n)\Big\}$$

$$=P\left\{\sup_{0\leqslant t\leqslant 1-\frac{a_{T_n}}{T_n}}\ \sup_{0\leqslant s\leqslant\frac{a_{T_n}}{T_n}}|Y_n(t+s)-Y_n(t)|\right.$$

$$\left.\geqslant\left(\left(\frac{a_{T_n}}{T_n}\right)^\alpha\left(\log\frac{T_n}{a_{T_n}}\right)^\gamma n^{-\delta}\right)\varepsilon n^\delta\frac{\left(2\log\dfrac{T_n}{a_{T_n}\log\log T_n}\right)^{1/2}}{(\log T_n/a_{T_n})^\gamma}\right\}$$

$$\leqslant c_\alpha\frac{T_n}{a_{T_n}}\exp\left(-c_{\alpha,\varepsilon}\frac{\log\dfrac{T_n}{a_{T_n}\log\log T_n}}{(\log T_n/a_{T_n})^{2\gamma}}n^{2\delta}\right)$$

$$\leqslant c_\alpha(\log\log T_n)^{r_0}\exp\left(-c_{\alpha,\varepsilon}\frac{\log\dfrac{T_n}{a_{T_n}\log\log T_n}}{(r_0\log\log\log T_n)^{2\gamma}}n^{2\delta}\right)$$

$$\leqslant c_\alpha(\log n)^{r_0}\exp\left(-c_{\alpha,\varepsilon}\frac{n^{2\delta}}{(\log\log n)^{2\gamma}}\right) \tag{2.2.28}$$

for n large enough. So

$$\sum_{n=1}^{\infty}J_n<\infty. \tag{2.2.29}$$

Then the Borel-Cantelli lemma implies

$$\limsup_{n\to\infty}\gamma(T_n)\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|\widetilde{X}_n(t+s)-\widetilde{X}_n(t)|\leqslant\varepsilon\ \text{a. s.} \tag{2.2.30}$$

From (2.2.26) and (2.2.29), it follows that

$$P\Big\{\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|X_n(t+s)-X_n(t)|\leqslant((1+\varepsilon)^{1/2}+\varepsilon)\gamma^{-1}(T_n)\Big\}$$

$$\geqslant P\Big\{\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|Z(t+s)-Z(t)|\leqslant(1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\Big\}$$

$$-P\Big\{\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|\widetilde{X}_n(t+s)-\widetilde{X}_n(t)|\geqslant\varepsilon\gamma^{-1}(T_n)\Big\}$$

$$\geqslant n^{-2/3}-J_n. \tag{2.2.31}$$

Noting that $\{X_n(t)\}, n=1,2,\cdots$ are independent, by (2.2.29), (2.2.31) and the Borel-Cantelli lemma we have

$$\liminf_{n\to\infty}\gamma(T_n)\sup_{0\leqslant t\leqslant T_n-a_{T_n}}\ \sup_{0\leqslant s\leqslant a_{T_n}}|X_n(t+s)-X_n(t)|$$

$$\leqslant(1+\varepsilon)^{1/2}+\varepsilon\quad\text{a. s.} \tag{2.2.32}$$

Combining (2.2.30) and (2.2.32) we obtain (2.2.22).

Step 2 Suppose that conditions (i), (ii) and (iv) are satisfied. Then we have

$$\liminf_{T\to\infty}\gamma(T)\sup_{0\leqslant t\leqslant T-a_T}|Z(t+a_T)-Z(t)|\geqslant 1 \text{ a. s.} \tag{2.2.33}$$

Proof If $0<\alpha\leqslant 1/2$, by Lemma 2.2.5 (where $\varepsilon=1/2$) we have

$$\begin{aligned}
&P\{\sup_{0\leqslant t\leqslant T-a_T}|Z(t+a_T)-Z(t)|<\gamma^{-1}(T)\}\\
&\leqslant\exp\left(-\frac{1}{2}H_{2\alpha}\frac{T}{a_T\sqrt{2\pi}}\left(2\log\frac{T}{a_T\log\log T}\right)^{1/\alpha-1}\frac{a_T\log\log T}{T}\right)\\
&\leqslant(\log T)^{-4}
\end{aligned}\tag{2.2.34}$$

for T large enough. Let $T_k=k^{\sqrt{k}}$ $(k=1,2,\cdots)$. The Borel-Cantelli lemma implies

$$\liminf_{k\to\infty}\gamma(T_k)\sup_{0\leqslant t\leqslant T_k-a_{T_k}}|Z(t+a_{T_k})-Z(t)|\geqslant 1 \text{ a. s.} \tag{2.2.35}$$

For $T_k\leqslant T\leqslant T_{k+1}$, we have

$$\begin{aligned}
&\gamma(T)\sup_{0\leqslant t\leqslant T-a_T}|Z(t+a_T)-Z(t)|\\
&\geqslant\left(2a_{T_{k+1}}^{2\alpha}\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}\right)^{-1/2}\left(\sup_{0\leqslant t\leqslant T_k-a_{T_k}}|Z(t+a_{T_k})-Z(t)|\right.\\
&\qquad\left.-\sup_{0\leqslant t\leqslant T_k-a_{T_k}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}-a_{T_k}}|Z(t+s)-Z(t)|\right)\\
&=:A_k\gamma(T_k)I(T_k)-J_k(T_k),
\end{aligned}\tag{2.2.36}$$

where

$$A_k=\left(\frac{a_{T_k}}{a_{T_{k+1}}}\right)^{\alpha}\left(\frac{\log((T_k/a_{T_k})/\log\log T_k)}{\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k)}\right)^{1/2}.$$

Noting that $T_k/T_{k+1}\to 1$ as $k\to\infty$, we have $a_{T_k}/a_{T_{k+1}}\to 1$ as $k\to\infty$. And then

$$A_k\to 1 \quad\text{as}\quad k\to\infty. \tag{2.2.37}$$

On the other hand, Theorem 2.2.2 implies

$$\limsup_{k\to\infty}\sup_{0\leqslant t\leqslant T_k-a_{T_k}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}-a_{T_k}}\beta(T_k)|Z(t+s)-Z(t)|\leqslant 1 \text{ a. s.}, \tag{2.2.38}$$

where

$$\beta(T_k)=\left(2(a_{T_{k+1}}-a_{T_k})^{2\alpha}\left(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log(T_k+a_{T_{k+1}})\right)\right)^{-1/2}.$$

It is easy to show that

$$a_{T_{k+1}}-a_{T_k}\leqslant a_{T_{k+1}}(1-T_k/T_{k+1})\leqslant 6a_{T_{k+1}}/k^{1/3},$$

which implies

$$\begin{aligned}
&\beta^{-2}(T_k)\left(2a_{T_{k+1}}^{2\alpha}\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}\right)^{-1}\leqslant\left(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\\
&\times\frac{\log\dfrac{2T_{k+1}}{a_{T_{k+1}}\log\log T_k}+\log\dfrac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+2\log\log(2T_k)}{\log\dfrac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}}\\
&\leqslant ck^{2\alpha/3}\log k+c\frac{\left(\dfrac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\log\dfrac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}}{\log\dfrac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}}\to 0\ (k\to\infty).
\end{aligned}$$

Hence we have

$$\limsup_{k\to\infty}J_k(T_k)=0 \text{ a. s.} \tag{2.2.39}$$

Combining (2.2.35)—(2.2.39) we prove (2.2.33).

If $1/2<\alpha<1$, with Lemma 2.2.6 instead of Lemma 2.2.5 and $(1-\varepsilon)\gamma^{-1}(T)$ instead of $\gamma^{-1}(T)$ in (2.2.34), the proof is similar.

Step 3 Suppose that conditions (i), (ii) and (iv)′ are satisfied. Then

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|Z(t+s)-Z(t)|\geqslant c_\alpha \text{ a. s.} \tag{2.2.40}$$

for some $c_\alpha > 0$.

Proof Condition (iv)$'$ implies $\gamma'(T)\left(\frac{T}{\log\log T}\right)^\alpha \to 1\ (T\to\infty)$. By lemma 2.2.7 we have

$$P\left\{\left(\frac{\log\log T}{T}\right)^\alpha \sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le x^\alpha\right\}$$
$$\le \exp\left(-\frac{T}{a_T}\frac{a_T\log\log T}{xT}c_\alpha\right)=(\log T)^{-\frac{c_\alpha}{x}}$$

for T large enough. Let $T_k = k^{\sqrt{k}}$ $(k=1,2,\cdots)$. The Borel-Cantelli lemma implies

$$\liminf_{k\to\infty}\left(\frac{\log\log T_k}{T_k}\right)^\alpha \sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Z(t+s)-Z(t)|\ge\left(\frac{c_\alpha}{2}\right)^\alpha. \tag{2.2.41}$$

Noting that

$$\left(\frac{\log\log T_k}{T_{k+1}}\right)^\alpha \beta^{-2}(T_k)$$
$$=2\left(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\left(\frac{a_{T_{k+1}}\log\log T_k}{T_{k+1}}\right)^{2\alpha}\left(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\right.$$
$$\left.+\log\log(T_k+a_{T_{k+1}})\right)$$
$$\le 2\left(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\right)^{2\alpha}\left(\frac{a_{T_{k+1}}\log\log T_k}{T_{k+1}}\right)^{2\alpha}\left(\log\frac{2T_{k+1}}{a_{T_{k+1}}\log\log T_k}\right.$$
$$\left.+\log\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+2\log\log(2T_k)\right)$$
$$\le ck^{-2\alpha/3}(\log k)^{2\alpha+1}+ck^{-2\alpha/3}(\log k)^{1/3}\to 0\ (k\to\infty),$$

where condition (iv)$'$ is used, the remainder proof is similar to that of (2.2.33).

Step 4 Suppose that condition (iv)$'$ is satisfied. Then

$$\liminf_{T\to\infty}\gamma_1(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|\le c_\alpha \quad \text{a.s.} \tag{2.2.42}$$

for some $c_\alpha>0$.

Proof The result follows from the Chung-type law of the iterated logarithm (cf. Theorem 4.3.2) and the following inequality

$$\liminf_{T\to\infty}\gamma'(T)\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Z(t+s)-Z(t)|$$
$$\le 2\liminf_{T\to\infty}\left(\frac{\log\log T}{T}\right)^\alpha\sup_{0\le t\le T}|Z(t)|.$$

Remark 2.2.3 Conditions (i) and (ii) of Theorem 2.2.4 can be removed with, however, a somewhat more complicated proof (cf. Zhang 1997b).

Remark 2.2.4 Zhang (1997a) also studied liminfs on Hanson-Russo type increments for $Z(\cdot)$.

2.2.4 A more general Gaussian process

Let $\{\Gamma(t);t\ge 0\}$ be a Gaussian process with $E\Gamma(t)=0$ and

$$\sigma^2(h)=E(\Gamma(t+h)-\Gamma(t))^2,$$

where $\sigma(s)$ is a non-decreasing function. Under some suitable conditions, such a Gaussian process has the moduli of continuity and large increment properties similar to a fractional Wiener process $Z(\cdot)$.

Theorem 2.2.5 *Assume that $\sigma(\cdot)$ is a regularly varying function at zero with a positive exponent α. Suppose*

$$E(\Gamma(d)-\Gamma(c))(\Gamma(b)-\Gamma(a))\le 0$$
$$\textit{for all } 0\le a<b\le c<d<h_0, \tag{2.2.43}$$

for some $h_0>0$. *Then*

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\frac{|\Gamma(t+s)-\Gamma(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}=1\ \text{a.s.},\qquad (2.2.44a)$$

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\frac{|\Gamma(t+h)-\Gamma(t)|}{\{2\sigma^2(h)\log h^{-1}\}^{1/2}}=1\ \text{a.s.}\qquad (2.2.44b)$$

Remark 2.2.5 The infinite series of independent Ornstein-Uhlenbeck processes $X(\cdot)$ defined as in (2.1.22) satisfies (2.2.43).

Remark 2.2.6 (2.2.44a) is also true if $\sigma(s)/s^\alpha$ is quasi-increasing on $(0,1)$ for some $\alpha>0$ and (2.2.43) holds (See Theorem 3.3.3).

Theorem 2.2.6 *Assume that* $\sigma(\cdot)$ *is a regularly varying function at infinity with a positive exponent* α. *Suppose*

$$E(\Gamma(d)-\Gamma(c))(\Gamma(b)-\Gamma(a))\le 0$$
$$\text{for all } 0<a<b\le c<d<\infty.\qquad (2.2.45)$$

Let $a_T(0<a_T\le T)$ *be a function of* T *satisfying* (i) *and* (ii) *of Theorem* 2.2.2. *Then*

$$\limsup_{T\to\infty}\sup_{0\le t\le a_T}\sup_{0\le s\le a_T}\beta_T|\Gamma(t+s)-\Gamma(t)|$$
$$=\limsup_{T\to\infty}\beta_T|\Gamma(T+a_T)-\Gamma(t)|=1\ \text{a.s.}$$

Furthermore, *if* (iii) *of Theorem* 2.2.2 *is fulfilled*, *then*

$$\lim_{T\to\infty}\sup_{0\le t\le a_T}\sup_{0\le s\le a_T}\beta_T|\Gamma(t+s)-\Gamma(t)|$$
$$=\lim_{T\to\infty}\sup_{0\le t\le a_T}\beta_T|\Gamma(t+a_T)-\Gamma(t)|=1\ \text{a.s.}$$

The proofs of Theorems 2.2.5 and 2.2.6 are similar to those of Theorems 2.2.1 and 2.2.2 in the case of $\alpha\le 1/2$, respectively.

For the liminfs, we also have a similar result to Theorem 2.2.4.

Theorem 2.2.7 *Suppose that* (2.2.45) *is fulfilled and* $\sigma(\cdot)$ *is a regularly varying function at infinity with a positive exponent. Let* $a_T(0<a_T\le T)$ *be a function of* T *satisfying* (i), (ii) *and* (iv) *of Theorem* 2.2.4. *Then*

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma(T)(\Gamma(t+s)-\Gamma(t))$$
$$=\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\gamma(T)(\Gamma(t+a_T)-\Gamma(t))=1\ \text{a.s.}\qquad (2.2.46)$$

Remark 2.2.7 It should be mentioned that (2.2.46) is for the one side values of the increments, and we don't know whether it holds or not if the one side values are taken by the absolute ones.

Proof First we show that

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\gamma(T)(\Gamma(t+s)-\Gamma(t))\le 1\ \text{a.s.}\qquad (2.2.47)$$

If (2.2.23) is fulfilled, the proof of (2.2.47) is similar to that of (2.2.22). If (2.2.25) is fulfilled, we let $T_n=e^{n^p}(p>1)$ and

$$A_n=\{\sup_{T_{n-1}<t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}(\Gamma(t+s)-\Gamma(t))\le(1+\varepsilon)^{1/2}\gamma^{-1}(T_n)\}.$$

Then $T_{n-1}/T_n\to 0(n\to\infty)$ and similar to (2.2.27) we have for n large enough

$$P(A_n)\ge n^{-2/3}.$$

It follows that $\sum_{n=1}^\infty P(A_n)=\infty$. On the other hand, by Slepian's inequality, (2.2.45) implies

$$P(A_jA_k)\le p(A_j)P(A_k)\quad\text{for } j\ne k.$$

It follows from the Borel-Cantelli lemma that $P(A_n,\ i.o.)=1$, which implies

$$\liminf_{n\to\infty}\gamma(T_n)\sup_{T_{n-1}<t\le T_n-a_{T_n}}\sup_{0\le s\le a_{T_n}}(\Gamma(t+s)-\Gamma(t))\le 1\ \text{a.s.}$$
$$(2.2.48)$$

Obviously

$$\sup_{0\leqslant t\leqslant T_n - a_{T_n}} \sup_{0\leqslant s\leqslant a_{T_n}} (\Gamma(t+s)-\Gamma(t))$$
$$\leqslant \sup_{T_{n-1}\leqslant t\leqslant T_n - a_{T_n}} \sup_{0\leqslant s\leqslant a_{T_n}} (\Gamma(t+s)-\Gamma(t)) \tag{2.2.49}$$
$$+ \sup_{0\leqslant u<v\leqslant T_{n-1}} |\Gamma(u)-\Gamma(v)|.$$

By Theorem 2.2.6 we have

$$\limsup_{n\to\infty} (2\sigma^2(T_{n-1})\log\log T_{n-1})^{-1} \sup_{0\leqslant u<v\leqslant T_{n-1}} |\Gamma(u)-\Gamma(v)| \leqslant 1 \text{ a.s.} \tag{2.2.50}$$

Since $\sigma(x)$ is regularly varying at infinity with exponent α, therefore $\sigma(x)/x^{\alpha/2}$ is quasi-increasing on $[1,\infty)$, it follows that by (2.2.25)

$$\frac{\sigma(T_{n-1})}{\sigma(a_{T_{n-1}})} \leqslant \frac{\sigma(T_{n-1})}{\sigma(T_n/(\log\log T_n)^{\gamma_0})} \leqslant c\left(\frac{T_{n-1}}{T_n/(\log\log T_n)^{\gamma_0}}\right)^{\alpha/2}$$
$$\leqslant c\exp(-cn^{p-1}),$$

which together with (iv) implies that

$$\gamma(T_n)(2\sigma^2(T_{n-1})\log\log T_{n-1})\to 0,\ n\to\infty. \tag{2.2.51}$$

Putting (2.2.48)—(2.2.51) together yields (2.2.47).

We are thus left with the proof of

$$\liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \gamma(T)(\Gamma(t+a_T)-\Gamma(t))\geqslant 1 \text{ a.s.},$$

which is similar to Step 2 of the proof of Theorem 2.2.4, if we note the following lamma.

Lemma 2.2.9 *Let $\{\Gamma(t);t\geqslant 0\}$ be a Gaussian process as in Theorem 2.2.7. Then*

$$P\{\sup_{0\leqslant t\leqslant T}(\Gamma(t+a)-\Gamma(t))\leqslant u\sigma(a)\}$$
$$\leqslant \exp\left(-\frac{1}{\sqrt{2\pi}}\frac{T}{a}\frac{1}{u}\left(1-\frac{1}{u^2}\right)e^{-u^2/2}\right)$$

for all $T>a>0$.

Proof By Slepian's inequality, we have

$$P\{\sup_{0\leqslant t\leqslant T}(\Gamma(t+a)-\Gamma(t))\leqslant u\sigma(a)\}$$
$$\leqslant P\{\sup_{0\leqslant k\leqslant [T/a]}(\Gamma(ka+a)-\Gamma(ka))\leqslant u\sigma(a)\}$$
$$\leqslant \prod_{k=0}^{[T/a]} P\{N(0,1)\leqslant u\}$$
$$\leqslant \exp\left(-\frac{1}{\sqrt{2\pi}}\frac{T}{a}\frac{1}{u}\left(1-\frac{1}{u^2}\right)e^{-u^2/2}\right)$$

as desired.

2.3 Large Increments of a Two-parameter Wiener Process

In This section and the sequel of this chapter, we study the sample path properties of multi-parameter stochastic processes, which are much more complicated than the one-parameter case. The simplest multi-parameter Gaussian process is the two-parameter Wiener process $\{W(x,y);(x,y)\in\mathbf{R}^2\}$.

A stochastic process $\{W(x,y);(x,y)\in\mathbf{R}^2\}$ is called a two-parameter Wiener process if

(1) $W(R)\in N(0,\lambda(R))$ for all $R=[x_1,x_2)\times[y_1,y_2)$, where $\lambda(R)=(x_2-x_1)(y_2-y_1)$ and $W(R)=W(x_2,y_2)-W(x_1,y_2)-W(x_2,y_1)+W(x_1,y_1)$,

(2) $W(0,y)=W(x,0)=0$ $(x,y\in\mathbf{R}^2)$,

(3) $\{W(x,y)\}$ is an independent increment process, that is $W(R_1),\cdots,W(R_n)$ $(n=2,3,\cdots)$ are independent r.v., if $R_1,\cdots,R_n$ are disjoint rectangles,

(4) the sample path function $W(x,y;\omega)$ is continuous in (x,y) with probability one.

The moduli of continuity of $W(\cdot,\cdot)$ were obtained by Pruitt and Orey (1973) (cf. also Section 2.5). Here we only study the problem on how large the increments over the rectangles with area a_T can be when $T\to\infty$.

Let $0<a_T\leqslant T$ and $b_T\geqslant T^{1/2}$ be two non-decreasing functions of T, $D_T=D_T(b_T)=\{(x,y);xy\leqslant T, 0\leqslant x,y\leqslant b_T\}$, and $L_T=L_T(a_T,b_T)$ (resp. $L_T^*=L_T^*(a_T,b_T)$) be the set of rectangles $R=[x_1,x_2)\times[y_1,y_2)\subset D_T(b_T)$ for which $\lambda(R)\leqslant a_T$ (resp. $\lambda(R)=a_T$). Define

$$\delta_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)+\log\log T)\}^{-1/2},$$

$$\gamma_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)-\log\log\log T)\}^{-1/2},$$

$$\lambda_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))\}^{-1/2}.$$

We say that a function of T $f(T)>0$ is properly non-increasing, if there exists a non-increasing function $g(T)>0$ such that

(a) $\lim_{T\to\infty}f(T)/g(T)=1$,

(b) For any $\varepsilon>0$, there exists $\theta_0=\theta_0(\varepsilon)>1$ such that for any $1<\theta\leqslant\theta_0$ and $k\geqslant1$,

$$\limsup_{T\to\infty}\frac{g(\theta^k)}{g(\theta^{k+1})}\leqslant1+\varepsilon.$$

Csörgő and Révész (1978) have discussed how big the increments of a two-parameter Wiener process are. They proved

Theorem 2.3.1 *Suppose that Ta_T^{-1} is a non-decreasing function of T and δ_T is a properly non-increasing function of T. Then*

$$\limsup_{T\to\infty}\sup_{R\in L_T^*}\delta_T|W(R)|=\limsup_{T\to\infty}\sup_{R\in L_T}\delta_T|W(R)|=1\ \text{a.s.} \tag{2.3.1}$$

If we also have

$$\lim_{T\to\infty}\frac{\log Ta_T^{-1}+\log(1+\log b_Ta_T^{-1/2})}{\log\log T}=\infty, \tag{2.3.2}$$

then $\limsup_{T\to\infty}$ *can be replaced by* $\lim_{T\to\infty}$ *in* (2.3.1).

The detailed proof of this result can be found in Csörgő and Révész (1981), we don't present it here. We pay more attention to the liminfs when (2.3.2) is weakened. Here are our main results due to Zhang (1997c).

Theorem 2.3.2 *Suppose that*

(i) *Ta_T^{-1} is a non-decreasing function of T,*

(ii) *γ_T is a properly non-increasing function of T,*

(iii) $\liminf_{T\to\infty}\dfrac{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)}{\log\log\log T}>1.$

Then

$$\liminf_{T\to\infty}\sup_{R\in L_T}\gamma_T|W(R)|=\liminf_{T\to\infty}\sup_{R\in L_T^*}\gamma_TW(R)=1\ \text{a.s.} \tag{2.3.3}$$

Before proving Theorem 2.3.2, here is an immediate consequence:

Corollary 2.3.1 *Suppose condition* (i) *in Theorem* 2.3.2 *is fulfilled and*

(ii)′ *λ_T is a properly non-increasing function of T,*

(iii)′ $\lim_{T\to\infty}\dfrac{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)}{\log\log\log T}=r\ (1\leqslant r\leqslant\infty).$

Then

$$\liminf_{T\to\infty}\sup_{R\in L_T}\lambda_T|W(R)|=\liminf_{T\to\infty}\sup_{R\in L_T^*}\lambda_TW(R)=\left(\frac{r-1}{r}\right)^{1/2}\ \text{a.s.}, \tag{2.3.4}$$

where $(r-1)/r=1$, if $r=\infty$.

Remark 2.3.1 Corollary 2.3.1 was first obtained by Lin

(1984) for $r=\infty$.

The proof of this theorem is based on an inequality, which is an analogue of Lemma 2.2.4 and is formulated in the following way.

Theorem 2.3.3 *For any $\varepsilon>0$, there exist constants $C=C(\varepsilon)>0$, $u_0=u_0(\varepsilon)>0$ and $T_0=T_0(\varepsilon)>0$ such that*

$$P\{\sup_{R\in L_T}|W(R)|\leqslant ua_T^{1/2}\}$$
$$\geqslant\exp\left(-CTa_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2})e^{-u^2/(2+\varepsilon)}\right) \tag{2.3.5}$$

holds for any $u\geqslant u_0$, $T\geqslant T_0$.

At first we introduce some notations and state two lemmas.

Let $\mu=\mu(T)$ be the smallest integer for which

$$\mu\geqslant\log b_Ta_T^{-1/2}$$

and, for any integer q, let $Q=Q(q)=2^q$. Define

$$z_i=z_i(q)=z_i(q,T)=a_T^{1/2}e^{i/Q}\quad(i=0,\pm1,\cdots,\pm Q\mu),$$
$$x_j(i)=x_j(i,T)=jz_iQ^{-1}\quad(j=0,1,\cdots),$$
$$y_j(i)=y_j(i,T)=ja_Tz_i^{-1}Q^{-1}\quad(j=0,1,\cdots),$$

and

$$R_i=R_i(q)=R_i(q,0,0)=[0,z_i]\times[0,a_Tz_i^{-1}],$$
$$R_i(j,l)=R_i(q,j,l)=R_i+(x_j(i),y_l(i))$$
$$=\{(x,y);(x-x_j(i),y-y_l(i))\in R_i\}.$$

Let $L_T^*(q)$ be the set of rectangles $R_i(q,j,l)$ contained in the domain $D_T(b_T)$. For any $R=[x_1,x_2]\times[y_1,y_2]\in L_T$ define the rectangle $R(q)\in L_T^*(q)$ as follows: let $i_0=i_0(R)$ denote the smallest integer for which $z_{i_0}\geqslant x_2-x_1$ and let $j_0=j_0(R)$, $l_0=l_0(R)$ denote the largest integers for which $x_{j_0}(i_0)\leqslant x_1$, $y_{l_0}(i_0)\leqslant y_1$ and now let

$$R(q)=R_{i_0}(q,j_0,l_0)=(x_{j_0}(i_0),y_{l_0}(i_0))+[0,z_{i_0}]\times[0,a_Tz_{i_0}^{-1}].$$

The following lemma is due to Csörgő and Révész (1981).

Lemma 2.3.1 *We have*

$$\text{Card }L_T^*(q)\leqslant 8Q^3Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2}), \tag{2.3.6}$$
$$\lambda(R\circ R(q))\leqslant 6a_TQ^{-1}\text{ for each }R\in L_T^*, \tag{2.3.7}$$

where λ is the Lebesgue measure and the operation "$\circ$" stands for symmetric difference,

$$\lambda(R)=a_T\text{ for each }R\in L_T^*(q). \tag{2.3.8}$$

In the proof of Theorem 2.3.3 the following result will be also used.

Lemma 2.3.2 *For any $\varepsilon>0$, there exist constants $C=C(\varepsilon)>0$, $u_0=u_0(\varepsilon)\geqslant1$ such that*

$$P\{\sup_{\substack{x_0\leqslant x<T_1+x_0\\ y_0\leqslant y\leqslant T_2+y_0}}\ \sup_{\substack{0\leqslant s<h_1\\ 0\leqslant t\leqslant h_2}}|W([x,x+s]\times[y,y+t])|\leqslant u(h_1h_2)^{1/2}\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-C\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)e^{-u^2/(2+\varepsilon)}\right\} \tag{2.3.9}$$

for any $x_0,y_0\geqslant0$, $0<h_1\leqslant T_1$, $0<h_2\leqslant T_2$ and $u\geqslant u_0$.

Particularly, for $u\geqslant u_0$, $T_1,T_2>0$ and $x_0,y_0\geqslant0$ we have

$$P\{\sup_{R\subset[x_0,x_0+T_1]\times[y_0,y_0+T_2]}|W(R)|\leqslant u(T_1T_2)^{1/2}\}$$
$$\overset{\text{KS}}{\geqslant}\exp\{-ce^{-u^2/(2+\varepsilon)}\}, \tag{2.3.10}$$

where $R=[x_1,x_2]\times[y_1,y_2]$.

Proof Without loss of generality we can assume $x_0=y_0=0$. For a positive real number s and integer r, let $s_r=h_1\times\left[s\frac{2^r}{h_1}\right]/2^r$. Also write $R=2^r$. Clearly, for each $\omega\in\Omega$ and x,y,s,r,t fixed, we have

$$|W([x,x+s]\times[y,y+t])|$$
$$\leqslant|W([x_r,(x+s)_r]\times[y,y+t])|$$

$$+\sum_{j=0}^{\infty}|W([(x+s)_{r+j},(x+s)_{r+j+1}]\times[y,y+t])|$$
$$+\sum_{j=0}^{\infty}|W([x_{r+j},x_{r+j+1}]\times[y,y+t])|. \qquad (2.3.11)$$

By Lemma 2.2.4, we have for any fixed x,s,

$$P\{\sup_{0\leqslant y\leqslant T_2}\sup_{0\leqslant t\leqslant h_2}|W([x,x+s]\times[y,y+t])|\leqslant us^{1/2}h_2^{1/2}\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-C\left(\frac{T_2}{h_2}+1\right)u^7e^{-\frac{u^2}{2}}\right\}\ (u\geqslant u_1\geqslant 1). \qquad (2.3.12)$$

Hence, by (2.3.11) and Lemma 2.2.2 we have that for any $u\geqslant u_0, x_j\geqslant u_1, 0<h_1\leqslant T_1$ and integers r,j,

$$P\left\{\sup_{\substack{0\leqslant x\leqslant T_1\\0\leqslant y\leqslant T_2}}\sup_{\substack{0\leqslant s\leqslant h_1\\0\leqslant t\leqslant h_2}}|W([x_r,(x+s)_r]\times[y,y+t])|\right.$$
$$\left.\leqslant uh_2^{1/2}\sqrt{h_1(1+1/R)}\right\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-CR^2\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)u^7e^{-\frac{u^2}{2}}\right\}, \qquad (2.3.13)$$

$$P\left\{\sup_{\substack{0\leqslant x\leqslant T_1\\0\leqslant y\leqslant T_2}}\sup_{\substack{0\leqslant s\leqslant h_1\\0\leqslant t\leqslant h_2}}|W([(x+s)_{r+j},(x+s)_{r+j+1}]\times[y,y+t])|\right.$$
$$\left.\leqslant x_jh_2^{1/2}\frac{h_1^{1/2}}{\sqrt{2^{r+j+1}}}\right\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-C2^{r+j+1}\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)x_j^7e^{-\frac{x_j^2}{2}}\right\} \qquad (2.3.14)$$

and

$$P\left\{\sup_{\substack{0\leqslant x\leqslant T_1\\0\leqslant y\leqslant T_2}}\sup_{\substack{0\leqslant s\leqslant h_1\\0\leqslant t\leqslant h_2}}|W([x_{r+j},x_{r+j+1}]\times[y,y+t])|\leqslant x_jh_2^{1/2}\frac{h_1^{1/2}}{\sqrt{2^{r+j+1}}}\right\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-C2^{r+j+1}\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)x_j^7e^{-\frac{x_j^2}{2}}\right\}. \qquad (2.3.15)$$

By (2.3.13), (2.3.14) and (2.3.15), using Lemma 2.2.2 again, we have

$$P\left\{\sup_{\substack{0\leqslant x\leqslant T_1\\0\leqslant y\leqslant T_2}}\sup_{\substack{0\leqslant s\leqslant h_1\\0\leqslant t\leqslant h_2}}|W([x,x+s]\times[y,y+t])|\right.$$
$$\left.\leqslant(h_1h_2)^{1/2}\left(u\sqrt{1+1/R}+2\sum_{j=0}^{\infty}\frac{x_j}{\sqrt{2^{r+j+1}}}\right)\right\}$$
$$\overset{\text{KS}}{\geqslant}\exp\left\{-CR^2\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)u^7e^{-\frac{u^2}{2}}\right.$$
$$\left.-8CR\left(\frac{T_1}{h_1}+1\right)\left(\frac{T_2}{h_2}+1\right)\sum_{j=0}^{\infty}2^jx_j^7e^{-\frac{x_j^2}{2}}\right\}. \qquad (2.3.16)$$

Let $x_j=\sqrt{2j+u^2}$. Taking R to be large enough implies

$$u\sqrt{1+\frac{1}{R}}+2\sum_{j=0}^{\infty}\frac{x_j^2}{\sqrt{2^{r+j+1}}}$$
$$\leqslant u\left(1+\left(\frac{1}{R}\right)^{1/2}\right)+2\left(\frac{1}{R}\right)^{1/2}\sum_{j=0}^{\infty}\frac{\sqrt{4ju^2}}{\sqrt{2^{j+1}}}$$
$$\leqslant u\left(1+\left(\frac{1}{R}\right)^{1/2}+2\left(\frac{1}{R}\right)^{1/2}\sum_{j=0}^{\infty}\frac{\sqrt{2j}}{\sqrt{2^j}}\right)$$
$$\leqslant u\left(1+A\left(\frac{1}{R}\right)^{1/2}\right)$$
$$\leqslant u\left(1+\frac{\varepsilon}{2}\right)^{1/2},$$

where $A=1+2\sum_{j=0}^{\infty}\sqrt{2j/2^j}$, and

$$\sum_{j=0}^{\infty}2^jx_j^7e^{-\frac{x_j^2}{2}}\leqslant e^{-\frac{u^2}{2}}\sum_{j=0}^{\infty}(4ju^2)^{7/2}(2/e)^j=u^7e^{-\frac{u^2}{2}}B,$$

where $B=\sum_{j=0}^{\infty}(4j)^{7/2}(2/e)^j$. Consequently (2.3.9) follows by taking $x=u(1+\varepsilon/2)^{1/2}$ and $u_0=2u_1$.

Proof of Theorem 2.3.3

For any $R\in L_T$, the symmetric difference $R(q)\circ R(q+1)$ is the sum of at most rectangles, say $R(q)\circ R(q+1)=R^{(1)}(q)+R^{(2)}(q)+R^{(3)}(q)+R^{(4)}(q)$. Denote this class of rectangles $R^{(i)}(q)(i=1,2,3,4)$ by $\widetilde{L}_T^*(q)$. Since $R(q)\to R$ as $q\to\infty$ for any

R in L_T^*, we have

$$\sup_{R\in L_T}|W(R)|\leqslant\sup_{R\in L_T^*(q)}\sup_{S\subset R}|W(S)|+4\sum_{i=0}^{\infty}\sup_{R\in \tilde{L}_T^*}\sup_{S\subset R}|W(S)|, \tag{2.3.17}$$

where S is a rectangle with edges parallel to the coordinate axes.

By Lemma 2.2.4, Lemma 2.3.1 and Lemma 2.3.2, we have that for any $\delta>0$, there exist constants $C_\delta>0$, $x_\delta\geqslant 1$ such that

$$P\{\sup_{R\in L_T^*(q)}\sup_{S\subset R}|W(S)|\leqslant xa_T^{1/2}\}$$
$$\overset{KS}{\geqslant}\exp\{-C_\delta \mathrm{Card} L_T^*(q)e^{-x^2/(2+\delta)}\}, \tag{2.3.18}$$
$$P\{\sup_{R\in \tilde{L}_T^*(q+i)}\sup_{S\subset R}|W(S)|\leqslant y_i(6a_TQ^{-1}2^{-i})^{1/2}\}$$
$$\overset{KS}{\geqslant}\exp\{-C_\delta \mathrm{Card}\tilde{L}_T^*(q+i)e^{-y_i^2/(2+\delta)}\} \tag{2.3.19}$$

for $x\geqslant x_\delta$, $y_i\geqslant x_\delta$. Noting that

$$\mathrm{Card}\ \tilde{L}_T^*(q+i)\leqslant 4\ \mathrm{Card}\ L_T^*(q+i),$$

by (2.3.17), (2.3.18), (2.3.19) and using Lemma 2.2.2 again, we have

$$P\Big\{\sup_{R\in L_T}|W(R)|\leqslant xa_T^{1/2}+4\sum_{j=0}^{\infty}y_i(6a_TQ^{-1}2^{-i})^{1/2}\Big\}$$
$$\overset{KS}{\geqslant}\exp\Big\{-C_\delta\Big(\mathrm{Card}L_T^*(q)e^{-x^2/(2+\delta)}$$
$$+4\sum_{i=0}^{\infty}\mathrm{Card}L_T^*(q+i)e^{-y_i^2/(2+\delta)}\Big)\Big\}. \tag{2.3.20}$$

Choose $y_i=(3i(2+\delta)+x^2)^{1/2}$ $(i=0,1,\cdots)$, we have

$$xa_T^{1/2}+4\sum_{j=0}^{\infty}y_i(6a_TQ^{-1}2^{-i})^{1/2}$$
$$\leqslant xa_T^{1/2}\Big\{1+4(6Q^{-1})^{1/2}\sum_{i=0}^{\infty}2^{-i/2}\Big\}+32a_T^{1/2}Q^{-1/2}\sum_{i=0}^{\infty}(i2^{-i})^{1/2}$$

$$\leqslant xa_T^{1/2}(1+Q^{-1/2}A)+a_T^{1/2}Q^{-1/2}B\leqslant(1+\delta)xa_T^{1/2}, \tag{2.3.21}$$

provided that Q is big enough and $x\geqslant 1$, where

$$A=4\sqrt{6}\ \sum\nolimits_{i=0}^{\infty}2^{-i/2},\ B=32\sum\nolimits_{i=0}^{\infty}(i2^{-i})^{1/2};$$

further, by Lemma 2.3.1.

$$\mathrm{Card}L_T^*(q)e^{-x^2/(2+\delta)}+4\sum_{i=0}^{\infty}\mathrm{Card}L_T^*(q+i)e^{-y_i^2/(2+\delta)}$$
$$\leqslant CTa_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2})e^{-x^2/(2+\delta)}. \tag{2.3.22}$$

Now given $u\geqslant(1+\delta)x_\delta$ we let $(1+\delta)x=u$, and from (2.3.10), (2.3.21) and (2.3.22) it follows that

$$P\Big\{\sup_{R\in L_T}|W(R)|\leqslant ua_T^{1/2}\Big\}$$
$$\overset{KS}{\geqslant}\exp\Big(-C_\delta Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2})e^{-\frac{u^2}{(2+\delta)(1+\delta)}}\Big).$$

The proof of Theorem 2.3.3 is now completed.

Proof of Theorem 2.3.2

This will be given in two steps.

Step 1 If

$$\text{(iv)}\ \Delta_T=\frac{Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_Ta_T^{-1/2})}{\log\log T}\to\infty\ (T\to\infty),$$

then

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\delta}_T|W(R)|\leqslant 1\ \text{a.s.}, \tag{2.3.23}$$

where $\tilde{\delta}_T=\{2a_T\log\Delta_T\}^{-1/2}$.

Proof If

$$\limsup_{T\to\infty}\frac{\log Ta_T^{-1}+\log(\log Ta_T^{-1}+1)+\log(\log b_Ta_T^{-1/2}+1)}{\log\log\log T}=\infty, \tag{2.3.24}$$

then there exists a subsequence $\{T_N\}$, $T_N\uparrow\infty$ such that

$$\limsup_{N\to\infty}\frac{\log T_N a_{T_N}^{-1}+\log(\log T_N a_{T_N}^{-1}+1)+\log(\log b_{T_N}a_{T_N}^{-1/2}+1)}{\log\log\log T_N}=\infty.$$
(2. 3. 25)

By Theorem 2. 3. 3, we have

$$P\{\sup_{R\in L_{T_N}}\tilde{\delta}_{T_N}|W(R)|\geqslant 1+\varepsilon\}$$
$$\leqslant cT_N a_{T_N}^{-1}(1+\log T_N a_{T_N}^{-1})(\log b_{T_N}a_{T_N}^{-1/2}+1)$$
$$\times\exp\left(-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\log\Delta_{T_N}\right)$$
$$\leqslant c\{T_N a_{T_N}^{-1}(1+\log T_N a_{T_N}^{-1})(\log b_{T_N}a_{T_N}^{-1/2}+1)\}^{-\varepsilon'}$$
$$\times(\log\log T_N)^{1+\varepsilon'}\to 0\ (N\to\infty),$$

where $\varepsilon'=\frac{2(1+\varepsilon)^2}{2+\varepsilon}-1>0$. It follows that

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\delta}_T|W(R)|\leqslant\liminf_{N\to\infty}\sup_{R\in L_{T_N}}\tilde{\delta}_{T_N}|W(R)|\leqslant 1+\varepsilon\text{ a. s.}$$

Now, suppose

$$\limsup_{T\to\infty}\frac{\log Ta_T^{-1}+\log(\log Ta_T^{-1}+1)+\log(\log b_T a_T^{-1/2}+1)}{\log\log\log T}<\infty.$$

Without loss of generality, we can assume that for some $0<r_0<\infty$,

$$Ta_T^{-1}(1+\log Ta_T^{-1})(1+\log b_T a_T^{-1/2})\leqslant(\log\log T)^{r_0}\quad(T>0).$$
(2. 3. 26)

Let $T_n=e^{n^p}(p>1)$, $D'_{T_{n+1}}=D_{T_{n+1}}\cap D^c_{T_n}$, $D''_{T_{n+1}}=\{(x,y);0\leqslant x,y\leqslant b_{T_{n+1}},xy\leqslant 2T_n\}$. Let l_n be a polygonal line between the curves $xy=T_n$, $xy=2T_n$ with edges parallel to the coordinate axes and vertices on $xy=T_n$, $xy=2T_n$, one of these vertices is $(\sqrt{2T_n},\sqrt{2T_n})$. The polygonal line l_n cut the plane into two parts. We denote the upper (resp. down) part by U_n(resp. V_n). For any $R=[x_1,x_2]\times[y_1,y_2]\subset D_{T_{n+1}}$, we have $R\cap U_n\subset D'_{T_{n+1}}$, $R\cap V_n\subset D''_{T_{n+1}}$ and there exist rectangles $R_1,\cdots,R_k\subset D'_{T_{n+1}}$, $\tilde{R}_1,\cdots,\tilde{R}_k\subset D''_{T_{n+1}}$ with disjoint interiors such that $R\cap U_n=\bigcup_{i=1}^k R_i$, $R\cap V_n=\bigcup_{i=1}^k\tilde{R}_i$. We define $W(R\cap U_n)=\sum_{i=1}^k W(R_i)$ and $W(R\cap V_n)=\sum_{i=1}^k W(\tilde{R}_i)$. Let

$$L'_{T_{n+1}}=\{R\cap U_n;R=[x_1,x_2]\times[y_1,y_2]\subset D_{T_{n+1}},\lambda(R)\leqslant a_{T_{n+1}}\}.$$

Obviously, $\{W(S);S\in L'_{T_{n+1}}\}_{n=1}^\infty$ are independent.

Now, for any $R=[x_1,x_2]\times[y_1,y_2]\subset D_{T_{n+1}}$, let $M_n(R)$ be the number of the vertices of $R\cap V_n$. If $R\subset D'_{T_{n+1}}$ or $R\subset D''_{T_{n+1}}$, then $M_n(R)\leqslant 6$. If $(x_1,y_1)\in D_{T_n}$, let $(u_1,v_1),\cdots,(u_k,v_k)$ be all the vertices of l_n on curve $xy=T_n$ and contained in R with $u_1<\cdots<u_k$, then $v_1=T_n/u_1$, $u_k=u_1 2^{k-1}$, $v_k=T_n/u_1 2^{-(k-1)}$. Since $(u_k-u_1)(v_1-v_k)\leqslant\lambda(R)$, i. e. $2^{k-1}(1-2^{-k+1})^2T_n\leqslant\lambda(R)$, we have $k\leqslant\frac{1}{\log 2}\log\frac{\lambda(R)}{T_n}+2$ (if $k\geqslant 3$). It follows that

$$M_n(R)\leqslant 2(k+4)\leqslant 4\log(1+\lambda(R)/T_n)+12.$$

So, in any case we have

$$M_n(R)\leqslant 4\log(1+\lambda(R)/T_n)+12.$$

Noting that $W(R)=W(R\cap U_n)+W(R\cap V_n)$, we have

$$\sup_{R\in L_{T_{n+1}}}|W(R)|$$
$$\leqslant\sup_{S\in L'_{T_{n+1}}}|W(S)|+\left\{4\log\left(\frac{a_{T_{n+1}}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D''_{T_{n+1}}}|W(x,y)|,$$
$$\sup_{S\in L'_{T_{n+1}}}|W(S)|$$
$$\leqslant\sup_{R\in L_{T_{n+1}}}|W(R)|+\left\{4\log\left(\frac{a_{T_{n+1}}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D''_{T_{n+1}}}|W(x,y)|.$$
(2. 3. 27)

By Theorem 2. 3. 3, we have that for n large enough,

$$J''_{n+1} := P\left\{\left(4\log\left(\frac{a_{T_{n+1}}}{T_n}+1\right)+12\right)\sup_{(x,y)\in D''_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(x,y)|>\varepsilon\right\}$$
$$\leqslant P\left\{\sup_{(x,y)\in D''_{T_{n+1}}}|W(x,y)|>2(2T_n)^{1/2}a^{1/2}_{T_{n+1}}/(n^pT_n^{1/2})\right\}$$
$$\leqslant c(1+\log(b_{T_{n+1}}(2T_n)^{-1/2}))\exp(-a_{T_{n+1}}/(n^{2p}T_n))$$
$$\leqslant c(1+\log b_{T_{n+1}}a^{-1/2}_{T_{n+1}}+\log a^{1/2}_{T_{n+1}}(2T_n)^{-1/2})$$
$$\times\exp\left\{-\frac{T_{n+1}}{T_nn^{2p}}\Big/(\log\log T_{n+1})^{r_0}\right\}$$
$$\leqslant c((\log\log T_{n+1})^{r_0}+\log T_{n+1})\exp\{-e^{(n+1)^p-n^p}/(n^{2p}(n+1)^{r_0})\}$$
$$\leqslant cn^pe^{-n}. \qquad (2.3.28)$$

Then the Borel-Cantelli lemma implies

$$\limsup_{n\to\infty}\left\{4\log\left(\frac{a_{T_{n+1}}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D''_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(x,y)|$$
$$\leqslant\varepsilon \text{ a.s.} \qquad (2.3.29)$$

By Theorem 2.3.3 and noting (iv), we have that for n large enough,

$$J_{n+1} := P\left\{\sup_{R\in L_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(R)|\leqslant 1+\varepsilon\right\}$$
$$\geqslant\exp\left\{-c\Delta_{T_{n+1}}\log\log T_{n+1}\cdot\exp\left(-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\log\Delta_{T_{n+1}}\right)\right\}$$
$$=\exp\{-c\Delta^{-\varepsilon'}_{T_{n+1}}\log\log T_{n+1}\}\geqslant(n+1)^{1/2}, \qquad (2.3.30)$$

where $\varepsilon'=\dfrac{2(1+\varepsilon)^2}{2+\varepsilon}-1>0$. From (2.3.28), (2.3.30) and (2.3.27), it follows that for n large enough,

$$J'_{n+1} := P\left\{\sup_{S\in L'_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(S)|\leqslant 1+2\varepsilon\right\}$$
$$\geqslant J_{n+1}-J''_{n+1}\geqslant(n+1)^{1/2}-cn^pe^{-n}. \qquad (2.3.31)$$

Hence $\sum_{n=1}^{\infty}J'_{n+1}=\infty$. Then the Borel-Cantelli lemma and the independence of $\{W(S);S\in L'_{T_{n+1}}\}_{n=1}^{\infty}$ imply

$$\liminf_{n\to\infty}\sup_{S\in L'_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(S)|\leqslant 1+2\varepsilon \text{ a.s.} \qquad (2.3.32)$$

From (2.3.27), (2.3.29) and (2.3.32), it follows that

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\delta}_T|W(R)|\leqslant\liminf_{n\to\infty}\sup_{R\in L_{T_{n+1}}}\tilde{\delta}_{T_{n+1}}|W(R)|\leqslant 1+3\varepsilon \text{ a.s.}$$

The proof of (2.3.23) is completed.

Step 2 Suppose condition (i) is fulfilled, and

(ii)′ $\tilde{\gamma}_T := \{2a_T(\log Ta_T^{-1}+\log(1+\log b_T^2T^{-1})$
$\qquad -\log\log\log T)\}^{-1/2}$

is a properly non-increasing function of T.

Let $\rho=\lim_{T\to\infty}a_T/T$. If

$$\text{(iv)}'\ \tilde{\Delta}_T=\frac{Ta_T^{-1}(\log b_T^2T^{-1}+1)}{\log\log T}\to\infty\ (T\to\infty),$$

then

$$\liminf_{T\to\infty}\sup_{R\in L_T}\tilde{\gamma}_TW(R)\geqslant 1 \text{ a.s.} \qquad (2.3.33)$$

Furthermore, if $\rho<1$ or (iii) holds then

$$\liminf_{T\to\infty}\sup_{R\in L_T^*}\tilde{\delta}_TW(R)\geqslant 1 \text{ a.s.} \qquad (2.3.34)$$

Proof Let $L=L(T)$ be the largest integer for which we have

$$\frac{T^{L+1}}{(T-a_T)^Lb_T}<b_T \quad \text{if } \rho<1,$$
$$a_T^{1/2}M^{L+1}=T^{1/2}M^{L+1}<b_T \quad \text{if } \rho=1\ (M>1).$$

Define the rectangles

$$S_i=S_i(T)=[x_1(i),x_2(i)]\times[y_1(i),y_2(i)]$$
$$=\begin{cases}\left[\left(\dfrac{T-a_T}{T}\right)^{i+1}b_T,\left(\dfrac{T-a_T}{T}\right)^{i}b_T\right]\times\left[0,\dfrac{T^{i+1}}{(T-a_T)^ib_T}\right] & \text{if } \rho<1,\\ [T^{1/2}M^i,T^{1/2}M^{i+1}]\times[0,T^{1/2}M^{-i-1}] & \text{if } \rho=1,\\ i=0,1,\cdots,L.\end{cases}$$

Then $S_i\subset D_T$ and

$$\lambda(S_i)=a_T(\rho<1),$$

$$\lambda(S_i)=a_T\left(1-\frac{1}{M}\right)=T\left(1-\frac{1}{M}\right)(\rho=1),\ i=0,1,\cdots,L.$$

If $\rho<1$, then

$$L=L(T)\geqslant(\log b_T^2T^{-1})/\log\frac{T}{T-a_T}\geqslant KTa_T^{-1}\log b_T^2T^{-1}.$$

It follows that

$$P\{\sup_{R\in L_T^*}\widetilde{\gamma}_TW(R)\leqslant1-\varepsilon\}\leqslant P\{\widetilde{\gamma}_T\sup_{0\leqslant i\leqslant L}W(S_i(T))\leqslant1-\varepsilon\}$$
$$\leqslant\{1-\exp(-(1-\varepsilon)\log\widetilde{\Delta}_T)\}^{L+1}$$
$$\leqslant\exp\{-c\widetilde{\Delta}_T^\varepsilon\log\log T\}. \tag{2.3.35}$$

If $\rho=1$, then $L\geqslant(\log b_TT^{-1/2})/\log M$. Let

$$L_T^{**}(M)=\left\{R\subset D_T;\lambda(R)=\left(1-\frac{1}{M}\right)T\right\},$$

$$L_T'(M)=\left\{R\subset D_T;\lambda(R)\leqslant\frac{1}{M}T\right\}.$$

We have that for M large enough,

$$P\{\sup_{R\in L_T^{**}(M)}\widetilde{\gamma}_TW(R)\leqslant1-\varepsilon\}\leqslant P\{\widetilde{\gamma}_T\sup_{0\leqslant i\leqslant L}W(S_i(T))\leqslant1-\varepsilon\}$$
$$\leqslant\left\{1-\Phi\left(-(1-\varepsilon)\left(\frac{M}{M-1}\right)^{1/2}(\log\widetilde{\Delta}_T)^{1/2}\right)\right\}^{L+1}$$
$$\leqslant\{1-\exp(-(1-\varepsilon)\log\widetilde{\Delta}_T)\}^{L+1}$$
$$\leqslant\exp\left\{-c\frac{1}{\log M}\widetilde{\Delta}_T^\varepsilon\log\log T\right\}. \tag{2.3.36}$$

Let $T_0=0$, $T_k=(1+k^{-1/2})^k(k=1,2,\cdots)$, then $T_k\uparrow\infty$, and if k is large enough we have $\log T_k=k\log(1+k^{-1/2})>k^{1/3}$. Noting that $\widetilde{\Delta}_T\to\infty$, by (2.3.35), (2.3.36) and the Borel-Cantelli lemma we get

$$\liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\widetilde{\gamma}_{T_k}W(R)\geqslant1-\varepsilon\ \text{a.s.}\ (\rho<1), \tag{2.3.37}$$

$$\liminf_{k\to\infty}\sup_{R\in L_{T_k}^{**}(M)}\widetilde{\gamma}_{T_k}W(R)\geqslant1-\varepsilon\ \text{a.s.}\ (\rho=1). \tag{2.3.38}$$

Let

$$L_{T_k}(k)=\{R;R\subset D_{T_{k+1}},\lambda(R)\leqslant a_{T_{k+1}}-a_{T_k}\}.$$

For any $T>0$, there exists k such that $T_k<T\leqslant T_{k+1}$, then

$$\sup_{R\in L_T^*}\widetilde{\gamma}_TW(R)\geqslant\sup_{R\in L_{T_k}^*}\widetilde{\gamma}_TW(R)-4\sup_{R\in L_{T_k}(k)}\widetilde{\gamma}_T|W(R)|.$$

Noting that $\widetilde{\gamma}_T$ is properly non-increasing and $T_k/T_{k+1}\to1$, we have

$$\liminf_{T\to\infty}\sup_{R\in L_T^*}\widetilde{\gamma}_TW(R)$$
$$\geqslant\liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\widetilde{\gamma}_{T_k}W(R)-4\limsup_{k\to\infty}\sup_{R\in L_{T_k}(k)}\widetilde{\gamma}_{T_{k+1}}|W(R)|. \tag{2.3.39}$$

Noting that $a_{T_{k+1}}/(a_{T_{k+1}}-a_{T_k})>\sqrt{k}$ if k is large enough, we have

$$(a_{T_{k+1}}-a_{T_k})\left\{\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}+\log\left(1+\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\right)\right.$$
$$\left.+\log\left(1+\frac{b_{T_{k+1}}}{\sqrt{a_{T_{k+1}}-a_{T_k}}}\right)\right\}\Big/(a_{T_{k+1}}\log\widetilde{\Delta}_{T_{k+1}})$$
$$\leqslant c\,\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_{k+1}}}\cdot\frac{\log\widetilde{\Delta}_{T_{k+1}}+\log\dfrac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log T_{k+1}}{\log\widetilde{\Delta}_{T_{k+1}}}$$
$$\to0\quad(k\to\infty).$$

By Theorem 2.3.3, we have that for k large enough

$$P\{\sup_{R\in L_{T_k}(k)}\widetilde{\gamma}_{T_{k+1}}|W(R)|>\varepsilon\}$$
$$\leqslant c\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\left(1+\log\frac{T_{k+1}}{a_{T_{k+1}}-a_{T_k}}\right)\left(1+\log\frac{b_{T_{k+1}}}{\sqrt{a_{T_{k+1}}-a_{T_k}}}\right)$$
$$\times\exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon}\,\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\log\widetilde{\Delta}_{T_{k+1}}\right\}$$

$$\leqslant \exp\left\{-\frac{3\varepsilon^2}{2+\varepsilon}\,\frac{a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\log\widetilde{\Delta}_{T_{k+1}}\right\}\leqslant \exp(-\sqrt{k}). \qquad (2.3.40)$$

Then the Borel-Cantelli lemma implies

$$\limsup_{k\to\infty}\sup_{R\in L_{T_k}(k)} \widetilde{\gamma}_{T_{k+1}}|W(R)|=0 \text{ a.s.} \qquad (2.3.41)$$

Combining (2.3.37), (2.3.39) and (2.3.41) yields

$$\liminf_{T\to\infty}\sup_{R\in L_T^*}\widetilde{\gamma}_T W(R)$$

$$\geqslant \liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\widetilde{\gamma}_{T_k}W(R)\geqslant 1-\varepsilon \text{ a.s. } (\rho<1). \qquad (2.3.42)$$

Similarly,

$$\liminf_{T\to\infty}\sup_{R\in L_T^{**}(M)}\widetilde{\gamma}_T W(R)$$

$$\geqslant \liminf_{k\to\infty}\sup_{R\in L_{T_k}^{**}(M)}\widetilde{\gamma}_{T_k}W(R)\geqslant 1-\varepsilon \text{ a.s. } (\rho=1). \qquad (2.3.43)$$

If $\rho<1$, (2.3.42) implies

$$\liminf_{T\to\infty}\sup_{R\in L_T}\widetilde{\gamma}_T W(R)\geqslant \liminf_{T\to\infty}\sup_{R\in L_T^*}\widetilde{\gamma}_T W(R)\geqslant 1 \text{ a.s.} \qquad (2.3.44)$$

If $\rho=1$, by (2.3.43) we have

$$\liminf_{T\to\infty}\sup_{R\in L_T}\widetilde{\gamma}_T W(R)\geqslant \liminf_{T\to\infty}\sup_{R\in L_T^{**}}\widetilde{\gamma}_T W(R)\geqslant 1-\varepsilon \text{ a.s.}$$

$$(2.3.45)$$

Hence (2.3.34)($\rho<1$) and (2.3.33) is proved.

Finally, we suppose $\rho=1$ and (iii) holds. We know that

$$\sup_{R\in L_T^*}\widetilde{\gamma}_T W(R)$$

$$\geqslant \sup_{R\in L_T^{**}(M)}\widetilde{\gamma}_T W(R)-4\sup_{R\in L_T'(M)}\widetilde{\gamma}_T|W(R)|. \qquad (2.3.46)$$

In (2.3.36), we choose $M=M_T=\exp\{\widetilde{\Delta}_T^{\varepsilon/2}\}$. Then

$$P\{\sup_{R\in L_T^{**}(M_T)}\widetilde{\gamma}_T W(R)\leqslant 1-\varepsilon\}$$

$$\leqslant \exp\{-c\widetilde{\Delta}_T^{\varepsilon/2}\log\log T\}. \qquad (2.3.47)$$

By (iii), there exists $r>1$ such that for T large enough,

$$\log b_T^2T^{-1}+1\geqslant (\log\log T)^r.$$

Theorem 2.3.3 implies that for T large enough

$$P\{\sup_{R\in L_T'(M_T)}\widetilde{\gamma}_T|W(R)|>\varepsilon\}$$

$$\leqslant cM_T(\log M_T+1)(\log b_TT^{-1/2}+\log M_T^{1/2}+1)$$

$$\times\exp\left(-\frac{2\varepsilon^2}{2+\varepsilon}M_T\log\widetilde{\Delta}_T\right)$$

$$\leqslant cM_T(\log M_T+1)(\widetilde{\Delta}_T^{\frac{r}{r-1}}+\log M_T)\exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon}M_T\log\widetilde{\Delta}_T\right\}$$

$$\leqslant \exp(-M_T)$$

$$\leqslant \exp\{-\exp((\log\log T)^{\frac{\varepsilon}{2}(r-1)})\}$$

$$\leqslant \exp\{-(\log\log T)^2\}. \qquad (2.3.48)$$

Combining (2.3.46)−(2.3.48) yields

$$P\{\sup_{R\in L_T^*}\widetilde{\gamma}_T W(R)\leqslant 1-5\varepsilon\}$$

$$\leqslant \exp\{-c\widetilde{\Delta}_T^{\varepsilon/2}\log\log T\}+\exp\{-(\log\log T)^2\}. \qquad (2.3.49)$$

Noting that $\widetilde{\Delta}_T\to\infty$ and $\log T_k\geqslant k^{\frac{1}{3}}$, (2.3.49) and the Borel-Cantelli lemma imply

$$\liminf_{k\to\infty}\sup_{R\in L_{T_k}^*}\widetilde{\gamma}_{T_k}W(R)\geqslant 1-5\varepsilon \quad \text{a.s.} \qquad (2.3.50)$$

From (2.3.42) and (2.3.50), it follows that (2.3.34) holds for $\rho=1$ if (iii) is satisfied.

To finish the proof of Theorem 2.3.2, it suffices to note that condition (iii) implies

$$\frac{\log(1+\log Ta_T^{-1})}{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)-\log\log\log T}\to 0\ (T\to\infty),$$

$$(2.3.51)$$

which together with the following fact

$$\log(1+\log b_Ta_T^{-1/2})\leqslant \log(1+\log b_T^2T^{-1})+\log(1+\log Ta_T^{-1})$$

implies that $\gamma_T/\widetilde{\gamma}_T\to 1$, $\gamma_T/\widetilde{\delta}_T\to 1$ and conditions (iv) and (iv)'.

Proof of Corollary 2. 3. 1 Suppose that (iii)$'$ holds. If $r>1$, then (2. 3. 4) follows from (2. 3. 3). If $r=1$, then with $\log\log T$ instead of Δ_T, (2. 3. 4) can be obtained along the lines of Step 1 of the proof of Theorem 2. 3. 2, where only (2. 3. 30) shall be modified as follows.

$$\begin{aligned}
&P\Big\{\sup_{R\in L_{T_{n+1}}}\{2a_{T_{n+1}}\log\log\log T_{n+1}\}^{1/2}|W(R)|\leqslant\varepsilon\Big\}\\
&\geqslant\exp\Big\{-c\cdot\exp\Big(-\varepsilon'\log\log\log T_{n+1}\\
&\quad+\frac{\log(\Delta_{T_{n+1}}\log\log T_{n+1})}{\log\log\log T_{n+1}}\log\log\log T_{n+1}\Big)\Big\}\\
&\geqslant\exp\Big\{-c\cdot\exp\Big(\Big(1-\frac{\varepsilon'}{2}\Big)\log\log\log T_{n+1}\Big)\Big\}\\
&\geqslant\exp\big(-c(\log\log T_{n+1})^{-\varepsilon'/2}\log\log T_{n+1}\big)\\
&\geqslant(n+1)^{1/2},
\end{aligned}\tag{2. 3. 30$'$}$$

where $\varepsilon'=\dfrac{2\varepsilon^2}{2+\varepsilon}>0$.

From the proof of Theorem 2. 3. 2, we get the following corollary.

Corollary 2. 3. 2 *Suppose that the conditions* (i) *and* (ii) *of Theorem* 2. 3. 2 *are fulfilled. If* (2. 3. 51) *holds, then*

$$\lim_{T\to\infty}\inf\sup_{R\in L_T}\gamma_T|W(R)|=1 \text{ a. s.}$$

Furthermore, if $\lim_{T\to\infty}a_T/T<1$ *or* (iii) *holds, then*

$$\lim_{T\to\infty}\inf\sup_{R\in L_T^*}\gamma_T W(R)=1 \text{ a. s.}$$

Conjecture We conjecture that if the conditions in Step 2 (i. e., (i), (ii)$''$ and (iv)$'$) are sastisfied, then

$$\lim_{T\to\infty}\inf\sup_{R\in L_T}\widetilde{\gamma}_T|W(R)|=\lim_{T\to\infty}\inf\sup_{R\in L_T^*}\widetilde{\gamma}_T|W(R)|=1 \text{ a. s.}$$

Theorems 2. 3. 1 and 2. 3. 2 can be extended to a two-parameter fractional Wiener process, and the two conditions that a_T and Ta_T^{-1} are non-descreasing functions of T and that δ_T or γ_T is a properly non-increasing function of T can be removed.

Let $\{Z(x,y);x,y\geqslant0\}$ be a two-parameter fractional Wiener process of order $\alpha(0<\alpha<1)$, i. e., it is a real valued Gaussian process with mean zero, $Z(0,0)=0$ a. s. and the covariance function

$$\begin{aligned}
&EZ(x_1,y_1)Z(x_2,y_2)\\
&=\{|x_1|^{2\alpha}+|x_2|^{2\alpha}-|x_2-x_1|^{2\alpha}\}\{|y_1|^{2\alpha}+|y_2|^{2\alpha}-|y_2-y_1|^{2\alpha}\}/4.
\end{aligned}$$

It is obvious that $Z(\cdot,\cdot)$ is a two-parameter Wiener process if $\alpha=1/2$. Now, redefined δ_T and γ_T by

$$\delta_T=\{2a_T^{2\alpha}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)+\log\log T)\}^{-1/2},$$
$$\gamma_T=\{2a_T^{2\alpha}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)-\log\log\log T)\}^{-1/2}.$$

Zhang, Lu and Wang (2001) obtained the following results on the increments of $Z(\cdot,\cdot)$.

Theorem 2. 3. 4 *Let* $0<a_T\leqslant T$ *and* $b_T\geqslant\sqrt{T}$ *be two functions of* T. *Suppose that* b_T *is quasi-increasing. Then*

$$\lim_{T\to\infty}\sup\sup_{R\in L_T^*}\delta_T Z(R)=\lim_{T\to\infty}\sup\sup_{R\in L_T}\delta_T|Z(R)|=1 \text{ a. s.}$$

Furthermore, if the condition (2. 3. 2) *in Theorem* 2. 3. 1 *is also satisfied, then*

$$\lim_{T\to\infty}\sup_{R\in L_T^*}\delta_T Z(R)=\lim_{T\to\infty}\sup_{R\in L_T}\delta_T|Z(R)|=1 \text{ a. s.}$$

Theorem 2. 3. 5 *Let* $0<a_T\leqslant T$ *and* $b_T\geqslant\sqrt{T}$ *be two functions of* T. *Suppose that* b_T *is quasi-increasing and the condition* (iii) *in Theorem* 2. 3. 2 *is satisfied. Then*

$$\lim_{T\to\infty}\inf\sup_{R\in L_T^*}\gamma_T Z(R)=\lim_{T\to\infty}\inf\sup_{R\in L_T}\gamma_T|Z(R)|=1 \text{ a. s.}$$

The idea of the proofs of Theorems 2. 3. 4 and 2. 3. 5 is something similar to that of the proofs of Theorems 2. 3. 1,

2.3.2 and 2.2.4. And the proofs are too complicated to be presented here.

2.4 Two-parameter Fractional Lévy-Wiener Process

Let $\{X(x,y);0\leqslant x,y<\infty\}$ be a two-parameter fractional Lévy-Wiener process of order α with $0<\alpha<1$, that is, let $\{X(x,y);0\leqslant x,y<\infty\}$ be an almost surely continuous, real-valued Gaussian process with mean zero, $X(0,0)=0$ a.s. and stationary increments

$$E\{X(x_1,y_1)-X(x_2,y_2)\}^2=\{(x_1-x_2)^2+(y_1-y_2)^2\}^{\alpha} \tag{2.4.1}$$

for any non-negative x_1,y_1,x_2,y_2. If $\alpha=1/2$, then $\{X(x,y);0\leqslant x,y<\infty\}$ is a two-parameter Lévy-Wiener process. A two-parameter Lévy-Wiener process is a stochastic process $\{L(x,y);0\leqslant x,y<\infty\}$ such that

(a) $L(0,0)=0$,

(b) $L(x,y)$ is a centered Gaussian variable,

(c) $E\{L(x_1,y_1)-L(x_2,y_2)\}^2=\{(x_1-x_2)^2+(y_1-y_2)^2\}^{1/2}$,

(d) the sample paths $(x,y)\to L(\omega;x,y)$ are almost surely continuous in (x,y).

Let us consider the rectangle $R:=R(s,t,u,v)=[s,s+t]\times[u,u+v]\subset\mathbf{R}_+^2$ for all $s,u\geqslant0$ and $t,v>0$, and define the increment $X(R)$ of the two-parameter fractional Lévy-Wiener process on R by

$$X(R):=X(R(s,t,u,v))$$

$$=X(s+t,u+v)-X(s,u+v)-X(s+t,u)+X(s,u).$$

Using the property (2.4.1), it is easy to see that the standard deviation of $X(R)$ has the translation invariance with respect to s and u. Put

$$S(t,v)=\{E(X(R(s,t,u,v)))^2\}^{1/2}.$$

For $0<T<\infty$, let A_T and B_T be non-decreasing continuous functions and let a_T and b_T be continuous functions with $0<a_T\leqslant A_T$ and $0<b_T\leqslant B_T$. Denote

$$G_T=\left(\frac{A_T-a_T}{a_T}\vee 1\right)\left(\frac{B_T-b_T}{b_T}\vee 1\right),$$

$$\beta_T=\{2(\log G_T+\log\log A_T+\log\log B_T)\}^{1/2}$$

and

$$D_1(A_T,B_T,a_T,b_T)$$

$$=\sup_{0\leqslant s\leqslant A_T-a_T}\ \sup_{0\leqslant t\leqslant a_T}\ \sup_{0\leqslant u\leqslant B_T-b_T}\ \sup_{0\leqslant v\leqslant b_T}\frac{|X(R(s,t,u,v))|}{S(a_T,b_T)\beta_T},$$

$$D_2(A_T,B_T,a_T,b_T)=\sup_{0\leqslant s\leqslant A_T-a_T}\ \sup_{0\leqslant u\leqslant B_T-b_T}\frac{|X(R(s,a_T,u,b_T))|}{S(a_T,b_T)\beta_T}.$$

Lin and Choi (1999) proved the following result.

Theorem 2.4.1 *Let $\{X(x,y);0\leqslant x,y<\infty\}$ be a two-parameter fractional Lévy-Wiener process of order α with $0<\alpha<1$. For $0<T<\infty$, let A_T and B_T be non-decreasing continuous functions, and let a_T and b_T be continuous functions for which*

(i) $0<a_T\leqslant A_T$, $0<b_T\leqslant B_T$,

(ii) *when A_T is bounded, a_T tends to zero, otherwise, $\liminf_{T\to\infty}a_T>0$, there is the same relation between B_T and b_T,*

(iii) *for some $0<c_1\leqslant c_2<\infty$,*

$$c_1\leqslant\liminf_{T\to\infty}\frac{a_T}{b_T}\leqslant\limsup_{T\to\infty}\frac{a_T}{b_T}\leqslant c_2.$$

Then, we have

$$\limsup_{T\to\infty} D_1(A_T,B_T,a_T,b_T)\leqslant 1 \quad \text{a. s.} \tag{2.4.2}$$

If, in addition, the following condition is also satisfied:

(iv) $\lim_{T\to\infty}\log G_T/(\log\log A_T+\log\log B_T)=\infty$,

then, we have

$$\liminf_{T\to\infty} D_2(A_T,B_T,a_T,b_T)\geqslant 1 \quad \text{a. s.} \tag{2.4.3}$$

Therefore, under conditions (i)—(iv), we have

$$\lim_{T\to\infty} D_1(A_T,B_T,a_T,b_T)=\lim_{T\to\infty} D_2(A_T,B_T,a_T,b_T)=1 \quad \text{a. s.}$$

Remark 2.4.1 From the above theorem, when A_T, B_T, a_T, b_T all tend to infinity, we obtain the large increment results for $\{X(x,y)\}$; when A_T, B_T are bounded and a_T, b_T tend to zero, we obtain the moduli of continuity for $\{X(x,y)\}$.

Example 2.4.1 Let $\{X(x,y);0\leqslant x,y<\infty\}$ be a two-parameter Lévy-Wiener process with $\alpha=1/2$. When $A_T=T$, $B_T=T^{3/2}$, $a_T=\sqrt{T}$, $b_T=\sqrt{T}/2$, conditions (i)—(iv) are satisfied and $\beta_T\sim(3\log T)^{1/2}$, $S(a_T,b_T)=\sqrt{3-\sqrt{5}}\,T^{1/4}$ (cf. (2.4.15) below). Hence we have the large increment results

$$\lim_{T\to\infty}\sup_{0\leqslant s\leqslant T-\sqrt{T}}\ \sup_{0\leqslant t\leqslant\sqrt{T}}\ \sup_{0\leqslant u\leqslant T^{3/2}-\sqrt{T}/2}\ \sup_{0\leqslant v\leqslant\sqrt{T}/2}\frac{|X(R(s,t,u,v))|}{T^{1/4}(\log T)^{1/2}}$$

$$=\lim_{T\to\infty}\sup_{0\leqslant s\leqslant T-\sqrt{T}}\ \sup_{0\leqslant u\leqslant T^{3/2}-\sqrt{T}/2}\frac{|X(R(s,\sqrt{T},u,\sqrt{T}/2))|}{T^{1/4}(\log T)^{1/2}}$$

$$=\sqrt{3(3-\sqrt{5})}\quad \text{a. s.}$$

When $A_T=B_T=1$, $a_T=1/T$, $b_T=1/2T$, conditions (i)—(iv) are also satisfied and $\beta_T\sim 2(\log T)^{1/2}$, $S(a_T,b_T)=\sqrt{3-\sqrt{5}}\,T^{-1/2}$. Hence we have the moduli of continuity

$$\lim_{T\to\infty}\sup_{0\leqslant s\leqslant 1-1/T}\ \sup_{0\leqslant t\leqslant 1/T}\ \sup_{0\leqslant u\leqslant 1-1/2T}\ \sup_{0\leqslant v\leqslant 1/2T}\frac{|X(R(s,t,u,v))|}{T^{-1/2}(\log T)^{1/2}}$$

$$=\lim_{T\to\infty}\sup_{0\leqslant s\leqslant 1-1/T}\ \sup_{0\leqslant u\leqslant 1-1/2T}\frac{|X(R(s,T^{-1},u,T^{-1}/2))|}{T^{-1/2}(\log T)^{1/2}}$$

$$=2\sqrt{3-\sqrt{5}}\quad \text{a. s.}$$

The following Lemmas 2.4.1—2.4.4 are essential to prove the theorem. Let $\mathscr{D}=\{t;t=(t_1,\cdots,t_N),a_i\leqslant t_i\leqslant b_i,i=1,2,\cdots,d\}$ be a real d-dimensional time parameter space. We assume that the space $\mathscr{D}$ has the usual Euclidean norm $\|\cdot\|$. Let $\{X(t);t\in\mathscr{D}\}$ be a real valued separable Gaussian process with $EX(t)=0$. Suppose that

$$0<\sup_{t\in\mathscr{D}}E\{X(t)\}^2=:\Gamma^2<\infty,\ \Gamma>0, \tag{2.4.4}$$

and

$$E\{X(t)-X(s)\}^2\leqslant\varphi^2(\|t-s\|), \tag{2.4.5}$$

where $\varphi(\cdot)$ is a non-decreasing continuous function which satisfies $\int_0^\infty\varphi(e^{-y^2})dy<\infty$.

The following Lemma is a version of Fernique's inequality (Theorem 1.1.3) (cf. Choi and Lin 1998).

Lemma 2.4.1 *Let $\{X(t);t\in\mathscr{D}\}$ be given as the above statements. Then, for $\lambda>0$, $x\geqslant 1$ and $A>\sqrt{2d\log 2}$, we have*

$$P\left\{\sup_{t\in\mathscr{D}}X(t)\geqslant x\left(\Gamma+(2\sqrt{2}+2)A\int_0^\infty\varphi(\sqrt{d}\,\lambda 2^{-y^2})dy\right)\right\}$$

$$\leqslant(2^d+B)\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2},$$

where $B=\sum_{n=1}^\infty\exp\{-2^{n-1}(A^2-2d\log 2)\}<\infty$.

Proof For each $n=0,1,2,\cdots$, set $\varepsilon_n=\lambda 2^{-2^n}$, $\lambda>0$. For $\boldsymbol{k}=(k_1,\cdots,k_d)$, where $k_i=0,1,\cdots,k_{in}:=[(b_i-a_i)/\varepsilon_n]$, $i=1,\cdots,d$, define $t_k^{(n)}=(t_{1k_1}^{(n)},\cdots,t_{dk_d}^{(n)})$ in $\mathscr{D}$, where

$$t_{ik_i}^{(n)}=a_i+k_i\varepsilon_n,\ i=1,\cdots,d.$$

Let

$$S_n=\{t_k^{(n)};k=0,\cdots,k_n:=(k_{1n},\cdots,k_{dn})\},$$

which contains $N_n:=\prod_{i=1}^d k_{in}$ points,

$$N_n\leqslant 2^{2^n d}\prod_{i=1}^d(b_i-a_i)/\lambda.$$

Then the set $\bigcup_{n=0}^\infty S_n$ is dense in $\mathscr{D}$ and $S_n\subset S_{n+1}$. For $x\geqslant 1$ and $A>\sqrt{2d\log 2}$, denote

$$x_m=xA\varphi(\sqrt{d}\,\varepsilon_{m-1})2^{m/2},\quad m\geqslant 1.$$

For $m\geqslant 1$, let $\delta_m=2^{(m-1)/2}$. Then

$$2^{m/2}=2(\sqrt{2}+1)(\delta_m-\delta_{m-1}).$$

Thus

$$\begin{aligned}\sum_{m=1}^\infty x_m&=xA\sum_{m=1}^\infty\varphi(\sqrt{d}\,\lambda 2^{-2^{m-1}})2^{m/2}\\&=xA\sum_{m=1}^\infty\varphi(\sqrt{d}\,\lambda 2^{-\delta_m^2})(2\sqrt{2}+2)(\delta_m-\delta_{m-1})\\&\leqslant(2\sqrt{2}+2)xA\sum_{m=1}^\infty\int_{\delta_{m-1}}^{\delta_m}\varphi(\sqrt{d}\,\lambda 2^{-y^2})\mathrm{d}y\\&\leqslant(2\sqrt{2}+2)xA\int_0^\infty\varphi(\sqrt{d}\,\lambda 2^{-y^2})\mathrm{d}y.\end{aligned}\tag{2.4.6}$$

Therefore, by (2.4.6) we have

$$\begin{aligned}&P\left\{\sup_{t\in\mathscr{D}}X(t)>x\left(\Gamma+(2\sqrt{2}+2)A\int_0^\infty\varphi(\sqrt{d}\,\lambda 2^{-y^2})\mathrm{d}y\right)\right\}\\&\leqslant P\left\{\sup_{t\in\mathscr{D}}X(t)\geqslant x\Gamma+\sum_{m=1}^\infty x_m\right\}\\&=P\left\{\max_{n\geqslant 0}\sup_{t\in S_n}X(t)\geqslant x\Gamma+\sum_{m=1}^\infty x_m\right\}\\&\leqslant\lim_{n\to\infty}P\left\{\sup_{t\in S_n}X(t)\geqslant x\Gamma+\sum_{m=1}^\infty x_m\right\}.\end{aligned}\tag{2.4.7}$$

For $n\geqslant 0$, let

$$A_n=\left\{\sup_{t\in S_n}X(t)\geqslant x\Gamma+\sum_{m=1}^\infty x_m\right\}.$$

By induction we have

$$\begin{aligned}P(A_n)&=P(A_n\cap A_{n-1})+P(A_n\cap A_{n-1}^c)\\&\leqslant P(A_{n-1})+P(A_n\cap A_{n-1}^c)\\&\leqslant P(A_{n-2})+P(A_{n-1}\cap A_{n-2}^c)+P(A_n\cap A_{n-1}^c)\\&\leqslant P(A_0)+\sum_{n=1}^\infty P(A_n\cap A_{n-1}^c)\\&\leqslant P(B_0)+\sum_{n=1}^\infty P(B_n\cap B_{n-1}^c),\end{aligned}\tag{2.4.8}$$

where

$$B_0=\{\sup_{t\in S_0}X(t)\geqslant x\Gamma\},\quad B_n=\left\{\sup_{t\in S_n}X(t)\geqslant\sum_{m=1}^\infty x_m\right\},\quad n\geqslant 1.$$

Now for $n\geqslant 1$, we have

$$\begin{aligned}&P(B_n\cap B_{n-1}^c)\\&=P\left\{\left\{\sup_{t\in S_n}X(t)\geqslant\sum_{m=1}^n x_m\right\}\cap\left\{\sup_{s\in S_{n-1}}X(s)<\sum_{m=1}^{n-1}x_m\right\}\right\}\\&\leqslant P\left\{\bigcup_{t\in S_n}\left\{X(t)\geqslant\sum_{m=1}^n x_m\right\}\cap\bigcup_{s\in S_{n-1}}\left\{X(s)<\sum_{m=1}^{n-1}x_m\right\}\right\}\\&\leqslant P\left\{\bigcup_{t\in S_n-S_{n-1}}\ \bigcup_{\substack{s\in S_{n-1}\\ \|t-s\|\leqslant\sqrt{d}\,\varepsilon_{n-1}}}\{X(t)-X(s)\geqslant x_n\}\right\}\\&\leqslant\sum_{t\in S_n}\ \sum_{\substack{s\in S_{n-1}\\ \|t-s\|\leqslant\sqrt{d}\,\varepsilon_{n-1}}}P\{X(t)-X(s)\geqslant x_n\}.\end{aligned}\tag{2.4.9}$$

But by the assumption (2.4.5), we get

$$\begin{aligned}&E\{X(t)-X(s)\}^2\\&\leqslant\varphi^2(\|t-s\|)\leqslant\varphi^2(\sqrt{d}\,\varepsilon_{n-1}),\quad n\geqslant 1.\end{aligned}\tag{2.4.10}$$

Hence, noting $A>\sqrt{2d\log 2}$, $x\geqslant 1$ and the fact that there is only one point s in the set $\{s\in S_{n-1};\|t-s\|\leqslant\sqrt{d}\,\varepsilon_{n-1}\}$ for any $t\in S_n$

$-S_{n-1}$, it follows from (2.4.10) that (2.4.9) implies

$$
\begin{aligned}
P(B_n\cap B_{n-1}^c) &\leqslant \sum_{t\in S_n}\ \sum_{\substack{s\in S_{n-1}\\ \|t-s\|\leqslant \sqrt{d}\,\varepsilon_{n-1}}} P\left\{N(0,1)\geqslant \frac{x_n}{\varphi(\sqrt{d}\,\varepsilon_{n-1})}\right\}\\
&\leqslant \left(\prod_{i=1}^d \frac{b_i-a_i}{\lambda}\right)(2^{2^n})^d P\left\{N(0,1)\geqslant \frac{x_n}{\varphi(\sqrt{d}\,\varepsilon_{n-1})}\right\}\\
&= 2^{2^n d}\left(\prod_{i=1}^d \frac{b_i-a_i}{\lambda}\right)P\{N(0,1)\geqslant Ax2^{n/2}\}\\
&\leqslant 2^{2^n d}\left(\prod_{i=1}^d \frac{b_i-a_i}{\lambda}\right)\frac{1}{2\sqrt{\pi}}e^{-A^2x^2 2^{n-1}}\\
&= \frac{1}{2\sqrt{\pi}}\left(\prod_{i=1}^d \frac{b_i-a_i}{\lambda}\right)2^{2^n d}e^{-(A^2 2^{n-1}-1/2)x^2}e^{-x^2/2}\\
&\leqslant \frac{1}{2\sqrt{\pi}}\left(\prod_{i=1}^d \frac{b_i-a_i}{\lambda}\right)e^{2^n d\log 2-2^{n-1}A^2+1/2}e^{-x^2/2}\\
&\leqslant e^{-2^n(A^2/2-d\log 2)}\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2},
\end{aligned}
$$

where $N(0,1)$ denotes the standard normal random variable. In particular, if $A>0$ is such that

$$\frac{A^2}{2}-d\log 2>0,$$

then

$$\sum_{n=1}^{\infty}P(B_n\cap B_{n-1}^c)\leqslant B\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2}, \qquad (2.4.11)$$

where

$$B=\sum_{n=1}^{\infty}\exp\{-2^{n-1}(A^2-2d\log 2)\}<\infty.$$

On the other hand,

$$
\begin{aligned}
P(B_0) &= P\{\sup_{t\in S_0}X(t)\geqslant x\Gamma\}\\
&\leqslant 2^d\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)P\{N(0,1)\geqslant x\}\\
&\leqslant 2^d\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2}. \qquad (2.4.12)
\end{aligned}
$$

From (2.4.8), (2.4.11) and (2.4.12) we obtain, for any $n\geqslant 0$,

$$P(A_n)\leqslant (2^d+B)\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2}. \qquad (2.4.13)$$

Thus by (2.4.13) the inequality (2.4.7) gives to

$$
\begin{aligned}
&P\left\{\sup_{t\in\mathscr{D}}X(t)>x\left(\Gamma+(2\sqrt{2}+2)A\int_0^{\infty}\varphi(\sqrt{d}\,\lambda 2^{-y^2})\mathrm{d}y\right)\right\}\\
&\leqslant (2^d+B)\left(\prod_{i=1}^d\left(\frac{b_i-a_i}{\lambda}\vee 1\right)\right)e^{-x^2/2}.
\end{aligned}
$$

This completes the proof of Lemma 2.4.1.

Lemma 2.4.2 *Let $p>0$ and let N, m and a be non-zero real numbers. Then there exists a constant c_0 such that*

$$
\begin{aligned}
&\left|\int_{\sqrt{a^2+N^2m^2p^2}}^{\sqrt{a^2+(Nm+1)^2p^2}}\mathrm{d}(x^{2\alpha})-\int_{\sqrt{a^2+(Nm-1)^2p^2}}^{\sqrt{a^2+N^2m^2p^2}}\mathrm{d}(x^{2\alpha})\right|\\
&\leqslant c_0\frac{\{a^2+(|Nm|+1)^2p^2\}^{\alpha}p^2}{a^2+(|Nm|-1)^2p^2}.
\end{aligned}
$$

Proof Set $b=(|Nm|-1)p$, $c=|Nm|p$ and $d=(|Nm|+1)p$. Then

$$
\begin{aligned}
&\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}}\mathrm{d}(x^{2\alpha})-\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}}\mathrm{d}(x^{2\alpha})\\
&=\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}}\mathrm{d}((x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha})\\
&\quad-\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}}\mathrm{d}(x^{2\alpha})\\
&=\int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}}\left(\frac{\mathrm{d}((x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha})}{\mathrm{d}x}\right.\\
&\quad\left.-\frac{\mathrm{d}(x^{2\alpha})}{\mathrm{d}x}\right)\mathrm{d}x+\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}}\frac{\mathrm{d}(x^{2\alpha})}{\mathrm{d}x}\mathrm{d}x
\end{aligned}
$$

$$= \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \left(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} \frac{\mathrm{d}^2(y^{2\alpha})}{\mathrm{d}y^2}\mathrm{d}y \right) \mathrm{d}x$$
$$+ \int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{\mathrm{d}(x^{2\alpha})}{\mathrm{d}x}\mathrm{d}x$$
$$=: I + J.$$

Let us estimate an upper bound for I. In fact there exists a constant $c_1>0$ such that

$$|I| = \left| \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \left(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} (2\alpha(2\alpha-1)\frac{y^{2\alpha}}{y^2})\mathrm{d}y \right) \mathrm{d}x \right|$$
$$\leqslant c_1 \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{(x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha}}{x^2}$$
$$\times (\sqrt{a^2+c^2}-\sqrt{a^2+b^2})\mathrm{d}x$$
$$\leqslant c_1 \frac{(a^2+d^2)^\alpha}{a^2+b^2}(\sqrt{a^2+d^2}-\sqrt{a^2+c^2})(\sqrt{a^2+c^2}-\sqrt{a^2+b^2})$$
$$= c_1 \frac{(a^2+d^2)^\alpha(d^2-c^2)(c^2-b^2)}{(a^2+b^2)(\sqrt{a^2+d^2}+\sqrt{a^2+c^2})(\sqrt{a^2+c^2}+\sqrt{a^2+b^2})}$$
$$\leqslant c_1 \frac{(a^2+d^2)^\alpha}{a^2+b^2}(d-c)(c-b)$$
$$= c_1 \frac{(a^2+(|Nm|+1)^2p^2)^\alpha p^2}{a^2+(|Nm|-1)^2p^2}.$$

As for J, we have, for some $c_2>0$,

$$J = \int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \left(2\alpha\frac{x^{2\alpha}}{x} \right) \mathrm{d}x$$
$$\leqslant c_2 \frac{(\sqrt{a^2+d^2})^{2\alpha}}{\sqrt{a^2+c^2}}(\sqrt{a^2+d^2}-\sqrt{a^2+c^2}$$
$$-(\sqrt{a^2+c^2}-\sqrt{a^2+b^2}))$$
$$= c_2 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}}\left(\frac{d^2-c^2}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}} \right.$$

$$\left. - \frac{c^2-b^2}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}} \right)$$
$$= c_2 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}}\left(\frac{2|Nm|+1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}} \right.$$
$$\left. - \frac{2|Nm|-1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}} \right) p^2$$
$$= c_2 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}}\left\{ \left(\frac{1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}} \right.\right.$$
$$\left. - \frac{1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}} \right)(2|Nm|)p^2$$
$$+ \left(\frac{1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}} \right.$$
$$\left.\left. + \frac{1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}} \right) p^2 \right\}$$
$$\leqslant c_2 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}} \frac{2p^2}{\sqrt{a^2+c^2}} = 2c_2 \frac{(a^2+d^2)^\alpha p^2}{a^2+c^2}$$
$$= 2c_2 \frac{(a^2+(|Nm|+1)^2p^2)^\alpha p^2}{a^2+N^2m^2p^2}.$$

Comparing the upper bounds of I and J, we have Lemma 2.4.2.

The following lemma can be found in Leadbetter et al. (1983).

Lemma 2.4.3 *Let $\{\xi_{ij}; i,j=1,2,\cdots,n\}$ be jointly standardized normal random variables with $Cov(\xi_{ij},\xi_{i'j'})=\Lambda_{ij}^{i'j'}$ such that*

$$\delta := \max_{(i,j)\neq(i'j')} |\Lambda_{ij}^{i'j'}| < 1.$$

Then for any real number u and integers $1\leqslant l_1<l_2<\cdots<l_f\leqslant n$ and $1\leqslant l'_1<l'_2<\cdots<l'_g\leqslant n$ with $f,g\leqslant n$,

$$P\{\max_{1\leqslant i\leqslant f}\max_{1\leqslant j\leqslant g}\xi_{l_i l'_j}\leqslant u\}$$
$$\leqslant \{\Phi(u)\}^{fg} + c\sum_{(i,j)\neq(i'j')}|\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right), \quad (2.4.14)$$

where $\lambda_{ij}^{i'j'}=\Lambda_{l_i l_j}^{l_{i'} l_{j'}}$, *and* $c=c(\delta)$ *is a constant independent of* n, u, f *and* g.

In order to establish the upper bound of the second term of the right hand side of (2.4.14), we establish the following lemma:

Lemma 2.4.4 *Let* $\{\xi_{ij}\}$, δ, f, g *and* $\lambda_{ij}^{i'j'}$ *be as in Lemma* 2.4.3. *Assume that*

$$|\lambda_{ij}^{i'j'}|<(|i-i'||j-j'|)^{-\nu},\ i\neq i', j\neq j',$$

and set $u=\sqrt{(2-\eta)\log(fg)}$, *where* ν *and* η *are positive constants such that* $0<\eta<(1-\delta)\nu/(1+\nu+\delta)$. *Then we have*

$$\sum := \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \leqslant c_0(fg)^{-\delta_0},$$

where $\delta_0=\{\nu(1-\delta)-\eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\}>0$ *and* c_0 *is a positive constant independent of* n, f *and* g.

Proof Let a be such that $0<a=(1+\eta\delta-\delta)/\{(1+\nu)(1+\delta)\}<1$. We split the sum $\sum$ into four parts as follows:

$$\begin{aligned}\sum = & \sum_{\substack{1\leqslant i,i'\leqslant f \\ 0<|i-i'|\leqslant[f^a]}} \sum_{\substack{1\leqslant j,j'\leqslant g \\ 0<|j-j'|\leqslant[g^a]}} |\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \\ & + \sum_{\substack{1\leqslant i,i'\leqslant f \\ |i-i'|>[f^a]}} \sum_{\substack{1\leqslant j,j'\leqslant g \\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \\ & + \sum_{\substack{1\leqslant i,i'\leqslant f \\ |i-i'|\leqslant[f^a]}} \sum_{\substack{1\leqslant j,j'\leqslant g \\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \\ & + \sum_{\substack{1\leqslant i,i'\leqslant f \\ |i-i'|>[f^a]}} \sum_{\substack{1\leqslant j,j'\leqslant g \\ |j-j'|\leqslant[g^a]}} |\lambda_{ij}^{i'j'}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\right) \\ =: & \sum{}^{(1)}+\sum{}^{(2)}+\sum{}^{(3)}+\sum{}^{(4)}.\end{aligned}$$

Now let us estimate each upper bound of the above four sums:

$$\begin{aligned}\sum{}^{(1)} & \leqslant c(fg)^{1+a}\exp\left(-\frac{2-\eta}{1+\delta}\log(fg)\right)=c(fg)^{1+a-\{(2-\eta)/(1+\delta)\}} \\ & =c(fg)^{\{\eta(1+\delta+\nu)-\nu(1-\delta)\}/\{(1+\nu)(1+\delta)\}}=c(fg)^{-\delta_0},\end{aligned}$$

$$\begin{aligned}\sum{}^{(2)} & \leqslant c(fg)^{2-a\nu}\exp(-(1-|\lambda_{ij}^{i'j'}|)u^2) \\ & \leqslant c(fg)^{2-a\nu}\exp(-(2-\eta)\log(fg)-(fg)^{-a\nu}(2-\eta)\log(fg)) \\ & \leqslant c(fg)^{\eta-a\nu}=c(fg)^{-\delta_0},\end{aligned}$$

$$\begin{aligned}\sum{}^{(3)} & \leqslant cf^{1+a}g^{2-a\nu}\exp\left(-\frac{(2-\eta)\log g+(2-\eta)\log f}{1+|\lambda_{ij}^{i'j'}|}\right) \\ & \leqslant cf^{1+a}\exp\left(-\frac{2-\eta}{1+\delta}\log f\right)g^{2-a\nu}\exp(-(1-g^{-a\nu})(2-\eta)\log g) \\ & \leqslant cf^{1+a-\{(2-\eta)/(1+\delta)\}}g^{\eta-a\nu}=c(fg)^{-\delta_0},\end{aligned}$$

$$\begin{aligned}\sum{}^{(4)} & \leqslant cf^{2-2a\nu}g^{1+a}\exp\left(-\frac{2-\eta}{1+\delta}\log g\right)\exp(-(2-\eta)(1-f^{-2a\nu})\log f) \\ & \leqslant cg^{1+a-\{(2-\eta)/(1+\delta)\}}f^{\eta-a\nu}=c(fg)^{-\delta_0}.\end{aligned}$$

We are now ready to prove Theorem 2.4.1.

Proof of Theorem 2.4.1 Using the relation $2ab=a^2+b^2-(a-b)^2$, we have, for all $t,v>0$,

$$S^2(t,v)=2\{t^{2a}+v^{2a}-(t^2+v^2)^a\}>0. \tag{2.4.15}$$

For integers k, j, l and r, let

$A_{kjlr}=\{T;\theta^{k-1}\leqslant A_T<\theta^k, \theta^{j-1}\leqslant a_T<\theta^j, \theta^{l-1}\leqslant B_T<\theta^l, \theta^{r-1}\leqslant b_T<\theta^r\}$

for any fixed $\theta>1$. We always consider such k, j, l and r that A_{kjlr} is non-empty. By condition (iii), we have

$$c_3\theta^r\leqslant\theta^j\leqslant c_4\theta^r \text{ and equivalently, } c_5\leqslant j-r\leqslant c_6$$

$$\text{for some } 0<c_3\leqslant c_4<\infty \text{ and } c_5\leqslant c_6<\infty. \tag{2.4.16}$$

By investigating the function $f(x)=(x^{2a}+1)/(x^2+1)^a$ and by (2.4.16), we have

$$c_7\theta^{2a(j+1)}\leqslant S^2(\theta^j,\theta^r)\leqslant c_8\theta^{2a(j+1)},$$

$$c_9\theta^{2a(r+1)}\leqslant S^2(\theta^j,\theta^r)\leqslant c_{10}\theta^{2a(r+1)} \tag{2.4.17}$$

for some $0<c_7\leqslant c_8<\infty$ and $0<c_9\leqslant c_{10}<\infty$. Moreover,

$$\inf_{T \in A_{kjlr}} \beta_T$$

$$\geqslant \left\{ 2\left(\log\left(\left(\frac{\theta^{k-1}-\theta^j}{\theta^j} \vee 1 \right) \left(\frac{\theta^{l-1}-\theta^r}{\theta^r} \vee 1 \right) \right) + \log\log\theta^{k-1} + \log\log\theta^{l-1} \right) \right\}^{1/2}$$

$$\geqslant \theta^{-1} \left\{ 2\left(\log\left(\left(\frac{\theta^{k-1}-\theta^j}{\theta^j} \vee 1 \right) \left(\frac{\theta^{l-1}-\theta^r}{\theta^r} \vee 1 \right) \right) + \log\log\theta^{k} + \log\log\theta^{l} \right) \right\}^{1/2}$$

$$=: \theta^{-1}\beta_{kjlr} \tag{2.4.18}$$

for all large $k \wedge l$. By conditions (ii) and (iii), A_T and B_T are both either bounded or unbounded. In the bounded case, both a_T and b_T tend to zero, and hence (and noting conditing (i)) we have

$$j \leqslant k+1 \leqslant d_1 \text{ and } r \leqslant l+1 \leqslant d_2 \tag{2.4.19}$$

for some positive integers d_1 and d_2. In the unbounded case, the limit inferiors of a_T and b_T are larger than zero, and hence (and also noting conditing (i)) we have

$$d_3 \leqslant j \leqslant k+1 \text{ and } d_4 \leqslant r \leqslant l+1 \tag{2.4.20}$$

for some positive integers d_3 and d_4. Next, we consider only the unbounded case. The bounded case can be dealed with in the same way. Noting that $S(t,v)$ is increasing on t and v and using (2.4.18), we can write

$$\limsup_{T\to\infty} D_1(A_T, B_T, a_T, b_T)$$

$$\leqslant \limsup_{\substack{k\to\infty \\ l\to\infty}} \sup_{\substack{d_3 \leqslant j \leqslant k+1 \\ d_4 \leqslant r \leqslant l+1 \\ c_5 \leqslant j-r \leqslant c_6}} \sup_{T \in A_{kjlr}} \sup_{\substack{0 \leqslant s \leqslant A_T - a_T \\ 0 \leqslant t \leqslant a_T}} \sup_{\substack{0 \leqslant u \leqslant B_T - b_T \\ 0 \leqslant v \leqslant b_T}} \frac{|X(R(s,t,u,v))|}{S(a_T, b_T)\beta_T}$$

$$\leqslant \limsup_{\substack{k\to\infty \\ l\to\infty}} \sup_{\substack{d_3 \leqslant j \leqslant k+1 \\ d_4 \leqslant r \leqslant l+1 \\ c_5 \leqslant j-r \leqslant c_6}} \sup_{\substack{0 \leqslant s \leqslant \theta^k - \theta^{j-1} \\ 0 \leqslant t \leqslant \theta^j}} \sup_{\substack{0 \leqslant u \leqslant \theta^l - \theta^{r-1} \\ 0 \leqslant v \leqslant \theta^r}} \frac{|X(R(s,t,u,v))|\theta^{1+\alpha}}{S(\theta^j, \theta^r)\beta_{kjlr}}. \tag{2.4.21}$$

Let $C_{kjlr} = \{(s,t,u,v); 0 \leqslant s \leqslant \theta^k - \theta^{j-1}, 0 \leqslant t \leqslant \theta^j, 0 \leqslant u \leqslant \theta^l - \theta^{r-1}, 0 \leqslant v \leqslant \theta^r\}$ be a four-dimensional set. In order to apply Lemma 2.4.1, we put

$$Y_{jr}(s,t,u,v) = \frac{X(R(s,t,u,v))}{S(\theta^j, \theta^r)}, \quad (t,s,u,v) \in C_{kjlr},$$

$$\varphi(z) = \frac{4(\sqrt{2}\,z)^\alpha}{S(\theta^j, \theta^r)}, \quad z > 0.$$

Clearly, $EY_{jr}(s,t,u,v) = 0$, $\Gamma^2 := \sup_{(s,t,u,v)\in C_{kjlr}} E\{Y_{jr}(s,t,u,v)\}^2 = 1$, and further

$$E\{X(R(s_1,t_1,u_1,v_1)) - X(R(s_2,t_2,u_2,v_2))\}^2$$
$$\leqslant 2E\{([X(s_1+t_1, u_1+v_1) - X(s_2+t_2, u_2+v_2)]$$
$$-[X(s_1, u_1+v_1) - X(s_2, u_2+v_2)])^2$$
$$+([X(s_2+t_2, u_2) - X(s_1+t_1, u_1)]$$
$$-[X(s_2, u_2) - X(s_1, u_1)])^2\}$$
$$\leqslant 4E\{[X(s_1+t_1, u_1+v_1) - X(s_2+t_2, u_2+v_2)]^2$$
$$+[X(s_1, u_1+v_1) - X(s_2, u_2+v_2)]^2$$
$$+[X(s_2+t_2, u_2) - X(s_1+t_1, u_1)]^2 + [X(s_2, u_2) - X(s_1, u_1)]^2\}$$
$$\leqslant 16\{(s_1+t_1-s_2-t_2)^2 + (u_1+v_1-u_2-v_2)^2\}^\alpha$$
$$\leqslant 16 \times 2^\alpha \left\{ \sqrt{(s_1-s_2)^2 + (t_1-t_2)^2 + (u_1-u_2)^2 + (v_1-v_2)^2} \right\}^{2\alpha}.$$

Thus we obtain

$$E\{Y_{jr}(s_1,t_1,u_1,v_1) - Y_{jr}(s_2,t_2,u_2,v_2)\}^2$$
$$\leqslant \varphi^2\left(\sqrt{(s_1-s_2)^2 + (t_1-t_2)^2 + (u_1-u_2)^2 + (v_1-v_2)^2} \right).$$

On the other hand, it follows from (2.4.17) that, for any given $\varepsilon' > 0$, there exists a small constant $\delta = \delta(\varepsilon') > 0$ such that

$$(2\sqrt{2} + 2)A\int_0^\infty \varphi(2\delta\theta^j 2^{-y^2})\,dy$$

$$= 8(\sqrt{2} + 1)A\int_0^\infty (2\sqrt{2}\,\delta\theta^j 2^{-y^2})^\alpha \frac{1}{S(\theta^j, \theta^r)}\,dy$$

$$\leqslant 4(\sqrt{2}+1)A\frac{(2\sqrt{2}\delta\theta^j)^\alpha}{\sqrt{c_7\theta^{\alpha(j+1)}}}\sqrt{\frac{\pi}{\alpha\log 2}}$$

$$<\frac{\varepsilon'}{8},$$

where A is defined in Lemma 2. 4. 1, c_7 is a positive constant. For any given $\varepsilon>0$, take $0<\varepsilon'<2\varepsilon$. Then it follows from Lemma 2. 4. 1 that

$$P\left\{\sup_{\substack{0\leqslant s\leqslant\theta^k-\theta^{j-1}\\0\leqslant t\leqslant\theta^j}}\sup_{\substack{0\leqslant u\leqslant\theta^l-\theta^{r-1}\\0\leqslant v\leqslant\theta^r}}\frac{|X(R(s,t,u,v))|}{S(\theta^j,\theta^r)\beta_{kjlr}}\geqslant 1+\varepsilon\right\}$$

$$\leqslant 2P\left\{\sup_{\substack{0\leqslant s\leqslant\theta^k-\theta^{j-1}\\0\leqslant t\leqslant\theta^j}}\sup_{\substack{0\leqslant u\leqslant\theta^l-\theta^{r-1}\\0\leqslant v\leqslant\theta^r}}Y_{jr}(s,t,u,v)\geqslant\sqrt{1+\varepsilon}\beta_{kjlr}\left(1+\frac{\varepsilon'}{8}\right)\right\}$$

$$\leqslant 2P\left\{\sup_{\substack{0\leqslant s\leqslant\theta^k-\theta^{j-1}\\0\leqslant t\leqslant\theta^j}}\sup_{\substack{0\leqslant u\leqslant\theta^l-\theta^{r-1}\\0\leqslant v\leqslant\theta^r}}Y_{jr}(s,t,u,v)\right.$$

$$\left.\geqslant\sqrt{1+\varepsilon}\beta_{kjlr}\left(1+(2\sqrt{2}+2)A\int_0^\infty\varphi(2\delta\theta^j 2^{-y^2})\mathrm{d}y\right)\right\}$$

$$\leqslant c\left(\frac{\theta^k-\theta^{j-1}}{\delta\theta^j}\vee 1\right)\frac{1}{\delta}\left(\frac{\theta^l-\theta^{r-1}}{\delta\theta^j}\vee 1\right)\left(\frac{\theta^r}{\delta\theta^j}\vee 1\right)\exp\left\{-\frac{1+\varepsilon}{2}\beta_{kjlr}^2\right\}$$

$$\leqslant c\delta^{-4}\theta^{k-j+l-j+r-j}(\theta^{k-j+l-r}kl)^{-(1+\varepsilon)}$$

$$\leqslant c\delta^{-4}c_3^{-2}\theta^{-\varepsilon(k-j+l-r)}(kl)^{-(1+\varepsilon)},$$

which implies

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=d_3}^{k+1}\sum_{r=d_4}^{l+1}P\left\{\sup_{\substack{0\leqslant s\leqslant\theta^k-\theta^{j-1}\\0\leqslant t\leqslant\theta^j}}\sup_{\substack{0\leqslant u\leqslant\theta^l-\theta^{r-1}\\0\leqslant v\leqslant\theta^r}}\frac{|X(R(s,t,u,v))|}{S(\theta^j,\theta^r)\beta_{kjlr}}\geqslant 1+\varepsilon\right\}$$

$$<\infty.$$

Then the Borel-Cantelli lemma and (2. 4. 21) yield (2. 4. 2) since θ is arbitrary.

Now we turn to prove (2. 4. 3) and consider only the case that A_T and B_T are both unbounded as well. Similar to (2. 4. 21), write

$$\liminf_{T\to\infty}D_2(A_T,B_T,a_T,b_T)$$

$$\geqslant\liminf_{\substack{k\to\infty\\l\to\infty}}\inf_{\substack{d_3\leqslant j\leqslant k+1\\d_4\leqslant r\leqslant l+1\\c_5\leqslant j-r\leqslant c_6}}\sup_{0\leqslant s\leqslant\theta^{k-1}-\theta^j}\sup_{0\leqslant u\leqslant\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)\theta\beta_{kjlr}}$$

$$-\limsup_{\substack{k\to\infty\\l\to\infty}}\sup_{\substack{d_3\leqslant j\leqslant k+1\\d_4\leqslant r\leqslant l+1\\c_5\leqslant j-r\leqslant c_6}}\sup_{0\leqslant s\leqslant\theta^{k-1}-\theta^j}\sup_{0\leqslant u\leqslant\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|\theta^{1+\alpha}}{S(\theta^j,\theta^r)\beta_{kjlr}}$$

$$=:J_1-J_2. \tag{2. 4. 22}$$

Imitating the proof of (2. 4. 2) and comparing the ranges of t, v in the right hand sides of (2. 4. 21) and J_2, we have, for any $\varepsilon>0$,

$$J_2\leqslant\varepsilon\quad\text{a. s.} \tag{2. 4. 23}$$

provided θ is near one enough. Consider J_1. For given large N, we define positive integers f_{kj} and g_{lr} by

$$f_{kj}=\left[\frac{\theta^{k-1}-\theta^j}{N\theta^j}\vee 1\right]\text{ and }g_{lr}=\left[\frac{\theta^{l-1}-\theta^r}{N\theta^r}\vee 1\right].$$

For $p=0,1,\cdots,f_{kj}$ and $q=0,1,\cdots,g_{lr}$, we also define incremental random variables

$$X_{jr}(R_{pq}):=X(R(Np\theta^j,\theta^j,Nq\theta^r,\theta^r)).$$

It follows from (iv) that, for any $0<\varepsilon'<\varepsilon<1$,

$$P\left\{\sup_{0\leqslant s\leqslant\theta^{k-1}-\theta^j}\sup_{0\leqslant u\leqslant\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)\beta_{kjlr}}<\sqrt{1-\varepsilon}\right\}$$

$$\leqslant P\left\{\sup_{0\leqslant s\leqslant\theta^{k-1}-\theta^j}\sup_{0\leqslant u\leqslant\theta^{l-1}-\theta^r}\frac{|X(R(s,\theta^j,u,\theta^r))|}{S(\theta^j,\theta^r)}\right.$$

$$\left.<\left(2(1-\varepsilon')\log(f_{kj}g_{lr})\right)^{1/2}\right\}$$

$$\leqslant P\left\{\max_{0\leqslant p\leqslant f_{kj}}\max_{0\leqslant q\leqslant g_{lr}}\frac{X_{jr}(R_{pq})}{S(\theta^j,\theta^r)}<\{2(1-\varepsilon')\log(f_{kj}g_{lr})\}^{1/2}\right\}, \tag{2. 4. 24}$$

provided $k\wedge l$ is large enough. Define the correlation function of

$X_{jr}(R_{pq})$ and $X_{jr}(R_{p'q'})$:

$$\lambda_{jr}(p,q,p',q')=\text{Correlation}(X_{jr}(R_{pq}),X_{jr}(R_{p'q'})),$$
$$p\neq p',q\neq q',$$

and let $h=p-p',m=q-q'$. By the relation $2ab=a^2+b^2-(a-b)^2$, we obtain

$$\begin{aligned}
&|\mathrm{Cov}(X_{jr}(R_{pq}),X_{jr}(R_{p'q'}))|\\
&\leqslant|\{[(Nh\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha-[(Nh\theta^j)^2+(Nm\theta^r)^2]^\alpha\}\\
&\quad-\{[(Nh\theta^j)^2+(Nm\theta^r)^2]^\alpha-[(Nh\theta^j)^2+(Nm\theta^r-\theta^r)^2]^\alpha\}|\\
&\quad+\frac{1}{2}|\{[(Nh\theta^j-\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha-[(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2]^\alpha\}\\
&\quad-\{[(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2]^\alpha-[(Nh\theta^j-\theta^j)^2+(Nm\theta^r-\theta^r)^2]^\alpha\}|\\
&\quad+\frac{1}{2}|\{[(Nh\theta^j+\theta^j)^2+(Nm\theta^r+\theta^r)^2]^\alpha-[(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2]^\alpha\}\\
&\quad-\{[(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2]^\alpha-[(Nh\theta^j+\theta^j)^2+(Nm\theta^r-\theta^r)^2]^\alpha\}|\\
&=\left|\int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^r+\theta^r)^2}}\mathrm{d}(x^{2\alpha})-\int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^r)^2}}\mathrm{d}(x^{2\alpha})\right|\\
&\quad+\frac{1}{2}\left|\int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r+\theta^r)^2}}\mathrm{d}(x^{2\alpha})-\int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^r)^2}}\mathrm{d}(x^{2\alpha})\right|\\
&\quad+\frac{1}{2}\left|\int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r+\theta^r)^2}}\mathrm{d}(x^{2\alpha})-\int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r-\theta^r)^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^r)^2}}\mathrm{d}(x^{2\alpha})\right|.
\end{aligned}$$

Without loss of generality, assume that $h>0$ and $m>0$. Applying Lemma 2.4.2 for $a=Nh\theta^j$, $Nh\theta^j-\theta^j$, $Nh\theta^j+\theta^j$ and $p=\theta^r$, respectively, we obtain

$$\begin{aligned}
&|\mathrm{Cov}(X_{jr}(R_{pq}),X_{jr}(R_{p'q'}))|\\
&\leqslant c\,\frac{\{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{2r}\}^\alpha\theta^{2r}}{(Nh-1)^2\theta^{2j}+(Nm-1)^2\theta^{2r}}.
\end{aligned}$$

Thus, by (2.4.16) and (2.4.17), we have, for all large $k\wedge l$ and N,

$$\begin{aligned}
|\lambda_{jr}(p,q,p',q')|&\leqslant c\,\frac{\{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{2r}\}^\alpha\theta^{2r}}{\{(Nh-1)^2\theta^{2j}+(Nm-1)^2\theta^{2r}\}S^2(\theta^j,\theta^r)}\\
&\leqslant c\{(Nh\theta^j)^2+(Nm\theta^r)^2\}^{\alpha-1}\theta^{2r}/S^2(\theta^j,\theta^r)\\
&\leqslant cc_9^{-1}\theta^{-2\alpha}\{(c_3Nh)^2+(Nm)^2\}^{\alpha-1}\\
&\leqslant(h^2+m^2)^{\alpha-1}\leqslant(2hm)^{\alpha-1}<(hm)^{-\nu},
\end{aligned}$$

where $\nu=1-\alpha>0$. In order to estimate an upper bound for the right hand side of (2.4.24), let us now apply Lemmas 2.4.3 and 2.4.4 for

$$\xi_{l_p l'_q}=X_{jr}(R_{pq})/S(\theta^j,\theta^r),\ p=0,1,\cdots,f_{kj},\quad q=0,1,\cdots,g_{lr},$$
$$|\lambda_{pq}^{p'q'}|=|\lambda_{jr}(p,q,p',q')|<(|hm|)^{-\nu},$$
$$h=p-p'\neq0,m=q-q'\neq0,$$
$$u=u_{kjlr}=\{(2-\eta)\log(f_{kj}g_{lr})\}^{1/2},\ \eta=2\varepsilon'<\frac{(1-\delta)\nu}{1+\nu+\delta},$$
$$f=f_{kj},\ g=g_{lr}.$$

Then the right hand side of (2.4.24) is less than or equal to

$$\{\Phi(u_{kjlr})\}^{(f_{kj}+1)(g_{lr}+1)}+c(f_{kj}g_{lr})^{-\delta_0}$$

for some $\delta_0>0$ and large $k\wedge l$. Thus we have, from (2.4.24),

$$\begin{aligned}
&P\left\{\sup_{0\leqslant s\leqslant\theta^{k-1}-\theta^j}\ \sup_{0\leqslant u\leqslant\theta^{l-1}-\theta^r}\frac{X(R(s,\theta^j,u,\theta^r))}{S(\theta^j,\theta^r)\beta_{kjlr}}<\sqrt{1-\varepsilon}\right\}\\
&\leqslant\exp\{-c_0((f_{kj}+1)(g_{lr}+1))^{\varepsilon'}\}+c(f_{kj}g_{lr})^{-\delta_0}\\
&\leqslant c(f_{kj}g_{lr})^{-\delta_0}\\
&\leqslant c\theta^{-\delta_0(k-j+l-r)}. \qquad (2.4.25)
\end{aligned}$$

Condition (iv) implies one of the following conditions:

(a) for any given $M>0$, $k-j\geqslant M\log k$ and $l-r\geqslant M\log l$, provided $k\wedge l$ is large enough;

(b) for any given $M>0$, $k-j\geqslant M\log k$ and $k\geqslant Ml$, provided $k\wedge l$ is large enough;

(c) for any given $M>0$, $l-r\geqslant M\log l$ and $l\geqslant Mk$, provided $k\wedge l$ is large enough.

If condition (a) is satisfied, we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=d_3}^{k-M\log k}\sum_{r=d_4}^{l-M\log l}\theta^{-\delta_0(k-j+l-r)}$$

$$\leqslant c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\theta^{-\delta_0 M(\log k+\log l)}<\infty$$

by taking M large enough. If condition (b) is satisfied, we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{[k/M]}\sum_{j=d_3}^{k-M\log k}\sum_{r=d_4}^{l+1}\theta^{-\delta_0(k-j+l-r)}\leqslant c\sum_{k=1}^{\infty}\frac{k}{M}\theta^{-\delta_0 M\log k}<\infty$$

by taking M large enough. For condition (c), we have the same convergence. Hence, by (2.4.25) and the Borel-Cantelli lemma, we obtain

$$J_1\leqslant\sqrt{1-\varepsilon}\quad \text{a.s.} \tag{2.4.26}$$

Combining (2.4.22), (2.4.23) with (2.4.26) yields (2.4.3). This completes the proof of Theorem 2.4.1.

2.5 Two-parameter Ornstein-Uhlenbeck Process

Let $X(\cdot)$ be an Ornstein-Uhlenbeck process with coefficients $\gamma\geqslant 0$ and $\lambda>0$. In Section 2.1.5, we mentioned that it is a stationary solution of the stochastic differential equation

$$\mathrm{d}X(t)=-\lambda X(t)+(2\gamma)^{1/2}\mathrm{d}W(t),$$

where $\{W(t);-\infty<t<\infty\}$ is a standard Wiener process. In general, the above stochastic differential equation with the boundary condition

$$X(0)=X_0$$

has an unique solution

$$X(t)=\mathrm{e}^{-\lambda t}\left\{X_0+(2\gamma)^{1/2}\int_0^t \mathrm{e}^{\lambda s}\mathrm{d}W(s)\right\},$$

where X_0 is a random variable independent of the process $W(\cdot)$. This is an Ornstein-Uhlenbeck process with coefficients γ and λ which starts at X_0. A two-parameter Ornstein-Uhlenbeck process (OUP_2) is an extension of this one-parameter process.

A two-parameter Ornstein-Uhlenbeck process $\{X(t,v);t\geqslant 0,v\geqslant 0\}$ is defined by

$$X(t,v)=\mathrm{e}^{-\alpha t-\beta v}\left\{X_0+\sigma\int_0^t\int_0^v \mathrm{e}^{\alpha x+\beta y}\mathrm{d}W(x,y)\right\},\ t\geqslant 0,v\geqslant 0, \tag{2.5.1}$$

where $\alpha>0$ and $\beta>0$ are two coefficients, $W(\cdot,\cdot)$ is a two-parameter Wiener process, X_0 is a random variable independent of $W(\cdot,\cdot)$. This defintion was introduced by Wang (1983). If X_0 is a Gaussian variable, then $X(\cdot,\cdot)$ is a Gaussian process. We denote

$$J(t,v)=\mathrm{e}^{-\alpha t-\beta v}\int_0^t\int_0^v \mathrm{e}^{\alpha x+\beta y}\mathrm{d}W(x,y).$$

Then it is easy to see that

$$\begin{aligned}&EJ(t_1,v_1)J(t_2,v_2)\\&=\mathrm{e}^{-\alpha(t_1+t_2)-\beta(v_1+v_2)}\cdot(\mathrm{e}^{2\alpha(t_1\wedge t_2)}-1)(\mathrm{e}^{2\beta(v_1\wedge v_2)}-1)/(4\alpha\beta).\end{aligned}$$

One can show that the OUP_2 is a process whose increments are neither independent nor stationary. On the other hand, for any fixed $c>0$, $\{X(t,c);t\geqslant 0\}$ is an one-parameter Ornstein-Uhlenbeck process which starts at $X(0,c)$. In fact,

$$X(t,c)=\mathrm{e}^{-\alpha t}X(0,c)+\sigma J(t,c),$$

and $EJ(t,c)=0$,

$$E[J(t_1,c)J(t_2,c)]=E[J(t_1)J(t_2)],$$

where

$$J(t) = \sqrt{\frac{1-e^{-2\beta c}}{2\beta}} e^{-\alpha t}\int_0^t e^{\alpha s} dW(s).$$

So, $\{X(t,c); t \geqslant 0\}$ has the same distribution as the distribution of the one-parameter Ornstein-Uhlenbeck process:

$$\left\{\widetilde{X}(t) := e^{-\alpha t}X(0,c) + \sigma\sqrt{\frac{1-e^{-2\beta c}}{2\beta}} e^{-\alpha t}\int_0^t e^{\alpha s} dW(s);\ t \geqslant 0\right\}.$$

Similarly, for any fixed $c > 0$, $\{X(c,s); s \geqslant 0\}$ is an one-parameter Ornstein-Uhlenbeck process which starts at $X(c,0)$. Wang (1983) investigated some Markov properties of OUP_2 in his paper. Chen (1989) studied the sample path properties of OUP_2 by giving Hausdoff dimension of the graph and image sets of this process. Lin (1995 a,b) gave some direct depictions of sample path properties of this process by establishing its Lévy's moduli of continuity and limit theorems on its large increments.

For simplicity, we assume that $\sigma = 1$, $EX_0 = 0$, $EX_0^2 = 1$, $E\exp(tX_0^2) < \infty$ for any $0 < t < \frac{1}{2}$. Consider the increments

$$\begin{aligned}
&X(t+s,v) - X(t,v) \\
&= e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})X_0 \\
&\quad + e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})\int_0^{t+s}\int_0^v e^{\alpha x+\beta y} dW(x,y) \\
&\quad + e^{-\alpha t-\beta v}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y} dW(x,y) \\
&=: \xi_1(t,s,v) + \xi_2(t,s,v) + \xi_3(t,s,v) \qquad (2.5.2)
\end{aligned}$$

and

$$\begin{aligned}
&X(R(t,s,v,u)) \\
&:= X(t+s,v+u) - X(t+s,v) - X(t,v+u) + X(t,v) \\
&= e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})(1-e^{\beta u})X_0
\end{aligned}$$

$$\begin{aligned}
&+ e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})\int_0^t\int_v^{v+u} e^{\alpha x+\beta y} dW(x,y) \\
&+ e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\alpha s})(1-e^{\beta u})\int_0^t\int_0^v e^{\alpha x+\beta y} dW(x,y) \\
&+ e^{-\alpha(t+s)-\beta(v+u)}\int_t^{t+s}\int_v^{v+u} e^{\alpha x+\beta y} dW(x,y) \\
&+ e^{-\alpha(t+s)-\beta(v+u)}(1-e^{\beta u})\int_t^{t+s}\int_0^v e^{\alpha x+\beta y} dW(x,y), \qquad (2.5.3)
\end{aligned}$$

where $R(t,s,v,u) = [t,t+s] \times [v,v+u]$. Hence

$$\begin{aligned}
&E(X(t+s,v) - X(t,v))^2 \\
&= e^{-2\alpha t-2\beta v}(1-e^{-\alpha s})^2 \\
&\quad + \frac{1}{4\alpha\beta}(1-e^{-2\beta v})\{(1-e^{-2\alpha t})(1-e^{-\alpha s})^2 + 1 - e^{-2\alpha s}\} \qquad (2.5.4)
\end{aligned}$$

and

$$\begin{aligned}
EX^2(R(t,s,v,u)) &= e^{-2\alpha t-2\beta v}(1-e^{-\alpha s})^2(1-e^{-\beta u})^2 \\
&\quad + \frac{1}{4\alpha\beta}(1-e^{-\alpha s})^2(1-e^{-2\alpha t})(1-e^{-2\beta u}) \\
&\quad + \frac{1}{4\alpha\beta}(1-e^{-\alpha s})^2(1-e^{-\beta u})^2(1-e^{-2\alpha t})(1-e^{-2\beta v}) \\
&\quad + \frac{1}{4\alpha\beta}(1-e^{-2\alpha s})(1-e^{-2\beta u}) \\
&\quad + \frac{1}{4\alpha\beta}(1-e^{-\beta u})^2(1-e^{-2\alpha s})(1-e^{-2\beta v}). \qquad (2.5.5)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&E(X(t+s,v) - X(t,v))^2 \\
&= \begin{cases} \alpha^2 s^2 e^{-2\alpha(t+s)-2\beta v} + \frac{s}{2\beta}(1-e^{-2\beta v}) + O(s^2) & \text{as } s \to 0, \\ e^{-2\alpha t-2\beta v} + \frac{1}{4\alpha\beta}(1-e^{-2\beta v})(2-e^{-2\alpha t}) + O(e^{-\alpha s}) & \text{as } s \to \infty \end{cases} \\
&=: \begin{cases} \sigma^2(t,s,v) + \sigma^2(s,v) + O(s^2) & \text{as } s \to 0, \\ \bar{\sigma}_1^2(t,v) + O(e^{-\alpha s}) & \text{as } s \to \infty \end{cases} \qquad (2.5.6)
\end{aligned}$$

and

$EX^2(R(t,s,v,u))$

$$=\begin{cases}\dfrac{1}{4\alpha\beta}(e^{2\alpha s}-1)(1-e^{-2\beta u})+O(su) & \text{as } s\to 0, u\to 0,\\ \bar{\sigma}_2^2(t,v)+O(e^{-\alpha s}+e^{-\beta u}) & \text{as } s\to\infty, u\to\infty,\end{cases} \tag{2.5.7}$$

where $\bar{\sigma}_2^2(t,v)=e^{-2\alpha t-2\beta v}+\dfrac{1}{4\alpha\beta}(2-e^{-2\alpha t})(2-e^{-2\beta v})$.

Also, it follows from (2.5.4) that for small s

$$E(X(t+s,v)-X(t,v))^2\leqslant\left(\alpha^2+\frac{\alpha}{4\beta}\right)s^2+\frac{1}{2\beta}s,$$

and by symmetry of $X(t,v)$ in t and v,

$$E(X(t,v+u)-X(t,v))^2\leqslant\left(\alpha^2+\frac{\alpha}{4\beta}\right)u^2+\frac{1}{2\beta}u$$

for small u. Hence

$$\begin{aligned}&E(X(t+s,v+u)-X(t,v))^2\\ &\leqslant 2E(X(t+s,v+u)-X(t+s,v))^2\\ &\quad+2E(X(t+s,v)-X(t,v))^2\\ &\leqslant 2\left\{\left(\alpha^2+\frac{\alpha}{4\beta}\right)(s^2+u^2)+\frac{1}{2\beta}(s+u)\right\},\end{aligned}$$

which together with Theorem 2.1.3 implies that $X(t,v)$ is almost surely continuous in (t,v).

2.5.1 The moduli of continuity of OUP_2

Put $\sigma_1(t,s,v)=\sigma(t,s,v)+\sigma(s,v)$, $\sigma_2(t,s,v)=\sigma(t,s,v)\wedge\sigma(s,v)$. It is easy to see that

$$\sigma(s,v)=o(\sigma(t,s,v)) \text{ as } v\to 0$$

for any fixed $t\geqslant 0$ and $s>0$.

As for moduli of continuity of $X(t,v)$ for each parameter, we have

Theorem 2.5.1 *Suppose that a_h is a function of h with $a_h=o(h^{-\delta})$ as $h\to 0$ for any $\delta>0$ and $\liminf_{h\to 0}a_h>0$. Then we have*

$$\lim_{h\to 0}\sup_{v>0}\sup_{0\leqslant t\leqslant a_h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\left\{2\left(\log\frac{1}{h}+\log\log\sigma_2^{-1}(t,h,v)\right)\right\}^{1/2}}=1 \text{ a.s.} \tag{2.5.8}$$

and

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\frac{|X(t+h,v)-X(t,v)|}{\sigma_1(t,h,v)\left\{2\left(\log\frac{1}{h}+\log\log\sigma_2^{-1}(t,h,v)\right)\right\}^{1/2}}=1 \text{ a.s.} \tag{2.5.9}$$

for any fixed $v>0$.

Remark 2.5.1 By symmetry of $X(t,v)$ in t and v. We can write alternatively

$$\lim_{h\to 0}\sup_{t>0}\sup_{0\leqslant v\leqslant a_h}\sup_{0\leqslant u\leqslant h}\frac{|X(t,v+u)-X(t,v)|}{\nu(t,v,h)}=1 \text{ a.s.}$$

and

$$\lim_{h\to 0}\sup_{0\leqslant v\leqslant a_h}\frac{|X(t,v+h)-X(t,v)|}{\nu(t,v,h)}=1 \text{ a.s.}$$

Where $\nu(t,v,h)$ is an analogue of the normalized factor in (2.5.8) and (2.5.9).

As for moduli of continuity of $X(t,v)$ for both parameters, we have

Theorem 2.5.2 *Suppose that a_h and b_h are functions of h with $\liminf_{h\to 0}a_hb_h>0$ and c_h is a continuous non-increasing function of h with $c_h\to 0$ and $a_hb_h=o((hc_h)^{-\delta})$ as $h\to 0$ for any $\delta>0$. Then we have*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b_h}\sup_{0<u\leqslant c_h}\frac{|X(R(t,s,v,u))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}=1 \text{ a.s.} \tag{2.5.10}$$

and

$$\lim_{h\to 0}\sup_{0\le t\le a_h}\sup_{0\le v\le b_h}\frac{|X(R(t,h,v,c_h))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}=1 \text{ a. s.} \tag{2.5.11}$$

In order to prove Theorems 2. 5. 1 and 2. 5. 2, we need some exponential inequalities.

Lemma 2. 5. 1 *For any $0<\varepsilon<1/2$, there exist $h=h(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for any fixed $t\geq 0$ and $0<s\leq h$*

$$P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,s,v)\{x^2+2\log\log\sigma_2^{-1}(t,s,v)\}^{1/2}}\geq 1+2\varepsilon\right\}$$

$$\leq C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}. \tag{2.5.12}$$

Proof Let $0<\theta<1$, $\delta>0$ be specified later on. Define v_k and v_k' by

$$\sigma^2(s,v_k)=\theta^k,\ k=k_0,\ k_0+1,\cdots,$$

where $k_0=[\log(\delta s/2\beta)/\log\theta]$, and

$$\sigma^2(t,s,v_k')=\theta^k,\ k=k_1,\ k_1+1,\cdots,$$

where $k_1=[\log\sigma^2(t,s,v_{k_0})/\log\theta]$. By the definition, it is easy to see that

$$v_k\to 0 \text{ and } v_k'\to\infty \text{ as } k\to\infty, \tag{2.5.13}$$

$$v_{k_1}'\leq v_{k_0}, \tag{2.5.14}$$

$$1-e^{-2\beta v_{k_0}+1}\leq\delta\leq 1-e^{-2\beta v_{k_0}} \tag{2.5.15}$$

and

$$\theta(1-e^{-2\beta v_k})=1-e^{-2\beta v_{k+1}}. \tag{2.5.16}$$

(2. 5. 15) and (2. 5. 16) imply that for $k\geq k_0$

$$e^{-2\beta(v_k-v_{k+1})}=1-(1-\theta)(1-e^{-2\beta v_k})e^{2\beta v_{k+1}}$$

$$\geq 1-\frac{1-\theta}{1-\delta}=\frac{\theta-\delta}{1-\delta}. \tag{2.5.17}$$

Moreover, obviously

$$e^{-2\beta(v_{k+1}'-v_k')}=\theta \tag{2.5.17$'$}$$

and

$$1-e^{-2\beta v_k'}=1-\frac{1}{\theta}e^{-2\beta v_{k+1}'}$$

$$=\frac{1}{\theta}(1-e^{-2\beta v_{k+1}'})-\left(\frac{1}{\theta}-1\right)$$

$$\geq\frac{1}{\sqrt{\theta}}(1-e^{-2\beta v_{k+1}'}) \tag{2.5.16$'$}$$

for $k\geq k_1$, provided that θ is close enough to 1 since

$$1-e^{-2\beta v_{k+1}'}\geq 1-e^{-2\beta v_{k_1+1}'}\geq 1-e^{-2\beta v_{k_0}}\geq\delta.$$

From (2. 5. 2) we have

$$P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,s,v)\{x^2+2\log\log\sigma_2^{-1}(t,s,v)\}^{1/2}}\geq 1+2\varepsilon\right\}$$

$$\leq\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geq(1+2\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'<v\leq v_{k+1}'}\frac{|\xi_1(t,s,v)|}{\sigma(t,s,v)}\geq(1+2\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geq\frac{\varepsilon}{2}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'\leq v<v_{k+1}'}\frac{|\xi_2(t,s,v)|}{\sigma(s,v)}\geq\frac{\varepsilon}{2}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$+\sum_{k=k_0}^{\infty}P\left\{\sup_{v_{k+1}<v\leq v_k}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geq\left(1+\frac{3\varepsilon}{2}\right)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$+\sum_{k=k_1}^{\infty}P\left\{\sup_{v_k'\leq v<v_{k+1}'}\frac{|\xi_3(t,s,v)|}{\sigma(s,v)}\geq\left(1+\frac{3\varepsilon}{2}\right)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$=:\sum_{j=1}^{6}p_j. \tag{2.5.18}$$

Estimate p_1 at first. By the assumption on X_0, for s small enough we have

$$p_1=\sum_{k=k_0}^{\infty}P\{(e^{\alpha s}-1)|X_0|\geq(1+2\varepsilon)\alpha s(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\}$$

$$\leqslant \sum_{k=k_0}^{\infty} P\{|X_0| \geqslant (1+\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\}$$

$$\leqslant \sum_{k=k_0}^{\infty} E\exp\left(\frac{1-\varepsilon/2}{2}X_0^2\right)$$

$$\times\exp\left\{-\frac{1}{2}(1+\varepsilon)(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\}\sum_{k=k_0}^{\infty} k^{-(1+\varepsilon)}$$

$$\leqslant c\exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\}. \tag{2.5.19}$$

For p_2 we have a similar estimation.

Consider p_3. Let

$$Y(v) = \int_0^{t+s}\int_0^{v} e^{\alpha x+\beta y}dW(x,y),$$

which is a Gaussian process with independent increments and

$$EY^2(v) = \frac{1}{4\alpha\beta}(e^{2\alpha(t+s)}-1)(e^{2\beta v}-1).$$

Noting (2.5.16) and (2.5.17), we have

$$p_3 \leqslant \sum_{k=k_0}^{\infty} P\left\{\sup_{v_{k+1}<v\leqslant v_k}|Y(v)| \geqslant \frac{\varepsilon}{2}e^{\alpha(t+s)+\beta v_{k+1}}(e^{\alpha s}-1)^{-1}\right.$$

$$\left.\times\sigma(s,v_{k+1})(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant 2\sum_{k=k_0}^{\infty} P\left\{|Y(v_k)|/(EY^2(v_k))^{\frac{1}{2}} \geqslant \frac{\varepsilon}{2}(EY^2(v_k))^{-\frac{1}{2}}e^{\alpha(t+s)+\beta v_{k+1}}\right.$$

$$\left.\times(e^{\alpha s}-1)^{-1}\sigma(s,v_{k+1})(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\varepsilon^2\theta}{8\alpha s}e^{-2\beta(v_k-v_{k+1})}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\varepsilon^2\theta(\theta-\delta)}{8\alpha s(1-\delta)}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\exp(-x^2) \tag{2.5.20}$$

provided that s is small enough. For p_4 we have a similar estimation by using (2.5.16)′ and (2.5.17)′ instead of (2.5.16) and (2.5.17).

We now turn to estimate p_5. Let

$$Z(v) = \int_t^{t+s}\int_0^{v} e^{\alpha x+\beta y}dW(x,y),$$

which is also a Gaussian process with independent increments and

$$EZ^2(v) = \frac{1}{4\alpha\beta}e^{2\alpha t}(e^{2\alpha s}-1)(e^{2\beta v}-1).$$

Similar to (2.5.20) we obtain

$$p_5 \leqslant \sum_{k=k_0}^{\infty} P\left\{\sup_{v_{k+1}<v\leqslant v_k}|Z(v)| \geqslant \left(1+\frac{3\varepsilon}{2}\right)e^{\alpha t+\beta v_{k+1}}\right.$$

$$\left.\times\sigma(s,v_{k+1})(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{1}{2}\left(1+\frac{3\varepsilon}{2}\right)\theta e^{-2\beta(v_k-v_{k+1})}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\sum_{k=k_0}^{\infty}\exp\left\{-\frac{\theta}{2}\left(1+\frac{3\varepsilon}{2}\right)\frac{\theta-\delta}{1-\delta}(x^2+2\log\log\theta^{-k})^{\frac{1}{2}}\right\}$$

$$\leqslant c\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\} \tag{2.5.21}$$

provided that θ is close enough to 1 and δ is small enough.

For p_6 we have a similar estimation.

Inserting these inequalities into (2.5.18), we obtain (2.5.12). Lemma 2.5.1 is proved.

Lemma 2.5.2 *Let $a>0$, $0<\varepsilon<1/2$. There exist $h=h(\varepsilon)>0$ and $C_1=C_1(\varepsilon)>0$ such that*

$$P\left\{\sup_{v>0}\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\}^{1/2}} \geqslant 1+4\varepsilon\right\}$$

$$\leqslant \frac{C_1 a}{h}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}. \tag{2.5.22}$$

Proof Without loss of generality, we assume that $x^2 \geqslant 2$. Let k be an integer specified later on and

$$t_j = [t2^j/h]h/2^j,\ j = k, k+1, \cdots,$$

for any $t \geqslant 0$. Since $X(t,v)$ is almost surely continuous in (t,v), we can write

$$\begin{aligned}|X(t+s,v)-X(t,v)| &\leqslant |X((t+s)_k,v)-X(t_k,v)| \\ &+ \sum_{j=0}^{\infty}|X((t+s)_{k+j+1},v) - X((t+s)_{k+j},v)| \\ &+ \sum_{j=0}^{\infty}|X(t_{k+j+1},v) - X(t_{k+j},v)|.\end{aligned} \tag{2.5.23}$$

By definitions, for h small enough, k large enough and $0 < s \leqslant h$,

$$\begin{aligned}\sigma^2(t_k,(t+s)_k - t_k, v) &\leqslant \alpha^2(1+2^{-k})^2 h^2 e^{-2\alpha(t-2^{-k}h)-2\beta v} \\ &\leqslant (1+\varepsilon/2)\sigma^2(t,h,v),\end{aligned}$$

$$\sigma^2((t+s)_k - t_k, v) \leqslant (1-2^{-k})\frac{h}{2\beta}(1-e^{-2\beta v}) \leqslant (1+\varepsilon/2)\sigma^2(h,v)$$

and

$$\begin{aligned}\sigma^2((t+s)_{k+j}, h/2^{k+j+1}, v) &\leqslant \alpha^2 2^{-2(k+j+1)} h^2 e^{-2\alpha t-2\beta v} \\ &\leqslant 2^{-(k+j+1)}\sigma^2(t,h,v),\end{aligned}$$

$$\sigma^2(h/2^{k+j+1}, v) \leqslant 2^{-(k+j+1)}\frac{h}{2\beta}(1-e^{-2\beta v}) \leqslant 2^{-(k+j+1)}\sigma^2(h,v).$$

From these inequalities and Lemma 2.5.1, we have

$$P\left\{\sup_{v>0}\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\frac{|X((t+s)_k,v)-X(t_k,v)|}{\sigma_1(t,h,v)\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\}^{1/2}} \geqslant 1+3\varepsilon\right\}$$

$$\leqslant c2^{2k}\frac{a}{h}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},$$

$$P\left\{\sup_{v>0}\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sum_{j=0}^{\infty}\frac{|X((t+s)_{k+j+1},v) - X((t+s)_{k+j},v)|}{\sigma_1(t,h,v)\{x^2+2\log\log\sigma_2^{-1}(t,h,v)\}^{1/2}}\right.$$

$$\left.\geqslant \frac{\varepsilon}{2}\right\}$$

$$\leqslant c\sum_{j=0}^{\infty}2^{2(k+j+1)}\frac{a}{h}\exp\left\{-\frac{(1+\varepsilon)\varepsilon^2}{8(1+2\varepsilon)^2}2^{k+j+1}x^2\right\}$$

$$\leqslant c\frac{a}{h}e^{-x^2}\sum_{j=0}^{\infty}2^{2(k+j+1)}\exp(-\varepsilon^2 2^{k+j+4})$$

$$\leqslant c\frac{a}{h}e^{-x^2}$$

provided that k is large enough, where we have used the inquality $bd \geqslant b+d$ for any $b \geqslant 2$ and $d \geqslant 2$. For the second series on the right hand side of (2.5.23), we have a similar estimation. Combining these inequalities with (2.5.23) yields (2.5.22).

Lemma 2.5.3 *Let $a>0$, $b>0$, $0<\varepsilon<1/2$. There exist $h=h(\varepsilon)>0$, $d=d(\varepsilon)>0$, $C_2=C_2(\varepsilon)>0$ such that*

$$P\left\{\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b}\sup_{0<u\leqslant d}\frac{|X(R(t,s,v,u))|}{(su)^{1/2}} \geqslant (1+2\varepsilon)x\right\}$$

$$\leqslant C_2\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\} \tag{2.5.24}$$

for any $x>0$.

Proof Without loss of generality, we assume that $x \geqslant \sqrt{2}$. Let k be an integer specified later on and

$$t_j = [t2^j/h]h/2^j,\ v_j' = [v2^j/d]d/2^j,\ j = k, k+1, \cdots,$$

for any $t \geqslant 0$, $v \geqslant 0$. Similarly to (2.5.23), we write

$$\begin{aligned}|X(R(t,s,v,u))| &\leqslant |X(R(t_k,(t+s)_k - t_k, v_k', (v+u)_k' - v_k'))| \\ &+ |X(R((t+s)_k,(t+s)-(t+s)_k, v_k', (v+u)_k' - v_k'))| \\ &+ |X(R(t_k, t-t_k, v_k', (v+u)_k' - v_k'))| + |X(R(t,s,v_k',v-v_k'))| \\ &+ |X(R(t,s,(v+u)_k',(v+u)-(v+u)_k'))| \\ &\leqslant |X(R(t_k,(t+s)_k - t_k, v_k', (v+u)_k' - v_k'))| \\ &+ \sum_{j=0}^{\infty}|X(R((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j}, v_k',\end{aligned}$$

$$(v+u)'_k - v'_k))|$$

$$+\sum_{j=0}^{\infty}|X(R(t_{k+j},t_{k+j+1}-t_{k+j},v'_k,(v+u)'_k-v'_k))|$$

$$+|X(R(t,s,v'_k,v-v'_k))|$$

$$+|X(R(t,s,(v+u)'_k,(v+u)-(v+u)'_k))|. \qquad (2.5.25)$$

Furthermore, by recalling (2.5.7), as $s\to 0$ and $u\to 0$,

$$EX^2(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))\leqslant(1+2^{-k})^2su+o(su),$$

$$EX^2(R((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k-v'_k))$$
$$=2^{-2(k+j+1)}su+o(2^{-2(k+j+1)}su),$$

and

$$EX^2(R(t,s,v'_k,v-v'_k))=2^{-k}su+o(su).$$

Therefore, for large k, small s and u,

$$P\left\{\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b}\sup_{0<u\leqslant d}\frac{|X(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))|}{(su)^{1/2}}\right.$$
$$\left.\geqslant(1+\varepsilon)x\right\}\leqslant 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},$$

$$P\left\{\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b}\sup_{0<u\leqslant d}\right.$$
$$\sum_{j=0}^{\infty}\frac{|X(R((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k-v'_k))|}{\sum_{j=0}^{\infty}(2^{-(j+1)}su)^{1/2}}$$
$$\left.\geqslant\frac{\sqrt{2}-1}{4}\varepsilon x\right\}$$

$$\leqslant\sum_{j=0}^{\infty}2^{4(k+j+1)}\frac{ab}{hd}\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b}\sup_{0<u\leqslant d}$$
$$P\left\{|X(R((t+s)_{k+j},(t+s)_{k+j+1}-(t+s)_{k+j},v'_k,(v+u)'_k\right.$$
$$\left.-v'_k))|\geqslant\frac{\sqrt{2}-1}{4}\varepsilon x2^{k+(j+1)/2}(2^{-2(k+j+1)}su)^{\frac{1}{2}}\right\}$$

$$\leqslant\frac{ab}{hd}\sum_{j=0}^{\infty}2^{4(k+j+1)}\exp\left\{-\frac{\varepsilon^2}{200}2^{2k+j+1}x^2\right\}$$

$$\leqslant c\,\frac{ab}{hd}e^{-x^2}.$$

For the second sum on the right hand side of (2.5.25), we have a similar estimation. As for $X(R(t,s,v'_k,v-v'_k))$, we have

$$P\left\{\sup_{0\leqslant t\leqslant a}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b}\sup_{0<u\leqslant d}\frac{|X(R(t,s,v'_k,v-v'_k))|}{(su)^{1/2}}\geqslant\frac{\varepsilon}{4}x\right\}$$
$$\leqslant 2^{4k}\frac{ab}{hd}\exp\left\{-\frac{\varepsilon}{40}2^kx^2\right\}\leqslant c\,\frac{ab}{hd}e^{-x^2}.$$

For the last term on the right hand side of (2.5.25), we have also a similar estimation. Combining these inequalities with (2.5.25) yields (2.5.24).

Proof of Theorem 2.5.1 First, we prove

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\leqslant t\leqslant a_h}\sup_{0<s\leqslant h}$$
$$\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\}^{1/2}}\leqslant 1 \text{ a.s.} \quad (2.5.26)$$

Without loss of generality, we assume that a_h is non-increasing for $0\leqslant h\leqslant 1$; otherwise we consider $a_h^*=\sup_{h\leqslant s\leqslant 1}a_s$ instead of a_h.

Let $0<\varepsilon<1/2$, $\theta=1-\varepsilon$. Define $h_j=\theta^j$. For j large enough, using Lemma 2.5.2 we obtain

$$P\left\{\sup_{v>0}\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h_j,v)\{2(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\}^{1/2}}\right.$$
$$\left.\geqslant 1+\varepsilon\right\}$$
$$\leqslant C_1\frac{a_{h_{j+1}}}{h_j}\exp\left\{-\left(1+\frac{\varepsilon}{4}\right)\log h_j^{-1}\right\}$$
$$\leqslant C_1\frac{(h_{j+1})^{-\varepsilon/8}}{h_j}h_j^{1+\varepsilon/4}\leqslant C_1\theta^{(j-1)\varepsilon/8},$$

which, in combination with the Borel-Cantelli lemma, implies

$$\limsup_{j\to\infty}\sup_{v>0}\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h_j,v)\{2(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\}^{1/2}}$$

$$\leqslant 1+\varepsilon \quad \text{a.s.}$$

Furthermore

$$\limsup_{h\to 0}\sup_{v>0}\sup_{0\leqslant t\leqslant a_h}\sup_{0<s\leqslant h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,h,v)\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\}^{1/2}}$$

$$\leqslant \limsup_{j\to\infty}\sup_{v>0}\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}$$

$$\frac{|X(t+s,v)-X(t,v)|}{\theta\sigma_1(t,h_j,v)\{2(\log h_j^{-1}+\log\log\sigma_2^{-1}(t,h_j,v))\}^{1/2}}$$

$$\leqslant (1-\varepsilon)^{-1}(1+\varepsilon) \quad \text{a.s.}$$

This proves (2.5.26) by the arbitrariness of ε.

Next, we prove that for fixed $v>0$

$$\liminf_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\frac{|X(t+h,v)-X(t,v)|}{\sigma_1(t,h,v)\{2(\log h^{-1}+\log\log\sigma_2^{-1}(t,h,v))\}^{1/2}}$$

$$\geqslant 1 \text{ a.s.} \tag{2.5.27}$$

Noting the fact that for any fixed $v>0$ and $t\geqslant 0$,

$$\sigma(t,h,v)=o(\sigma(h,v)) \text{ as } h\to 0$$

and recalling the proof of Lemma 2.5.1, we find that (2.5.27) is equivalent to

$$\liminf_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\frac{|\xi_3(t,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}\geqslant 1 \text{ a.s.} \tag{2.5.28}$$

Put $t_i=ih$, $i=0,1,\cdots$, $i_h:=[a_h/h]$. Since $\xi_3(t_i,h,v)(i=0,1,\cdots,i_h)$ are independent, we have for any $\varepsilon>0$

$$P\left\{\max_{0\leqslant i\leqslant i_h}\frac{|\xi_3(t_i,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}\leqslant 1-\varepsilon\right\}$$

$$=\prod_{i=0}^{i_h}\left\{1-P\left\{\frac{|\xi_3(t_i,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}>1-\varepsilon\right\}\right\}$$

$$\leqslant \prod_{i=0}^{i_h}\{1-\exp(-(1-\varepsilon)\log h^{-1})\}$$

$$\leqslant \exp(-i_h h^{1-\varepsilon})\leqslant \exp(-h^{\varepsilon/2}). \tag{2.5.29}$$

Let $h_k=k^{-1}$, (2.5.29) implies

$$\liminf_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\frac{|\xi_3(t,h,v)|}{\sigma(h,v)(2\log h^{-1})^{1/2}}$$

$$\geqslant \liminf_{k\to\infty}\max_{0\leqslant i\leqslant i_{h_k}}\frac{|\xi_3(t_i,h_k,v)|}{\sigma(h_k,v)(2\log h_k^{-1})^{1/2}}$$

$$\geqslant 1-\varepsilon \quad \text{a.s.}$$

Hence (2.5.27) is proved. Combining (2.5.26) and (2.5.27) yields the conclusion of Theorem 2.5.1.

Proof of Theorem 2.5.2 At first, we prove

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b_h}\sup_{0<u\leqslant c_h}\frac{|X(R(t,s,v,u))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}\leqslant 1 \text{ a.s.} \tag{2.5.30}$$

We also assume that a_h and b_h are non-increasing, otherwise we consider $a_h^*=\sup_{h\leqslant s\leqslant 1}a_s$ and $b_h^*=\sup_{h\leqslant s\leqslant 1}b_s$. Let $0<\varepsilon<1/2$, $\theta=1-\varepsilon$. Define h_j by $h_jc_{h_j}=\theta^j$, $j=0,1,\cdots$. Then by Lemma 2.5.3

$$P\left\{\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}\sup_{0\leqslant v\leqslant b_{h_{j+1}}}\sup_{0<u\leqslant c_{h_j}}\frac{|X(R(t,s,v,u))|}{(2h_jc_{h_j}\log(h_jc_{h_j})^{-1})^{\frac{1}{2}}}\geqslant 1+2\varepsilon\right\}$$

$$\leqslant C_2\frac{a_{h_{j+1}}b_{h_{j+1}}}{h_jc_{h_j}}\exp\{-(1+\varepsilon)\log(h_jc_{h_j})^{-1}\}$$

$$\leqslant C_2\frac{(h_{j+1}c_{h_{j+1}})^{-\varepsilon/2}}{h_jc_{h_j}}(h_jc_{h_j})^{1+\varepsilon}=C_2\theta^{(j-1)\varepsilon/2},$$

which implies

$$\limsup_{j\to\infty}\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}\sup_{0\leqslant v\leqslant b_{h_{j+1}}}\sup_{0<u\leqslant c_{h_j}}\frac{|X(R(t,s,v,u))|}{(2h_jc_{h_j}\log(h_jc_{h_j})^{-1})^{\frac{1}{2}}}$$

$$\leqslant 1+2\varepsilon \quad \text{a.s.}$$

Furthermore

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\sup_{0<s\leqslant h}\sup_{0\leqslant v\leqslant b_h}\sup_{0<u\leqslant c_h}\frac{|X(R(t,s,v,u))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}$$

$$\leqslant \limsup_{j\to\infty}\sup_{0\leqslant t\leqslant a_{h_{j+1}}}\sup_{0<s\leqslant h_j}\sup_{0\leqslant v\leqslant b_{h_{j+1}}}\sup_{0<u\leqslant c_{h_j}}\frac{|X(R(t,s,v,u))|}{(2h_jc_{h_j}\log(h_jc_{h_j})^{-1})^{\frac{1}{2}}}$$

$$\leqslant (1-\varepsilon)^{-\frac{1}{2}}(1+2\varepsilon) \text{ a. s.}$$

This proves (2. 5. 30) by the arbitrariness of ε.

Next we prove

$$\liminf_{h\to 0} \sup_{0\leqslant t\leqslant a_h} \sup_{0\leqslant v\leqslant b_h} \frac{|X(R(t,h,v,c_h))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}} \geqslant 1 \text{ a. s.} \tag{2. 5. 31}$$

Put $t_i=ih$, $i=0,1,\cdots$, $i_h:=[a_h/h]$, $v_j=jc_h$, $j=0,1,\cdots$, $j_h:=[b_h/c_h]$. Then for any given $\varepsilon>0$,

$$P\left\{\max_{0\leqslant i\leqslant i_h}\max_{0\leqslant j\leqslant j_h}\frac{|X(R(t_i,h,v_j,c_h))|}{(2hc_h\log(hc_h)^{-1})^{1/2}}\leqslant 1-\varepsilon\right\}$$

$$\leqslant \prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\left\{1-\left\{\frac{|X(R(t_i,h,v_j,c_h))|}{(2hc_h\log(hc_h)^{-1})^{1/2}}>1-\varepsilon\right\}\right\}$$

$$\leqslant \prod_{i=0}^{i_h}\prod_{j=0}^{j_h}\{1-\exp(-(1-\varepsilon)\log(hc_h)^{-1})\}$$

$$\leqslant \exp\{-i_h j_h (hc_h)^{1-\varepsilon}\}$$

$$\leqslant \exp\{-a_h b_h (hc_h)^{-\varepsilon}/2\}$$

$$\leqslant \exp\{-c(hc_h)^{-\varepsilon}\} \tag{2. 5. 32}$$

provided that h is small enough. Define h_k by $h_k c_{h_k}=k^{-1}$. Then (2. 5. 32) implies

$$\liminf_{h\to 0}\sup_{0\leqslant t\leqslant a_h}\sup_{0\leqslant v\leqslant b_h}\frac{|X(R(t,h,v,c_h))|}{(2hc_h\log(hc_h)^{-1})^{\frac{1}{2}}}$$

$$\geqslant \liminf_{k\to\infty}\max_{0\leqslant i\leqslant i_{h_k}}\max_{0\leqslant j\leqslant j_{h_k}}\frac{|X(R(t_i,h_k,v_j,c_{h_k}))|}{(2h_k c_{h_k}\log(h_k c_{h_k})^{-1})^{\frac{1}{2}}}$$

$$\geqslant 1-\varepsilon \quad \text{a. s.},$$

i. e., (2. 5. 31) holds true. (2. 5. 30) and (2. 5. 31) together yield the conclusion of Theorem 2. 5. 2.

2. 5. 2 On large increments of OUP_2

In the above subsection, we saw how big the increments of OUP_2 would be when one or two parameters get small increments. Now, we are going to study how big the increments of this process will be when one or two parameters get large increments.

Theorem 2. 5. 3 *Let a_T be a function of T with $0<a_T\leqslant T$ and $a_T\to\infty$ as $T\to\infty$. Then we have*

$$\limsup_{T\to\infty}\sup_{v>0}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,v)\{2[\log((T-a_T)a_T)+\log\hat{v}]\}^{1/2}}$$

$$\leqslant 1 \text{ a. s.} \tag{2. 5. 33}$$

where $\hat{v}=v\vee\log v^{-1}$ and $\log((T-a_T)a_T)$ means $\log(T-a_T)+\log a_T$. If, in addition, there exists $0\leqslant b<1$ such that

$$a_T=o(T^{b+\varepsilon}) \tag{2. 5. 34}$$

for any $\varepsilon>0$ as $T\to\infty$, then for any fixed $v>0$,

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\sigma_1(t,v)\{2\log((T-a_T)a_T)\}^{1/2}}\geqslant\left(\frac{1-b}{1+b}\right)^{1/2} \text{ a. s.} \tag{2. 5. 35}$$

and

$$\limsup_{T\to\infty}\frac{|X(T,v)-X(T-a_T,v)|}{\sigma_1(T,v)\{2\log((T-a_T)a_T)\}^{1/2}}\geqslant\left(\frac{1-b}{1+b}\right)^{1/2} \text{ a. s.} \tag{2. 5. 36}$$

Let $b=0$ in (2. 5. 34). Then an immediate consequence of Theorem 2. 5. 3 is

Corollary 2. 5. 1 *Suppose that $a_T=o(T^\varepsilon)$ for any $\varepsilon>0$ as $T\to\infty$. Then we have*

$$\lim_{T\to\infty}\sup_{v>0}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2(\log T+\log v)\}^{1/2}}=1 \text{ a.s.}$$

and for any fixed $v>0$,

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2\log T\}^{1/2}}=1 \text{ a.s.},$$

$$\lim_{T\to\infty}\sup\frac{|X(T,v)-X(T-a_T,v)|}{\bar{\sigma}_1(T,v)\{2\log T\}^{1/2}}=1 \text{ a.s.}$$

Remark 2.5.2 By symmetry of $X(t,v)$ in t and v, we can write an alternative result for $X(t,v+u)-X(t,v)$ as we do in Remark 2.5.1.

The following theorem is about the limit behavior of large increments of $X(t,v)$ for both two parameters.

Theorem 2.5.4 *Let a_T, b_T and V_T be functions of T with $0<a_T\leqslant T$, $0<b_T\leqslant V_T$ and $a_T\to\infty$, $b_T\to\infty$ as $T\to\infty$. Then we have*

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant v\leqslant V_T}\sup_{0\leqslant u\leqslant b_T}\frac{|X(R(t,s,v,u))|}{\bar{\sigma}_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}$$

$$\leqslant 1 \text{ a.s.} \tag{2.5.37}$$

If, in addition, there exists $0\leqslant b<1$ such that

$$a_Tb_T=o((TV_T)^{b+\varepsilon}) \tag{2.5.38}$$

for any $\varepsilon>0$ as $T\to\infty$, then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant v\leqslant V_T}\frac{|X(R(t,a_T,v,b_T))|}{\bar{\sigma}_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\geqslant\left(\frac{1-b}{1+b}\right)^{1/2} \text{ a.s.} \tag{2.5.39}$$

and

$$\lim_{T\to\infty}\sup\frac{|X(R(T,a_T,V_T,b_T))|}{\bar{\sigma}_2(T,V_T)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\geqslant\left(\frac{1-b}{1+b}\right)^{1/2} \text{ a.s.} \tag{2.5.40}$$

Similar to Corollary 2.5.1, we have

Corollary 2.5.2 *If $b=0$ in (2.5.38), then we have*

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant v\leqslant V_T}\sup_{0\leqslant u\leqslant b_T}\frac{|X(R(t,s,v,u))|}{\bar{\sigma}_2(t,v)\{2\log(TV_T)\}^{1/2}}=1 \text{ a.s.},$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant v\leqslant V_T}\frac{|X(R(t,a_T,v,b_T))|}{\bar{\sigma}_2(t,v)\{2\log(TV_T)\}^{1/2}}=1 \text{ a.s.},$$

$$\lim_{T\to\infty}\sup\frac{|X(R(T,a_T,V_T,b_T))|}{\bar{\sigma}_2(T,V_T)\{2\log(TV_T)\}^{1/2}}=1 \text{ a.s.}$$

In order to prove Theorems 2.5.3 and 2.5.4, we need the following continuity moduli results which can be proved along the lines of the proofs of (2.5.26) and (2.5.30).

Lemma 2.5.4 *For any positive functions h_T, w_T and V_T with $V_T\geqslant w_T$, $h_T\to 0$ and $w_T\to 0$ as $T\to\infty$, we have*

$$\lim_{T\to\infty}\sup\sup_{v>0}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h_T}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,h_T,v)\left\{2\left(\log\dfrac{T}{h_T}+\log\log\dfrac{1}{\bar{\sigma}_2(t,h_T,v)}\right)\right\}}$$

$$\leqslant 1 \text{ a.s.}, \tag{2.5.41}$$

$$\lim_{T\to\infty}\sup\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h_T}\sup_{0\leqslant v\leqslant V_T}\sup_{0\leqslant u\leqslant w_T}\frac{|X(R(t,s,v,u))|}{\{2h_Tw_T\log(TV_T/h_Tw_T)\}^{1/2}}$$

$$\leqslant 1 \text{ a.s.} \tag{2.5.42}$$

The following lemma is a version of Lemma 2.5.1.

Lemma 2.5.5 *For any $0<\varepsilon<1/2$, there exist $b=b(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for any fixed $t\geqslant 0$ and $s\geqslant b$,*

$$P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\geqslant 1+2\varepsilon\right\}\leqslant C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\}.$$

Proof Observe that $\bar{\sigma}_1(t,v)$ defined by (2.5.6) is increasing (non-increasing) on v if $(2-e^{-2\alpha t})/2\alpha>2\beta e^{-2\alpha t}$ ($(2-e^{-2\alpha t})/2\alpha\leqslant 2\beta e^{-2\alpha t}$). Without loss of generality, we assume that $\bar{\sigma}_1(t,v)$ is increasing on v. For any $\varepsilon>0$, let $\theta=\theta(\varepsilon)>1$. Define v_k by

$$\bar{\sigma}_1(t,v_k)=\theta^{-k}.$$

The number of v_k is finite (say k is from k_0 to k_1), since $\bar{\sigma}_1(t,v)$

increases from $e^{-2\alpha t}$ to $(2-e^{-2\alpha t})/4\alpha\beta$ as v changes from 0 to ∞. Moreover, define v_k' by $e^{-2\beta v_k'}=\theta^{-k}(k=0,1,\cdots)$. We put $\{v_k;k=k_0,\cdots,k_1\}$ and $\{v_k';k=0,1,\cdots\}$ together and get a new increasing sequence denoted by $\{v_k'';k=0,1,\cdots\}$. By the definition, $v_k''\leqslant\frac{k}{2\beta}\times\log\theta$. Let $K=[2\beta/\log\theta]+1$ and

$$\begin{aligned}Y_{t,s}(v)\ :=\ & e^{-\alpha(t+s)}(1-e^{\alpha s})X_0+e^{-\alpha(t+s)}(1-e^{\alpha s})\\ &\times\int_0^t\int_0^v e^{\alpha x+\beta y}dW(x,y)+e^{-\alpha(t+s)}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y),\end{aligned}$$

which is an independent increment Gaussian process for any fixed t and s. Obviously

$$e^{-2\beta v_{k-1}''}EY_{t,s}^2(v_k'')\leqslant\theta\bar{\sigma}_1^2(t,v_k'')\leqslant\theta^2\bar{\sigma}_1^2(t,v_{k-1}'').$$

Hence, recalling $\log x=\log(x\vee e)$, we have

$$\begin{aligned}&P\left\{\sup_{v>0}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\geqslant 1+2\varepsilon\right\}\\
&\leqslant\sum_{k=1}^{\infty}P\left\{\sup_{v_{k-1}''<v\leqslant v_k''}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)(x^2+2\log v)^{1/2}}\geqslant 1+2\varepsilon\right\}\\
&\leqslant\sum_{k=1}^{\infty}P\left\{\sup_{v_{k-1}''<v\leqslant v_k''}\frac{e^{-\beta v_{k-1}''}|Y_{t,s}(v)|}{\bar{\sigma}_1(t,v_{k-1}'')(x^2+2\log v_k'')^{1/2}}\geqslant 1+2\varepsilon\right\}\\
&\leqslant\sum_{k=1}^{\infty}\exp\left\{-\frac{1}{2}(1+2\varepsilon)\theta^{-2}(x^2+2\log v_{k+1}'')\right\}\\
&\leqslant\exp\left\{-\frac{1}{2}(1+\varepsilon)x^2\right\}\left\{K+\sum_{k=K}^{\infty}\exp\left(-(1+\varepsilon)\log\left(\frac{k}{2\beta}\log\theta\right)\right)\right\}\\
&\leqslant C\exp\left\{-\frac{1+\varepsilon}{2}x^2\right\},\end{aligned}$$

provided that θ is near one enough.

Proof of Theorem 2.5.3 First, we prove (2.5.33). For any given $h>0$ small enough, we have

$$\sup_{v>0}\sup_{0\leqslant t\leqslant T}\frac{\sigma_1(t,h,v)}{\bar{\sigma}_1(t,v)}\leqslant ah+(2ah)^{1/2}\leqslant ch^{1/2},$$

and further

$$\sup_{v>0}\sup_{0\leqslant t\leqslant T}\frac{\sigma_1(t,h,v)\left\{\log\frac{T}{h}+\log\log\frac{1}{\sigma_2(t,h,v)}\right\}^{1/2}}{\bar{\sigma}_1(t,v)\{\log((T-a_T)a_T)+\log v\}^{1/2}}\leqslant ch^{1/3}. \tag{2.5.43}$$

Hence, from (2.5.41) of Lemma 2.5.4, we obtain

$$\limsup_{T\to\infty}\sup_{v>0}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s,v)-X(t,v)|}{\bar{\sigma}_1(t,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}}\leqslant ch^{1/3}\ \text{a.s.} \tag{2.5.44}$$

Consequently, it suffices for the proof of (2.5.33) to show that for any $\varepsilon>0$,

$$\limsup_{T\to\infty}\sup_{v>0}\max_{0\leqslant j\leqslant j_T}\max_{0\leqslant i\leqslant i_T}\frac{|X((j+i)h,v)-X(jh,v)|}{\bar{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}}\leqslant 1+\varepsilon\ \text{a.s.} \tag{2.5.45}$$

where $j_T=[(T-a_T)/h]$, $i_T=[a_T/h]$. Set $A_k=\{T;\theta^k\leqslant a_T\leqslant\theta^{k+1}\}$ for some $\theta>1$, $\mathscr{A}=\{k;A_k\neq\varnothing\}$, T_k be such T that $T_k-a_{T_k}=\max\{T-a_T;T\in A_k\}$. Then

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{v>0}\max_{0\leqslant j\leqslant j_T}\max_{0\leqslant i\leqslant i_T}\frac{|X((j+i)h,v)-X(jh,v)|}{\bar{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}}\\
&\leqslant\limsup_{\substack{k\to\infty\\ k\in\mathscr{A}}}\sup_{v>0}\sup_{T\in A_k}\max_{0\leqslant j\leqslant j_T}\max_{0\leqslant i\leqslant i_T}\\
&\qquad\frac{|X((j+i)h,v)-X(jh,v)|}{\bar{\sigma}_1(jh,v)\{2(\log((T-a_T)a_T)+\log v)\}^{1/2}}\\
&\leqslant\limsup_{\substack{k\to\infty\\ k\in\mathscr{A}}}\sup_{v>0}\max_{0\leqslant j\leqslant j_{T_k}}\max_{0\leqslant i\leqslant\theta^{k+1}/h}\\
&\qquad\frac{|(X((j+i)h,v)-X(jh,v)|}{\bar{\sigma}_1(jh,v)\{2(\log((j+1)h\theta^k)+\log v)\}^{1/2}}.\end{aligned} \tag{2.5.46}$$

Using Lemma 2.5.5 and noting

$$E(X(t+s,v)-X(t,v))^2\leqslant\bar{\sigma}_1^2(t,v),$$

we obtain that for large k,

$$P\left\{\sup_{v>0}\max_{0\le j\le j_{T_k}}\max_{0\le i\le \theta^{k+1}/h}\frac{|X((j+i)h,v)-X(jh,v)|}{\sigma_1(jh,v)\{2(\log((j+1)h\theta^k)+\log v)\}^{1/2}}\right.$$

$$\left.\ge 1+2\varepsilon\right\}$$

$$\le \sum_{j=0}^{j_{T_k}}\sum_{i=0}^{[\theta^{k+1}/h]}P\left\{\sup_{v>0}\frac{|X((j+i)h,v)-X(jh,v)|}{\sigma_1(jh,v)\{2(\log((j+1)h\theta^k)+\log v)\}^{1/2}}\right.$$

$$\left.\ge 1+2\varepsilon\right\}$$

$$\le c\theta^{k+1}h^{-1}\sum_{j=0}^{j_{T_k}}\exp\{-(1+\varepsilon)\log((j+1)h\theta^k)\}$$

$$\le ch^{-2-\varepsilon}\theta^{1-k\varepsilon}\sum_{j=0}^{\infty}(j+1)^{-1-\varepsilon},\tag{2.5.47}$$

which, in combination with (2.5.46), implies (2.5.45). Thus (2.5.43) is proved.

Next, we prove (2.5.35). Using condition (2.5.34), we have

$$\limsup_{T\to\infty}\frac{\log((T-a_T)a_T)}{\log(T/a_T)}\le\frac{1+b}{1-b}.\tag{2.5.48}$$

Therefore, in order to prove (2.5.35), it is enough to show

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\sigma_1(t,v)(2\log(T/a_T))^{1/2}}\ge 1-\varepsilon \text{ a.s.}\tag{2.5.49}$$

for any $0<\varepsilon<1/4$. Let $B_{nk}=\{T;kh\le a_T<(k+1)h,n-1\le T<n\}$, $a_n'=\inf\{a_T;n-1\le T<n\}$, $a_n^*=\sup\{a_T;n-1\le T<n\}$. Then

$$\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\sigma_1(t,v)(2\log(T/a_T))^{1/2}}$$

$$\ge\liminf_{n\to\infty}\min_{a_n'/h-1\le k<a_n^*/h}\inf_{T\in B_{nk}}\sup_{0\le t\le T-a_T}\frac{|X(t+a_T,v)-X(t,v)|}{\sigma_1(t,v)(2\log(T/a_T))^{1/2}}$$

$$\ge\liminf_{n\to\infty}\min_{a_n'/h-1\le k<a_n^*/h}\sup_{0\le t\le n/2}\frac{|X(t+kh,v)-X(t,v)|}{\sigma_1(t,v)(2\log(n/kh))^{1/2}}$$

$$-\limsup_{n\to\infty}\sup_{0\le t\le n}\sup_{0\le s\le h}\frac{|X(t+s,v)-X(t,v)|}{\sigma_1(t,v)(2\log((n-1)/a_n^*))^{1/2}}$$

$$=: I_1-I_2.\tag{2.5.50}$$

From (2.5.44) and condition (2.5.34), we have

$$I_2\le ch^{1/3}\quad\text{a.s.}\tag{2.5.51}$$

Moreover,

$$I_1\ge\liminf_{n\to\infty}\min_{a_n'/h-1\le k<a_n^*/h}\max_{0\le j\le n/2kh}\frac{X((j+1)kh,v)-X(jkh,v)}{\sigma_1(jkh,v)(2\log(n/kh))^{1/2}}.\tag{2.5.52}$$

It is easy to verify that for large k

$$E\{X((j+1)kh,v)-X(jkh,v)\}\{X((i+1)kh,v)-X(ikh,v)\}\le 0\tag{2.5.53}$$

for $j\ne i$. Therefore, using Slepian's inequality and letting $G_j(j=0,1,\cdots)$ be independent standard normal random variables, we obtain

$$P\left\{\min_{a_n'/h-1\le k<a_n^*/h}\max_{0\le j\le n/2kh}\frac{X((j+1)kh,v)-X(jkh,v)}{\sigma_1(jkh,v)(2\log(n/kh))^{1/2}}\le 1-\varepsilon\right\}$$

$$\le\sum_{k=[a_n'/h]-1}^{[a_n^*/h]}P\left\{\max_{0\le j\le n/2kh}G_j\le(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}$$

$$\le\sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\left(P\left\{G_1\le(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}\right)^{[n/2kh]+1}$$

$$\le\sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\left(1-\exp\left\{-(1-\varepsilon)\log\frac{n}{kh}\right\}\right)^{n/2kh}$$

$$\le n^{b+\delta}h^{-1}\exp\left\{-\frac{1}{2}\left(\frac{n}{a_n^*}\right)^{\varepsilon}\right\}$$

$$\le n^{b+\delta}h^{-1}\exp\{-n^{(1-b-\delta)\varepsilon}\}\tag{2.5.54}$$

for $0<\delta<1-b$, which, in combination (2.5.52), implies $I_1\ge 1$

$-\varepsilon$. Inserting the estimations for I_1 and I_2 into (2.5.50) and by the arbitrary h, we obtain (2.5.49), and hence (2.5.35) is proved.

Finally, we show (2.5.36). Let $t_0=1$. Define t_k by $t_k=t_{k-1}+a_{t_{k-1}}$, $k=1,2,\cdots$. Put $D_n=\{k;\frac{1}{2}n\leqslant t_k\leqslant n-1\}$. Obviously, by condition (2.5.34), for $k\in D_n$ and for any $0<\delta<1-b$, we have $a_{t_k}=o(n^{b+\delta})$ and further

$$\sum_{k\in D_n}a_{t_k}\geqslant n-1-\frac{n}{2}-\max_{k\in D_{n-1}}a_{t_k}\geqslant\frac{1}{3}n$$

for all large n. Moreover, we have a relation similar to (2.5.53). Therefore, letting $t'=t'(t)$ be a solution of $t'-a_{t'}=t$, we have for $n-1<T\leqslant n$,

$$\begin{aligned}
&P\left\{\sup_{T/2\leqslant t\leqslant T}\frac{|X(t+a_{t'},v)-X(t,v)|}{\bar\sigma_1(t,v)(2\log T)^{1/2}}\leqslant(1-\varepsilon)(1-b)^{1/2}\right\}\\
&\leqslant P\left\{\max_{k\in D_n}\frac{X(t_k+a_{t_k'},v)-X(t_k,v)}{\bar\sigma_1(t_k,v)(2\log(n/a_{t_k}))^{1/2}}\leqslant 1-\varepsilon/2\right\}\\
&\leqslant\prod_{k\in D_n}\left(1-\exp\left\{-(1-\varepsilon/2)\log\frac{n}{a_{t_k}}\right\}\right)\\
&\leqslant\exp\left\{-\sum_{k\in D_n}(a_{t_k}/n)^{1-\varepsilon/2}\right\}\\
&\leqslant\exp\left\{-c\left(n/\max_{k\in D_n}a_{t_k}\right)^{\varepsilon/2}\right\}\\
&\leqslant\exp(-cn^{(1-b-\delta)\varepsilon/2})\to 0 \qquad (2.5.55)
\end{aligned}$$

as $n\to\infty$. Hence, noting that

$$\bar\sigma_1^2(t,v)\to(1-e^{-2\beta v})/2\alpha\beta=:\bar\sigma_1^2(v)$$

as $t\to\infty$, we obtain

$$\begin{aligned}
&P\left\{\sup_{\frac{1}{2}T\leqslant t-a_t\leqslant T}\frac{|X(t,v)-X(t-a_t,v)|}{\bar\sigma_1(t,v)\{2\log((t-a_t)a_t)\}^{1/2}}\geqslant(1-\varepsilon)\left(\frac{1-b}{1+b}\right)^{1/2}\right\}\\
&\geqslant P\left\{\sup_{\frac{1}{2}T\leqslant t\leqslant T}\frac{|X(t+a_{t'},v)-X(t,v)|}{\bar\sigma_1(v)(2\log T)^{1/2}}\geqslant\left(1-\frac{\varepsilon}{2}\right)(1-b)^{1/2}\right\}\\
&\to 1
\end{aligned}$$

as $T\to\infty$. Hence the proof of (2.5.36) is completed.

Proof of Theorem 2.5.4 The proof is similar to that of Theorem 2.5.3. We outline only the difference.

From (2.5.42) of Lemma 2.5.4 and noting $\inf_{t,v}\bar\sigma_2(t,v)>0$, we obtain

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\sup_{0\leqslant v\leqslant V_T}\sup_{0\leqslant u\leqslant h}\frac{|X(R(t,s,v,u))|}{\bar\sigma_2(t,v)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\\
&\leqslant ch^{2/3}\ \text{a.s.} \qquad (2.5.56)
\end{aligned}$$

Therefore (2.5.37) follows from

$$\begin{aligned}
&\limsup_{T\to\infty}\max_{0\leqslant j\leqslant j_T}\max_{0\leqslant i\leqslant i_T}\max_{0\leqslant r\leqslant r_T}\max_{0\leqslant l\leqslant l_T}\frac{|X(R(jh,ih,rh,lh))|}{\bar\sigma_2(jh,rh)\{2\log(Ta_TV_Tb_T)\}^{1/2}}\\
&\leqslant 1+\varepsilon\ \text{a.s.} \qquad (2.5.57)
\end{aligned}$$

for any $\varepsilon>0$, where $j_T=[T/h]$, $i_T=[a_T/h]$, $r_T=[V_T/h]$, $l_T=[b_T/h]$. Let $A_{kl}=\{T;\theta^k\leqslant a_T<\theta^{k+1},\theta^l\leqslant b_T<\theta^{l+1}\}$ for some $\theta>1$, $\mathscr{A}=\{(k,l);A_{kl}\neq\varnothing\}$, $T_{kl}=\sup\{T;T\in A_{kl}\}$, $V_{T'_{kl}}=\sup\{V_T;T\in A_{kl}\}$. Then the left hand side of (2.5.57) does not exceed

$$\limsup_{\substack{k,l\to\infty\\(k,l)\in\mathscr{A}}}\max_{0\leqslant j\leqslant j_{T_{kl}}}\max_{0\leqslant i\leqslant\theta^{k+1}/h}\max_{0\leqslant r\leqslant r_{T'_{kl}}}\max_{0\leqslant w\leqslant\theta^{l+1}/h}\frac{|X(R(jh,ih,rh,wh))|}{\bar\sigma_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}}.$$

Noting that $EX^2(R(t,s,v,u))\leqslant\bar\sigma_2^2(t,v)$, we obtain

$$P\left\{\max_{0\leqslant j\leqslant j_{T_{kl}}}\max_{0\leqslant i\leqslant\theta^{k+1}/h}\max_{0\leqslant r\leqslant r_{T'_{kl}}}\max_{0\leqslant w\leqslant\theta^{l+1}/h}\frac{|X(R(jh,ih,rh,wh))|}{\bar\sigma_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}}\geqslant 1+\varepsilon\right\}$$

$$\leqslant \sum_{j=0}^{j_{T_{kl}}} \sum_{i=0}^{[\theta^{k+1}/h]} \sum_{r=0}^{r_{T'_{kl}}} \sum_{w=0}^{[\theta^{l+1}/h]} P\left\{\frac{|X(R(jh,ih,rh,wh))|}{\bar{\sigma}_2(jh,rh)\{2\log((j+1)(r+1)h^2\theta^{k+l})\}^{1/2}} \geqslant 1+\varepsilon\right\}$$

$$\leqslant c\theta^{k+l+2}h^{-2}\sum_{j=0}^{j_{T_{kl}}}\sum_{r=0}^{r_{T'_{kl}}}\exp\{-(1+\varepsilon)\log((j+1)(r+1)h^2\theta^{k+l})\}$$

$$\leqslant ch^{-4-2\varepsilon}\theta^{2-\varepsilon(k+l)}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}(j+1)^{-(1+\varepsilon)}(r+1)^{-(1+\varepsilon)}.$$

Thus, using the Borel-Cantelli lemma, whose generalization to the case of two indices is trivial, we obtain (2. 5. 57), and hence (2. 5. 37).

We turn to (2. 5. 39). From condition (2. 5. 38), (2. 5. 39) is implied by

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant v\leqslant V_T}\frac{|X(R(t,a_T,v,b_T))|}{\bar{\sigma}_2(t,v)\{2\log(TV_T/(a_Tb_T))\}^{1/2}}\geqslant 1-\varepsilon \quad \text{a. s.}$$

Let $B_{mnkl}=\{T;kh\leqslant a_T\leqslant(k+1)h, lh\leqslant b_T\leqslant(l+1)h, m-1\leqslant T<m, n-1\leqslant V_T<n\}$, $a_m^*=\sup\{a_T;m-1\leqslant T<m\}$, $a'_m=\inf\{a_T;m-1\leqslant T<m\}$, $b_n^*=\sup\{b_T;n-1\leqslant V_T<n\}$, $b'_n=\inf\{b_T;n-1\leqslant V_T<n\}$. Then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant v\leqslant V_T}\frac{|X(R(t,a_T,v,b_T))|}{\bar{\sigma}_2(t,v)\{2\log(TV_T/(a_Tb_T))\}^{1/2}}$$

$$\geqslant\liminf_{m,n\to\infty}\min_{\substack{a'_m/h-1\leqslant k<a_m^*/h\\ b'_n/h-1\leqslant l<b_n^*/h}}\inf_{T\in B_{mnkl}}\sup_{\substack{0\leqslant t\leqslant T\\ 0\leqslant v\leqslant V_T}}\frac{|X(R(t,kh,v,lh))|}{\bar{\sigma}_2(t,v)\{2\log(mn/(klh^2))\}^{1/2}}$$

$$-\limsup_{m,n\to\infty}\sup_{\substack{0\leqslant t\leqslant m\\ 0\leqslant s\leqslant h}}\sup_{\substack{0\leqslant v\leqslant n\\ 0\leqslant u\leqslant h}}\frac{|X(R(t,s,v,u))|}{\bar{\sigma}_2(t,v)\{2\log((m-1)(n-1)/(a_m^*b_m^*))\}^{1/2}}$$

$$=: J_1-J_2.$$

From (2. 5. 56) and (2. 5. 38) we have

$$J_2\leqslant ch^{2/3} \quad \text{a. s.}$$

As for J_1,

$$J_1\geqslant\liminf_{m,n\to\infty}\min_{\substack{a'_m/h-1\leqslant k<a_m^*/h\\ b'_n/h-1\leqslant l<b_n^*/h}}\max_{\substack{0\leqslant j\leqslant m/2kh\\ 0\leqslant i\leqslant n/2lh}}\frac{X(R(jkh,kh,ilh\ lh))}{\bar{\sigma}_2(jkh,ilh)\{2\log(mn/(klh^2))\}^{1/2}}.$$

We can verify that

$$EX(R(jkh,kh,ilh,lh))X(R(pkh,kh,qlh,lh))\leqslant 0 \quad (2.5.58)$$

for large k and/or l, $j\neq p$ and/or $i\neq q$ via elementary calculations. The following proof is similar to that of Theorem 2. 5. 3 (see (2. 5. 54)), and hence omitted.

Imitating the proof of (2. 5. 36), we can show (2. 5. 40) by using the non-positive correlation similar to (2. 5. 58). It will not be presented here.

2.6 Kernel Generated Two-parameter Gaussian Processes

In the previous section, we studied the two-parameter Ornstein-Uhlenbeck process (OUP_2)$X(t,v)$ defined by

$$X(t,v)=\mathrm{e}^{-\alpha t-\beta v}\left\{X_0+\sigma\int_0^t\int_0^v\mathrm{e}^{\alpha x+\beta y}\mathrm{d}W(x,y)\right\}. \quad (2.6.1)$$

If $X_0=0$, the OUP_2 $X(t,v)$ can be rewritten as

$$X(t,v)=\int_0^t\int_0^v\sigma\mathrm{e}^{-\alpha t-\beta v}\mathrm{e}^{\alpha x+\beta y}\mathrm{d}W(x,y). \quad (2.6.2)$$

In Section 2. 1. 5, we have studied the continuity of the infinite series of independent Ornstein-Uhlenbeck processes $X(\cdot)$ defined by (2. 1. 22), which is the limit of the processes $X(\cdot,n)$

$= \sum_{k=1}^{n} X_k(\cdot)$ as $n \to \infty$. Integrating the equation (2.1.22) from $-\infty$ to t we obtain that the Ornstein-Uhlenbeck processes $X_i(\cdot)$ can be rewritten as

$$X_i(t) = \int_{-\infty}^{t} \exp(-\lambda_i |t-s|)(2\gamma_i)^{1/2} \mathrm{d}W_i(s), \; i = 1,2,\cdots, \tag{2.6.3}$$

where $\{W_i(t); -\infty < t < \infty\}$ are independent Wiener processes, and hence we have also

$$\begin{aligned} X(t,n) &= \sum_{k=1}^{n} X_k(t) \\ &= \sum_{k=1}^{n} \int_{-\infty}^{t} \exp(-\lambda_k |t-s|)(2\gamma_k)^{1/2} \mathrm{d}W_k(s). \end{aligned} \tag{2.6.4}$$

The latter has led us to study also the two-parameter Gaussian process

$$X(t,v) = \int_0^v \int_{-\infty}^{t} \exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2} \mathrm{d}W(x,y), \tag{2.6.5}$$

where $\gamma(y)$ and $\lambda(y)$ are assumed to be positive continuous functions on $[0,\infty)$, and $\{W(x,y); -\infty < x, y < \infty\}$ is a standard two-parameter Wiener process (cf. Section 2.3).

This brings us to the study of two-parameter Gaussian processes $\{X(t,v); t \in \mathbf{R}, v \in \mathbf{R}_+\}$ of the form

$$X(t,v) = \int_0^\infty \int_{-\infty}^{\infty} \Gamma(t,v,x,y) \mathrm{d}W(x,y), \tag{2.6.6}$$

where the kernel function $\Gamma(t,v,x,y)$ is assumed to be square integrable in (x,y) on $\mathbf{R}_+ \times \mathbf{R}$ and $W(x,y)$ is a standard two-parameter Wiener process. Thus $X(t,v)$ is a Gaussian process with mean zero and covariance function

$$\begin{aligned} &\mathrm{Cov}(X(t,v), X(s,u)) \\ &= \int_0^\infty \int_{-\infty}^{\infty} \Gamma(t,v,x,y)\Gamma(s,u,x,y) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.6.7}$$

We let

$$H_1^2(t,s,v) = E\{X(t+s,v) - X(t,v)\}^2, \tag{2.6.8}$$

$$X(R(t,s,v,u)) = X(t+s,v+u) - X(t,v+u) - X(t+s,v) + X(t,v),$$

$$H_2^2(t,s,v,u) = EX^2(R(t,s,v,u)), \tag{2.6.9}$$

where $R(t,s,v,u) = [t,t+s] \times [v,v+u]$. It is easy to see that

$$\begin{aligned} &H_1^2(t,s,v) \\ &= \int_0^\infty \int_{-\infty}^{\infty} (\Gamma(t+s,v,x,y) - \Gamma(t,v,x,y))^2 \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{2.6.10}$$

$$\begin{aligned} H_2^2(t,s,v,u) = \int_0^\infty \int_{-\infty}^{\infty} &(\Gamma(t+s,v+u,x,y) \\ &-\Gamma(t,v+u,x,y) - \Gamma(t+s,v,x,y) \\ &+\Gamma(t,v,x,y))^2 \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.6.11}$$

The following examples are immediate.

Example 2.6.1 If $\Gamma(t,v,x,y) = I_{(-\infty,t]\times[0,v]}(x,y)$, $-\infty < t < \infty$, $0 \leqslant v < \infty$, then

$$X(t,v) = W(t,v),$$

$$H_1^2(t,s,v) = sv, 0 \leqslant s < \infty,$$

$$H_2^2(t,s,v,u) = su, \; 0 \leqslant s,u < \infty.$$

Example 2.6.2 If $\Gamma(t,v,x,y) = I_{[0,t]\times[0,v]}(x,y) - tI_{[0,1]\times[0,v]}(x,y)$, $0 \leqslant t \leqslant 1$, $0 \leqslant v < \infty$, then $X(t,v) = W(t,v) - tW(1,v)$, a Kiefer process (cf. Section 1.15 in Csörgő and Révész (1981)),

$$H_1^2(t,s,v) = s(1-s)v, \; 0 \leqslant s \leqslant 1, \; 0 \leqslant v < \infty,$$

$$H_2^2(t,s,v,u) = s(1-s)u, 0 \leqslant s \leqslant 1, \; 0 \leqslant u < \infty.$$

Example 2.6.3 If, with $-\infty < t < \infty$, $0 < v < \infty$,

$$\Gamma(t,v,x,y)=I_{(-\infty,t]\times(0,v]}(x,y)\exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2},$$

where $\lambda(y)$ and $\gamma(y)$ are positive continuous functions on $[0,\infty)$, then $X(t,v)$ is the two-parameter Gaussian process of (2.6.5) with

$$H_1^2(t,s,v)=2\int_0^v\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))\mathrm{d}x,$$

$$H_2^2(t,s,v,u)=2\int_v^{v+u}\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))\mathrm{d}x.$$

Example 2.6.4 If, with $-\infty<t<\infty$, $0<v<\infty$,

$$\Gamma(t,v,x,y)$$
$$=\sum_{k=0}^{\infty}\varphi_k(v)I_{(-\infty,t]\times(k,k+1]}(x,y)\exp(-\lambda_k(t-x))(2\gamma_k)^{1/2},$$

then

$$H_1^2(t,s,v)=2\sum_{k=0}^{\infty}\varphi_k^2(v)(1-\mathrm{e}^{-\lambda_k s})\left(\frac{\gamma_k}{\lambda_k}\right),$$

$$H_2^2(t,s,v,u)=2\sum_{k=0}^{\infty}(\varphi_k(v+u)-\varphi_k(v))^2(1-\mathrm{e}^{-\lambda_k s})\frac{\gamma_k}{\lambda_k},$$

$$X(t,v)=\sum_{k=0}^{\infty}\varphi_k(v)X_k(t),$$

where $\{X_k(t);-\infty<t<\infty\}$ are independent Ornstein-Uhlenbeck processes with coefficients $\gamma_k\geqslant 0$ and $\lambda_k>0$.

The path properties of the process $X(t,v)$ as in (2.6.6) were studied by Csörgő and Lin (1991), Lin (1991) and Csörgő, Lin and Shao (1994 b). In this section we present some large deviation results for some increments of the process $X(t,v)$. Using these results we obtain some theorems concerning path properties of the process $X(t,v)$. For an early study of such processes we refer to Csörgő and Lin (1991) and Lin (1991). For more general results we refer to Csörgő, Lin and Shao (1994 b).

Put

$$H^2(t,s,v,u)=E(X(t+s,v+u)-X(t,v))^2$$
$$=\int_0^\infty\int_{-\infty}^\infty(\Gamma(t+s,v+u,x,y)-\Gamma(t,v,x,y))^2\mathrm{d}x\mathrm{d}y \tag{2.6.12}$$

and let

$$\varphi(h,B)=\sup_{0\leqslant s,u\leqslant h,|t|\leqslant B,|v|\leqslant B}H(t,s,v,u).$$

We note that, by Theorem 2.1.3, if we have

$$\int_0^\infty\varphi(\mathrm{e}^{-y^2},B)\mathrm{d}y<\infty\text{ for every }B>0, \tag{2.6.13}$$

then $X(t,v)$ is almost surely continuous. Since we are mainly interested in studying moduli of continuity and other path properties of increments of $X(t,v)$, for the convenience of the statements, and for that of the proofs later on, we assume throughout the whole section that $X(t,v)$ is a.s. continuous. Also, further we assume that $H_1(t,s,v)$ is non-decreasing in s, $H_2(t,s,v,u)$ is non-decreasing in s and u, and that a_T, b_T, c_T and d_T, $H_1(t,s,T)$ and $H_2(t,s,v,u)$ are continuous functions of T,s,u.

2.6.1 Large deviations

Proposition 2.6.1 *Let $A\subset\mathbf{R}_+$, $s_0>0$, $b_{1,T}\leqslant b_{2,T}$. Assume that*

$$E(X(t+s,v)-X(t,v))(X(t+s,u)-X(t,u))$$
$$\geqslant E(X(t+s,u)-X(t,u))^2 \tag{2.6.14}$$

for each $v\geqslant u$ and each t,s and that there exist positive numbers c_0 and α such that

$$\frac{H_1(t,s,T)}{s^\alpha}\leqslant c_0\frac{H_1(t,s_1,T)}{s_1^\alpha} \tag{2.6.15}$$

for each $T\in A$, $b_{1,T}\leqslant t\leqslant b_{2,T}$, $0\leqslant s\leqslant s_1\leqslant s_0$. *Then, for every* $0<\varepsilon<1/(1+c_0^{1+\alpha})$, *there exists a positive constant* $C(\varepsilon)$ *depending only on* α, c_0, ε *such that*

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\leqslant t\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t+s,T)-X(t,T)|}{x\{H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\}}\geqslant 1+\varepsilon\right\}$$

$$\leqslant C(\varepsilon)\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp\left(-\frac{x^2}{2}\right)\qquad(2.6.16)$$

for each $x\geqslant 1$, *where* $T^*=\exp\{T;T\in A\}$.

Proof Before proving (2.6.16), we show that for every fixed t,s

$$P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T^*)y}\geqslant 1\right\}\leqslant 8\exp(-y^2/2)\qquad(2.6.17)$$

holds ture for every $y>0$ and $H_1^*(t,s,T^*)\geqslant H_1(t,s,T^*)$.

Let $Y(T)$ be a process of independent increments with $Y(T)\overset{\mathscr{D}}{=}X(t+s,T)-X(t,T)$. Then $EY^2(T)=H_1^2(t,s,T)$ and

$$EY(T)Y(T')=H_1^2(t,s,T')$$
$$\leqslant E(X(t+s,T)-X(t,T))(X(t+s,T')-X(t,T'))$$

for each $T>T'$ by (2.6.14). Also (2.6.14) implies that $H_1(t,s,T)$ is non-decreasing in T. Applying Slepian's inequality, we have

$$P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{H_1^*(t,s,T^*)y}\geqslant 1\right\}$$
$$\leqslant P\left\{\sup_{T\in A}\frac{X(t+s,T)-X(t,T)}{H_1^*(t,s,T^*)y}\geqslant 1\right\}$$
$$+P\left\{\sup_{T\in A}\frac{-(X(t+s,T)-X(t,T))}{H_1^*(t,s,T^*)y}\geqslant 1\right\}$$
$$\leqslant P\left\{\sup_{T\in A}\frac{Y(T)}{H_1^*(t,s,T^*)y}\geqslant 1\right\}+P\left\{\sup_{T\in A}\frac{-Y(T)}{H_1^*(t,s,T^*)y}\geqslant 1\right\}$$

$$\leqslant 2P\left\{\sup_{T\in A}\frac{|Y(T)|}{H_1^*(t,s,T^*)y}\geqslant 1\right\}$$
$$\leqslant 8\exp(-y^2/2),$$

as required. We now turn to the proof of (2.6.16). Let $K=2^{2^k}$ and

$$t_{j+k}=\left(\left[\frac{t2^{2^{j+k}}}{s_0}\right]+1\right)s_0/2^{2^{j+k}},\quad j=0,1,2,\cdots.$$

Since we assumed $X(\cdot,\cdot)$ to be almost surely continuous, we can write

$$|X(t+s,T)-X(t,T)|\leqslant|X((t+s)_k,T)-X(t_k,T)|$$
$$+\sum_{j=0}^{\infty}|X((t+s)_{k+j+1},T)-X((t+s)_{k+j},T)|$$
$$+\sum_{j=0}^{\infty}|X(t_{k+j+1},T)-X(t_{k+j},T)|.\qquad(2.6.18)$$

By the definition of $H_1(t,s,T)$ and t_{k+j}, it is clear that

$$H_1(t_k,(t+s)_k-t_k,T)\leqslant H_1(t,t_k-t,T)+H_1(t,(t+s)_k-t,T)$$
$$\leqslant H_1(t,s_0,T)+H_1(t,s_0/K,T)+H_1(t+s,s_0/K,T),\qquad(2.6.19)$$

$$H_1((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},T)$$
$$\leqslant 2H_1(t+s,s_0/2^{2^{k+j}},T),\qquad(2.6.20)$$

$$H_1(t_{k+j+1},t_{k+j}-t_{k+j+1},T)\leqslant 2H_1(t,s_0/2^{2^{k+j}},T).\qquad(2.6.21)$$

From (2.6.19) and (2.6.17) it follows that

$$P\Big\{\sup_{T\in A}\sup_{b_{1,T}\leqslant T\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}|X((t+s)_k,T)-X(t_k,T)|/$$
$$\{(H_1(t,s_0,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*))x\}\geqslant 1\Big\}$$
$$\leqslant 8\times 2^{2^{k+1}}\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x^2/2).\qquad(2.6.22)$$

Similarly, by (2.6.20), (2.6.21) and (2.6.17) we have for each $x_j>0$

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\leqslant T\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}|X((t+s)_{k+j+1},T)-X((t+s)_{k+j},T)|/\right.$$

$$\left.\{2H_1(t+s,s_0/2^{2^{k+j}},T^*)x_j\}\geqslant 1\right\}$$

$$\leqslant 8\times 2^{2^{k+j+1}}\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x_j^2/2), \tag{2.6.23}$$

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\leqslant T\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}|X(t_{k+j+1},T)-X(t_{k+j},T)|/\right.$$

$$\left.\{2H_1(t,s_0/2^{2^{k+j}},T^*)x_j\}\geqslant 1\right\}$$

$$\leqslant 8\times 2^{2^{k+j+1}}\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x_j^2/2). \tag{2.6.24}$$

From (2.6.22)−(2.6.24) it follows that

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\leqslant T\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}|X(t+s,T)-X(t,T)|/\right.$$

$$\{(H_1(t,s_0,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*))x$$

$$\left.+2\sum_{j=0}^{\infty}(H_1(t+s,s_0/2^{2^{k+j}},T^*)+H_1(t,s_0/2^{2^{k+j}},T^*))x_j\}\geqslant 1\right\}$$

$$\leqslant 8\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)$$

$$\times\left(2^{2^{k+1}}\exp(-x^2/2)+2\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(-x_j^2/2)\right). \tag{2.6.25}$$

Let $x_j^2=x^2+2^{k+j+2}$. Then by (2.6.15), for k large enough and $x\geqslant 1$ we have

$$((H_1(t,s_0,T^*)+H_1(t,s_0/K,T^*)+H_1(t+s,s_0/K,T^*))x$$

$$+2\sum_{j=0}^{\infty}(H_1(t+s,s_0/2^{2^{k+j}},T^*)+H_1(t,s_0/2^{2^{k+j}},T^*))x_j$$

$$\leqslant\left(H_1(t,s_0,T^*)+c_0\left(\frac{1}{K}\right)^{\alpha}H_1(t,s_0,T^*)\right.$$

$$\left.+\left(\frac{1}{\varepsilon K}\right)^{\alpha}c_0H_1(t+s,\varepsilon s_0,T^*)\right)x$$

$$+2c_0\sum_{j=0}^{\infty}\left(\frac{1}{\varepsilon 2^{2^{k+j}}}\right)^{\alpha}H_1(t+s,\varepsilon s_0,T^*)(x+2^{\frac{k+j+1}{2}})$$

$$+2c_0\sum_{j=0}^{\infty}\left(\frac{1}{2^{2^{k+j}}}\right)^{\alpha}H_1(t,s_0,T^*)(x+2^{\frac{k+j+1}{2}})$$

$$\leqslant(H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*))$$

$$\times\left(x(1+3c_0(1+\frac{1}{\varepsilon^{\alpha}}))\sum_{j=0}^{\infty}2^{-\alpha 2^{k+j}}+2c_0\sum_{j=0}^{\infty}(\frac{1}{\varepsilon^{\alpha}}+1)2^{-\alpha 2^{k+j}+\frac{k+j+1}{2}}\right)$$

$$\leqslant(H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*))(1+\varepsilon)x \tag{2.6.26}$$

and

$$\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(-x_j^2/2)$$

$$=\exp(-x^2/2)\sum_{j=0}^{\infty}(2/e)^{2^{k+j+1}}$$

$$\leqslant C(\varepsilon)\exp(-x^2/2). \tag{2.6.27}$$

Combining (2.6.25)−(2.6.27) yields

$$P\left\{\sup_{T\in A}\sup_{b_{1,T}\leqslant t\leqslant b_{2,T}}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t+s,T)-X(t,T)|}{x\{H_1(t,s_0,T^*)+H_1(t+s,\varepsilon s_0,T^*)\}}\right.$$

$$\left.\geqslant 1+\varepsilon\right\}$$

$$\leqslant C(\varepsilon)\sup_{T\in A}\left(\frac{b_{2,T}-b_{1,T}}{s_0}+1\right)\exp(-x^2/2),$$

as required.

To study the increments of $X(t,v)$ in t and v, we give below another large deviation result for $X(t,v)$. Put

$$H_{21}(t,s,v,u,K)=2H_2\left(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\right)$$

$$+2H_2\left(t,s,v+u-\frac{c_T}{K},\frac{2c_T}{K}\right)+H_2\left(t,\frac{a_T}{K},v,c_T\left(1+\frac{2}{K}\right)\right)$$

$$+2H_2\left(t+s,\frac{a_T}{K},v,c_T\left(1+\frac{2}{K}\right)\right).$$

Here and in the sequel, we write $H_2(t,s,v,u)=H_2(t,s,0,u)$ if $v<0$.

Proposition 2.6.2 *Assume that $H_2(t,s,v,u)$ is non-decreasing in s and u and that for each $t,s,a\geqslant v\geqslant v'>0$*

$$\begin{aligned}&EX(R(t,s,v',a-v'))X(R(t,s,v,a-v))\\&\geqslant EX^2(R(t,s,v,a-v)),\end{aligned}\tag{2.6.28}$$

and that there exist positive numbers c_0 and α such that

$$\frac{H_2(t,s,v,u)}{s^\alpha}\leqslant c_0\frac{H_2(t,s_1,v,u)}{s_1^\alpha}\tag{2.6.29}$$

for each $0\leqslant s\leqslant s_1\leqslant a_T$, $0\leqslant v\leqslant d_T+c_T$, $0\leqslant u\leqslant 2c_T$, $|t|\leqslant b_T$. Then, for every $0<\varepsilon<1$, there exists $C(\varepsilon)$ depending only on ε, c_0, α such that

$$\begin{aligned}&P\Big\{\sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant c_T}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant a_T}|X(R(t,s,v,u))|/\\&\quad\{H_2(t,a_T,v,c_T(1+\varepsilon))+3H_2(t,a_T,v-\varepsilon c_T,2\varepsilon c_T)\\&\quad+3H_2(t,a_T,v+u-\varepsilon c_T,2\varepsilon c_T)+H_2(t,\varepsilon a_T,v-\varepsilon c_T,c_T(1+\varepsilon))\\&\quad+H_2(t,\varepsilon a_T,v+u-\varepsilon c_T,c_T(1+\varepsilon))\\&\quad+H_2(t+s,\varepsilon a_T,v-\varepsilon a_T,c_T(1+\varepsilon))\\&\quad+H_2(t+s,\varepsilon a_T,v+u-\varepsilon a_T,c_T(1+\varepsilon))\}\geqslant(1+\varepsilon)x\Big\}\\&\leqslant C(\varepsilon)\left(\frac{d_T}{c_T}+1\right)\left(\frac{b_T}{a_T}+1\right)e^{-x^2/2}\end{aligned}\tag{2.6.30}$$

for each $x\geqslant 1$.

Proof Let

$$t_{k+j}=\left(\left[\frac{t2^{2^{k+j}}}{a_T}\right]+1\right)a_T/2^{2^{k+j}},$$

$$v'_{k+j}=\left(\left[\frac{v2^{2^{k+j}}}{c_T}\right]+1\right)c_T/2^{2^{k-j}}.$$

We have

$$\begin{aligned}&|X(R(t,s,v,u))|\\&\leqslant|X(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))|\\&\quad+|X(R(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k))|\\&\quad+|X(R(t,t_k-t,v'_k,(v+u)'_k-v'_k))|\\&\quad+|X(R(t,s,v,v'_k-v))|\\&\quad+|X(R(t,s,v+u,(v+u)'_k-(v+u)))|\\&\leqslant|X(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))|\\&\quad+\sum_{j=0}^{\infty}|X(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},\\&\qquad v'_k,(v+u)'_k-v'_k))|\\&\quad+\sum_{j=0}^{\infty}|X(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k))|\\&\quad+|X(R(t,s,v,v'_k-v))|\\&\quad+|X(R(t,s,v+u,(v+u)'_k-(v+u)))|.\end{aligned}\tag{2.6.31}$$

From (2.6.28) it follows that we have

$$H_2(t,s,v,u)\geqslant H_2(t,s,v',u')\tag{2.6.32}$$

for each $v'\geqslant v, v+u\geqslant v'+u'$, and with the help of (2.6.32), we get

$$\begin{aligned}&H_2(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)\leqslant H_2(t,s,v,u)\\&\quad+H_2(t+s,(t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k)\\&\quad+H_2(t,t_k-t,v'_k,(v+u)'_k-v'_k)+H_2(t,s,v,v'_k-v)\\&\quad+H_2(t,s,v+u,(v+u)'_k-(v+u))\\&\leqslant H_2(t,s,v,u)+H_{21}(t,s,v,u,K).\end{aligned}$$

Also, we have

$$\begin{aligned}&H_2((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k)\\&\leqslant 2H_2\left(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{K}\right)\right),\\&H_2(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k)\end{aligned}$$

$$\leqslant 2H_2\left(t,\frac{a_T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{K}\right)\right).$$

Then, for every $x>0$ and $x_j>0$,

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k'))|}{x(H_2(t,s,v,u)+H_{21}(t,s,v,u,K))}\geqslant 1\right\}$$

$$\leqslant 4\times 2^{2^{k+2}}\left(\frac{d_T}{c_T}+1\right)\left(\frac{b_T}{a_T}+1\right)e^{-x^2/2}, \qquad (2.6.33)$$

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\sum_{j=0}^{\infty}|X(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v_k',(v+u)_k'-v_k'))|}{2\sum_{j=0}^{\infty}x_jH_2(t+s,a_T/2^{2^{k+j}},v,c_T(1+2/K))}\geqslant 1\right\}$$

$$\leqslant 4\left(\frac{d_T}{c_T}+1\right)\left(\frac{b_T}{a_T}+1\right)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2}, \qquad (2.6.34)$$

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\sum_{j=0}^{\infty}|X(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v_k',(v+u)_k'-v_k'))|}{2\sum_{j=0}^{\infty}x_jH_2(t,a_T/2^{2^{k+j}},v,c_T(1+2/K))}\geqslant 1\right\}$$

$$\leqslant 4\left(\frac{d_T}{c_T}+1\right)\left(\frac{b_T}{a_T}+1\right)\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2}. \qquad (2.6.35)$$

Let $x_j^2=x^2+2^{k+j+2}$. It follows that

$$\sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2}\leqslant 2K^2e^{-x^2/2}. \qquad (2.6.36)$$

We now deal with $X(R(t,s,v,v_k'-v))$ and $X(R(t,s,v+u,(v+u)_k'-(v+u)))$. For each $y>0$ we have

$$P\{\sup_{0\leqslant v\leqslant d_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}|X(R(t,s,v,v_k'-v))|\geqslant y\}$$

$$\leqslant P\left\{\max_{0\leqslant i\leqslant\frac{d_T}{c_T}K}\sup_{\frac{ic_T}{K}\leqslant v\leqslant\frac{(i+1)c_T}{K}}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}|X(R(t,s,v,v_k'-v))|\geqslant y\right\}$$

$$\leqslant\sum_{i=0}^{\left[\frac{d_T}{c_T}K\right]}P\left\{\sup_{\frac{ic_T}{K}\leqslant v\leqslant\frac{(i+1)c_T}{K}}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\left|X\left(R\left(t,s,v,\frac{(i+1)c_T}{K}-v\right)\right)\right|\geqslant y\right\}.$$

$$(2.6.37)$$

Let $d_i=(i+1)c_T/K$. We show below that for each fixed t,s

$$P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{|X(R(t,s,v,d_i-v))|}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$\leqslant 4\exp(-y^2/2) \qquad (2.6.38)$$

for each $y>0$.

Let $Y(v)$ be an independent increment process with $Y(d_i-v)\overset{\mathscr{D}}{=}X(R(t,s,v,d_i-v))$ for $ic_T/K\leqslant v\leqslant(i+1)c_T/K$. Then for each $v>v'$

$$EY(d_i-v)Y(d_i-v')=EY^2(d_i-v)$$
$$=EX^2(R(t,s,v,d_i-v))$$
$$\leqslant EX(R(t,s,v,d_i-v))X(R(t,s,v',d_i-v')),$$

where the last inequality is from (2.6.28). By (2.6.32) we find that

$$H_2\left(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\right)\geqslant H_2\left(t,s,\frac{ic_T}{K},\frac{c_T}{K}\right) \qquad (2.6.39)$$

for each $ic_T/K\leqslant v\leqslant(i+1)c_T/K$. Using Slepian's inequality and (2.6.39), we obtain

$$P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{|X(R(t,s,v,d_i-v))|}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$\leqslant P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{X(R(t,s,v,d_i-v))}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$+P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{-X(R(t,s,v,d_i-v))}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$\leqslant P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{Y(d_i-v)}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$+P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\frac{-Y(d_i-v)}{H_2(t,s,v-c_T/K,2c_T/K)}\geqslant y\right\}$$

$$\leqslant 2P\left\{\frac{|Y(d_i-ic_T/K)|}{H_2(t,s,ic_T/K,c_T/K)}\geqslant y\right\}$$

$$=2P\left\{\frac{X(R(t,s,ic_T/K,d_i-ic_T/K))}{H_2(t,s,ic_T/K,c_T/K)}\geqslant y\right\}$$

$$\leqslant 4\exp(-y^2/2),$$

which implies (2.6.38).

Along the lines of the proof of (2.6.25)—(2.6.26), by (2.6.38) we can get

$$P\left\{\sup_{ic_T/K\leqslant v\leqslant(i+1)c_T/K}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,d_i-v))|}{xI_1(t,s,v,K)+I_2(t,s,v,K)}\geqslant 1\right\}$$

$$\leqslant 8\times 2^{2^{k+1}}\left(\frac{b_T}{a_T}+1\right)\exp(-x^2/2), \tag{2.6.40}$$

where

$$I_1(t,s,v,K)=H_2\left(t,s,v-\frac{c_T}{K},\frac{2c_T}{K}\right)$$

$$+16\sum_{j=0}^{\infty}H_2\left(t,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T\left(1+\frac{1}{K}\right)\right)$$

$$+16\sum_{j=0}^{\infty}H_2\left(t+s,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T\left(1+\frac{1}{K}\right)\right),$$

$$I_2(t,s,v,K)=40\sum_{j=0}^{\infty}H_2\left(t,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T\left(1+\frac{1}{K}\right)\right)2^{\frac{k+j+1}{2}}$$

$$+\sum_{j=0}^{\infty}H_2\left(t+s,\frac{a_T}{2^{2^{k+j}}},v-\frac{c_T}{K},c_T\left(1+\frac{1}{K}\right)\right)2^{\frac{k+j+1}{2}}.$$

Combining (2.6.37) and (2.6.40) yields

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,v_k'-v))|}{xI_1(t,s,v,K)+I_2(t,s,v,K)}\geqslant 1\right\}$$

$$\leqslant 16\times 2^{2^{k+2}}\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\exp(-x^2/2). \tag{2.6.41}$$

Similarly, we have

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\right.$$

$$\left.\frac{|X(R(t,s,v+u,(v+u)_k'-(v+u)))|}{xI_1(t,s,v+u,K)+I_2(t,s,v+u,K)}\geqslant 1\right\}$$

$$\leqslant 16\times 2^{2^{k+2}}\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\exp(-x^2/2). \tag{2.6.42}$$

Combining (2.6.33)—(2.6.36) and (2.6.41)—(2.6.42) yields

$$P\left\{\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}|X(R(t,s,v,v))|/\right.$$

$$\left\{x(H_2(t,s,v,u)+H_{21}(t,s,v,u,K))\right.$$

$$+\sum_{j=0}^{\infty}2x_jH_2\left(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{K}\right)\right)$$

$$+\sum_{j=0}^{\infty}2x_jH_2\left(t,\frac{a^T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{k}\right)\right)$$

$$+xI_1(t,s,v,K)+I_2(t,s,v,K)+xI_1(t,s,v+u,K)$$

$$\left.\left.+I_2(t,s,v+u,K\right\}\geqslant 1\right\}$$

$$\leqslant 52\times 2^{2^{k+2}}\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\exp(-x^2/2). \tag{2.6.43}$$

Along the lines of the proof of (2.6.26), by (2.6.29) we can get that for k large enough and all $x\geqslant 1$,

$$x(H_2(t,s,v,u)+H_{21}(t,s,v,u,K))$$

$$+\sum_{j=0}^{\infty}2x_jH_2\left(t+s,\frac{a_T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{K}\right)\right)$$

$$+\sum_{j=0}^{\infty}2x_jH_2\left(t,\frac{a_T}{2^{2^{k+j}}},v,c_T\left(1+\frac{2}{K}\right)\right)$$

$$
\begin{aligned}
&+ xI_1(t,s,v,K) + I_2(t,s,v,K) + xI_1(t,s,v+u,K) \\
&+ I_2(t,s,v+u,K) \\
&\leqslant (1+\varepsilon)x\{H_2(t,a_T,v,c_T(1+\varepsilon)) \\
&+ 3H_2(t,a_T,v-\varepsilon c_T,2\varepsilon c_T) + 3H_2(t,a_T,v+u-\varepsilon c_T,2\varepsilon c_T) \\
&+ H_2(t,\varepsilon a_T,v-\varepsilon c_T,c_T(1+\varepsilon)) \\
&+ H_2(t,\varepsilon a_T,v+u-\varepsilon c_T,c_T(1+\varepsilon)) \\
&+ H_2(t+s,\varepsilon a_T,v-\varepsilon a_T,c_T(1+\varepsilon)) \\
&+ H_2(t+s,\varepsilon a_T,v+u-\varepsilon a_T,c_T(1+\varepsilon))\},
\end{aligned} \tag{2.6.44}
$$

and (2.6.30) now follows from (2.6.43) and (2.6.44).

With a similar proof, we can get

Proprosition 2.6.3 *Let* $A\in \mathbf{R}_+$, $a_0>0$, $c_0>0$. *Assume that* $H_2(t,s,v,u)$ *is non-decreasing in* s *and* u *and that for each* $t,s,a\geqslant v\geqslant v'>0$

$$
\begin{aligned}
&EX(R(t,s,v',a-v'))X(R(t,s,v,a-v)) \\
&\geqslant EX^2(R(t,s,v,a-v)),
\end{aligned}
$$

and that there exist positive numbers c_1 *and* α *such that*

$$
\frac{H_2(t,s,v,u)}{s^\alpha}\leqslant c_1\frac{H_2(t,s_1,v,u)}{s_1^\alpha}
$$

for each $T\in A$, $0\leqslant s\leqslant s_1\leqslant a_0$, $0\leqslant v\leqslant d_T+c_0$, $0\leqslant u\leqslant 2c_0$, $|t|\leqslant b_T$. *Then, for every* $0<\varepsilon<1$, *there exists* $C(\varepsilon)$ *depending only on* ε, c_1, α *such that*

$$
\begin{aligned}
P\Big\{&\sup_{T\in A}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_0}\sup_{0\leqslant s\leqslant b_T}\sup_{|t|\leqslant a_0}|X(R(t,s,v,u))|/ \\
&\{H_2(t,a_0,v,c_0(1+\varepsilon))+3H_2(t,a_0,v-\varepsilon c_0,2\varepsilon c_0) \\
&+3H_2(t,a_0,v+u-\varepsilon c_0,2\varepsilon c_0)+H_2(t,\varepsilon a_0,v-\varepsilon c_0,c_0(1+\varepsilon)) \\
&+H_2(t,\varepsilon a_0,v+u-\varepsilon c_0,c_0(1+\varepsilon)) \\
&+H_2(t+s,\varepsilon a_0,v-\varepsilon a_0,c_0(1+\varepsilon)) \\
&+H_2(t+s,\varepsilon a_0,v+u-\varepsilon a_0,c_0(1+\varepsilon))\}\geqslant(1+\varepsilon)x\Big\}
\end{aligned}
$$

$$
\leqslant C(\varepsilon)\sup_{T\in A}\left(\frac{d_T}{c_0}+1\right)\left(\frac{b_T}{a_0}+1\right)e^{-x^2/2}
$$

for each $x\geqslant 1$.

2.6.2 Path properties

We now use the large deviation results for increments of the process $X(t,v)$ as in (2.6.6) to establish its path properties.

Theorem 2.6.1 *Assume that* $H_1(t,s,T)=H_1(0,s,T)=:H_0(s,T)$ *for each* $s>0$, $|t|\leqslant b_T+a_T$, *and that*

$$
\begin{aligned}
&E(X(t+s,v)-X(t,v))(X(t+s,u)-X(t,u)) \\
&\geqslant E(X(t+s,u)-X(t,u))^2
\end{aligned} \tag{2.6.45}
$$

for each $v\geqslant u$ *and each* t,s *and that there exist positive numbers* c_0 *and* α *such that*

$$
\frac{H_0(s,T)}{s^\alpha}\leqslant c_0\frac{H_0(s_1,T)}{s_1^\alpha} \tag{2.6.46}
$$

for each $0\leqslant s\leqslant s_1\leqslant a_T$. *Moreover, assume that*

$$
\log\log\left(a_T+\frac{1}{b_T}\right)=o\left(\log\frac{b_T}{a_T}\right)\quad \text{as } T\to\infty, \tag{2.6.47}
$$

$$
\log\log\left(H_0(a_T,T)+\frac{1}{H_0(a_T,T)}\right)=o\left(\log\frac{b_T}{a_T}\right)\quad \text{as } T\to\infty, \tag{2.6.48}
$$

$$
E(X((j+1)s,v)-X(js,v))(X((l+1)s,v)-X(ls,v))\leqslant 0 \tag{2.6.49}
$$

for each $j\neq l$, $s>0$, *then, we have*

$$
\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}=1\quad \text{a.s.}, \tag{2.6.50}
$$

$$
\lim_{T\to\infty}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}=1\quad \text{a.s.} \tag{2.6.51}
$$

Proof We first prove that

$$\limsup_{T\to\infty}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\le 1 \quad \text{a. s.} \tag{2.6.52}$$

For every $0<\varepsilon<1/2$, by (2.6.46) there exists a constant N such that

$$\frac{H_0(a_T/N,T)}{H_0(a_T,T)}\le\varepsilon. \tag{2.6.53}$$

Let $1<\theta<\min\left(1+\frac{1}{N},1+\frac{\varepsilon^2}{38}\right)$. Put

$$A_i=\{T;\theta^i<a_T\le\theta^{i+1}\},\quad -\infty<i<\infty,$$

$$B_j=\{T;\theta^j<1+b_T/a_T\le\theta^{j+1}\},\quad j=0,1,2,\cdots,$$

$$C_{ki}=\{T;\theta^k<H_0(\theta^{i+1},T)\le\theta^{k+1}\},\quad -\infty<k<\infty.$$

Clearly, (2.6.47) implies $b_T/a_T\to\infty$ as $T\to\infty$ and $A_iB_j=\varnothing$ if $|i|\ge\theta^{\varepsilon j}$, where j is sufficiently large. Also, (2.6.48) implies $A_iB_jC_{ki}=\varnothing$ if $|k|\ge\theta^{\varepsilon j}$, if j is sufficiently large. Let $T'_{kij}=\inf\{T;T\in A_iB_jC_{ki}\}$ and $T^*_{kij}=\sup\{T;T\in A_iB_jC_{ki}\}$. Hence we have

$$\limsup_{T\to\infty}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}}$$

$$\le\limsup_{j\to\infty}\sup_{T\in B_j}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}}$$

$$\le\limsup_{j\to\infty}\max_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log\frac{b_T}{a_T})^{1/2}}$$

$$\le\limsup_{j\to\infty}\max_{|i|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_i}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{i+1}}\frac{(1+\varepsilon)|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T)(2\log\theta^j)^{1/2}}$$

$$\le\limsup_{j\to\infty}\max_{|i|\le\theta^{\varepsilon j}}\max_{|k|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_iC_{ki}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{i+1}}$$

$$\frac{(1+\varepsilon)|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T'_{kij})(2\log\theta^j)^{1/2}}$$

$$\le\limsup_{j\to\infty}\max_{|i|\le\theta^{\varepsilon j}}\max_{|k|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_iC_{ki}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{i+1}}\frac{(1+\varepsilon)^2|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}. \tag{2.6.54}$$

Using Proposition 2.6.1 and (2.6.53) and noting that $H_1(t,s,T)=H_0(s,T)$, we get

$$P\left\{\sup_{T\in B_jA_iC_{ki}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{i+1}}\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\ge(1+2\varepsilon)^4\right\}$$

$$\le C(\varepsilon)\sup_{T\in B_jA_iC_{ki}}\left(\frac{b_T}{\theta^{i+1}}+1\right)\exp(-(1+2\varepsilon)^2\log\theta^j)$$

$$\le C(\varepsilon)\theta^{-4\varepsilon j}, \tag{2.6.55}$$

and hence

$$P\{\max_{|i|\le\theta^{\varepsilon j}}\max_{|k|\le\theta^{\varepsilon j}}\sup_{T\in B_jA_iC_{ki}}\sup_{|t|\le b_T}\sup_{0\le s\le\theta^{i+1}}$$

$$\left.\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{i+1},T^*_{kij})(2\log\theta^j)^{1/2}}\ge(1+2\varepsilon)^4\right\}\le C(\varepsilon)\theta^{-\varepsilon j}. \tag{2.6.56}$$

It follows from (2.6.54), (2.6.56) and the Borel-Cantelli lemma that

$$\limsup_{T\to\infty}\sup_{|t|\le b_T}\sup_{0\le s\le a_T}\frac{|X(t+s,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}$$

$$\le(1+2\varepsilon)^6\quad\text{a. s.} \tag{2.6.57}$$

This proves (2.6.52) by the arbitrariness of ε.

Next, we prove that

$$\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\ge 1\quad\text{a. s.} \tag{2.6.58}$$

Note that

$$\liminf_{T\to\infty}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}$$

$$\ge\liminf_{j\to\infty}\inf_{|i|\le\theta^{\varepsilon j}}\inf_{T\in B_jA_i}\sup_{0\le t\le b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}$$

$$\geqslant \liminf_{j\to\infty}\inf_{|i|\leqslant\theta^j}\inf_{T\in B_jA_i}\sup_{0\leqslant t\leqslant b_T}\frac{|X(t+\theta^{j+1},T)-X(t,T)|}{(1+\varepsilon)H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty}\sup_{|i|\leqslant\theta^j}\sup_{T\in B_jA_i}\sup_{0\leqslant t\leqslant b_T}\frac{|X(t+\theta^{j+1},T)-X(t+a_T,T)|}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}$$

$$\geqslant \liminf_{j\to\infty}\inf_{|i|\leqslant\theta^j}\inf_{T\in B_jA_i}\sup_{0\leqslant l\leqslant\theta^{j-2}}\frac{|X((l+1)\theta^{j+1},T)-X(l\theta^{j+1},T)|}{(1+\varepsilon)H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty}\sup_{|i|\leqslant\theta^j}\sup_{T\in B_jA_i}\sup_{a_T\leqslant t\leqslant b_T+a_T}\sup_{0\leqslant s\leqslant(\theta-1)\theta^j}\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}. \qquad (2.6.59)$$

Along the lines of the proof of (2.6.57), by (2.6.46), we have

$$\limsup_{j\to\infty}\sup_{|i|\leqslant\theta^j}\sup_{T\in B_jA_i}\sup_{a_T\leqslant t\leqslant b_T+a_T}\sup_{0\leqslant s\leqslant(\theta-1)\theta^j}\frac{|X(t+s,T)-X(t,T)|}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}$$

$$\leqslant\limsup_{j\to\infty}\sup_{|i|\leqslant\theta^j}\sup_{T\in B_jA_i}\sup_{a_T\leqslant t\leqslant b_T+a_T}\sup_{0\leqslant s\leqslant(\theta-1)\theta^j}\frac{\varepsilon|X(t+s,T)-X(t,T)|}{H_0((\theta-1)\theta^j,T)(2\log\theta^j)^{1/2}}$$

$$\leqslant\varepsilon\quad \text{a.s.} \qquad (2.6.60)$$

Put $Y(l,T)=X((l+1)\theta^{j+1},T)-X(l\theta^{j+1},T)$. Let $Z(l,T)$ be a two-parameter Gaussian process such that for each fixed l, $Z(l,T)$ is an independent increment process with $Z(l,T)\overset{\mathscr{D}}{=}Y(l,T)$, and $EZ(l,T)Z(n,T')=EY(l,T)Y(n,T')$ for $l\neq n$. Then, by (2.6.45) we have

$$EY^2(l,T)=EZ^2(l,T),$$
$$EY(l,T)Y(n,T)=EZ(l,T)Z(n,T),$$
$$EY(l,T)Y(n,T')=EZ(l,T)Z(n,T'),\quad \text{for } l\neq n,$$
$$EY(l,T)Y(l,T')\geqslant EY^2(l,T\wedge T')=EZ(l,T)Z(l,T').$$

Thus, we can use Corollary 1.2.2 and obtain

$$P\left\{\inf_{T\in B_jA_i}\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Y(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$=1-P\left\{\bigcap_{T\in B_jA_i}\bigcup_{0\leqslant l\leqslant\theta^{j-2}}\left(\frac{Y(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}\right)\right\}$$

$$\leqslant 1-P\left\{\bigcap_{T\in B_jA_i}\bigcup_{0\leqslant l\leqslant\theta^{j-2}}\left(\frac{Z(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}\right)\right\}$$

$$=P\left\{\inf_{T\in B_jA_i}\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}\right\}. \qquad (2.6.61)$$

From (2.6.48) it is easy to see that $C_{ki}B_j=\varnothing$, if $|k|\geqslant\theta_j$, when j is sufficiently large. Hence

$$P\left\{\inf_{T\in B_jA_i}\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\sum_{|k|\leqslant\theta^j}P\left\{\inf_{T\in B_jA_iC_{ki}}\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Z(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\sum_{|k|\leqslant\theta^j}P\left\{\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Z(l,T^*_{kij})}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\leqslant\frac{\theta}{(1+\varepsilon)^2}\right\}$$

$$+\sum_{|k|\leqslant\theta^j}P\left\{\max_{0\leqslant l\leqslant\theta^{j-2}}\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\right\}.$$

$$(2.6.62)$$

Noting that $Z(l,T)$ is an independent increment process for l fixed, we have

$$E(Z(l,T^*_{kij})-Z(l,T'_{kij}))^2=EZ^2(l,T^*_{kij})-EZ^2(l,T'_{kij})$$
$$=EY^2(l,T^*_{kij})-EY^2(l,T'_{kij})$$
$$\leqslant\theta^{2(k+1)}-\theta^{2k}$$
$$\leqslant(\theta^2-1)EY^2(l,T^*_{kij})$$
$$=(\theta^2-1)H_0^2(\theta^{j+1},T^*_{kij})$$

and hence

$$\sum_{|k|\leqslant\theta^j}P\left\{\max_{0\leqslant l\leqslant\theta^{j-2}}\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\sum_{|k|\leqslant\theta^j}\sum_{l=0}^{\theta^{j-2}}P\left\{\sup_{T\in B_jA_iC_{ki}}\frac{|Z(l,T^*_{kij})-Z(l,T)|}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\geqslant\frac{\varepsilon}{2}\right\}$$

$$\leqslant 2\sum_{|k|\leqslant\theta^j}\sum_{l=0}^{\theta^{j-2}}P\left\{\frac{|Z(l,T^*_{kij})-Z(l,T'_{kij})|}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\geqslant\frac{\varepsilon}{2}\right\}$$

$$\leqslant 4\sum_{|k|\leqslant\theta^j}\sum_{l=0}^{\theta^{j-2}}\exp\left(-\frac{\varepsilon^2\log\theta^j}{4(\theta^2-1)}\right)\leqslant 4\theta^{-2j}. \tag{2.6.63}$$

Here $1<\theta<1+\varepsilon/32$ is used. By (2.6.49) and Slepian's inequality, we have

$$P\left\{\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Z(l,T^*_{kij})}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\leqslant\frac{\theta}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\prod_{l=0}^{[\theta^{j-2}]}P\left\{\frac{Z(l,T^*_{kij})}{H_0(\theta^{j+1},T^*_{kij})(2\log\theta^j)^{1/2}}\leqslant\frac{\theta}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\prod_{j=0}^{[\theta^{j-2}]}\left(1-\exp\left(-\frac{\theta^2}{1+\varepsilon}\log\theta^j\right)\right)$$

$$\leqslant\exp(-\theta^{\varepsilon j/4})\leqslant\theta^{-4j}. \tag{2.6.64}$$

Therefore, we conclude from (2.6.61)—(2.6.64) that

$$P\left\{\inf_{T\in B_jA_i}\max_{0\leqslant l\leqslant\theta^{j-2}}\frac{Y(l,T)}{H_0(\theta^{j+1},T)(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}\leqslant 5\theta^{-2j} \tag{2.6.65}$$

for every sufficiently large j.

Combining (2.6.59), (2.6.60), (2.6.65) with the Borel-Cantelli lemma yields

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H_0(a_T,T)(2\log(b_T/a_T))^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}-\varepsilon\quad\text{a.s.} \tag{2.6.66}$$

This proves (2.6.58) by the arbitrariness of ε, and so completes the proof of Theorem 2.6.1.

Theorem 2.6.2 *Assume that* (2.6.28) *is satisfied and that* (2.6.29) *holds for each* $0\leqslant s\leqslant s_1\leqslant a_T$, $0\leqslant v\leqslant d_T$, $0\leqslant u\leqslant 2c_T$, $|t|\leqslant b_T+a_T$. *Moreover, assume that*

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{|t|\leqslant b_T+a_T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant v\leqslant d_T+c_T}\sup_{-\delta c_T\leqslant u\leqslant c_T}\frac{H_2(t+s,a_T,v+u,\delta c_T)+H_2(t+s,\delta a_T,v+u,c_T)}{H_2(t,a_T,v,c_T)}=0, \tag{2.6.67}$$

$$\log\log\left(a_T+c_T+\frac{1}{d_T}+\frac{1}{b_T}\right)=o\left(\log\left(\frac{b_T}{a_T}+1\right)\left(1+\frac{d_T}{c_T}\right)\right) \tag{2.6.68}$$

as $T\to\infty$. *Then*

$$\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}|X(R(t,s,v,u))|/H_2(t,a_T,v,c_T)$$

$$\times\left(2\left(\log\left(\frac{b_T}{a_T}+1\right)\left(1+\frac{d_T}{c_T}\right)+\log\log\widetilde{H}_2(t,a_T,v,c_T)\right)\right)^{1/2}\leqslant 1\ \text{a.s.}, \tag{2.6.69}$$

where and in the sequel of this section $\widetilde{x}=x+1/x$. *If, in addition, the following conditions are satisfied*

$$\log\log\widetilde{H}_2(t,a_T,v,c_T)=o(\log(b_T/a_T+1)(1+d_T/c_T)) \tag{2.6.70}$$

uniformly in $|t|\leqslant b_T$ *and* $0\leqslant v\leqslant d_T$ *as* $T\to\infty$,

$$EX(R(js,s,ku,u))X(R(ms,s,lu,u))\leqslant 0 \tag{2.6.71}$$

for each $s>0$, $u>0$, $j+k\neq m+l$, *then*

$$\lim_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant t\leqslant b_T}\frac{|X(R(t,a_T,v,c_T))|}{H_2(t,a_T,v,c_T)\left(2\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}=1\quad\text{a.s.} \tag{2.6.72}$$

$$\lim_{T\to\infty}\sup_{\substack{0\leqslant v\leqslant d_T\\0\leqslant u\leqslant c_T}}\sup_{\substack{0\leqslant t\leqslant b_T\\0\leqslant s\leqslant a_T}}\frac{|X(R(t,s,v,u))|}{H_2(t,a_T,v,c_T)\left(2\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}=1\quad\text{a.s.} \tag{2.6.73}$$

Proof For every $0<\varepsilon<1/2$, by (2.6.67) there exists a positive N such that

$$\sup_{\substack{|t|\leqslant b_T+a_T\\0\leqslant s\leqslant a_T}}\ \sup_{\substack{0\leqslant v\leqslant d_T+c_T\\-c_T/N\leqslant u\leqslant c_T}} \frac{H_2\left(t+s,a_T,v+u,\frac{c_T}{N}\right)+H_2\left(t+s,\frac{a_T}{N},v+u,c_T\right)}{H_2(t,a_T,v,c_T)}\leqslant\varepsilon \quad (2.6.74)$$

for $T\geqslant N$. Let $1<\theta<1+1/N$. Put

$$A_k=\{T;\theta^k<a_T\leqslant\theta^{k+1}\},\quad -\infty<k<\infty,$$

$$B_i=\{T;\theta^i<c_T\leqslant\theta^{i+1}\},\quad -\infty<i<\infty,$$

$$G_j=\left\{T;\theta^j<\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\leqslant\theta^{j+1}\right\},\quad j=0,1,2\cdots.$$

Clearly (2.6.68) implies that

$$\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\to\infty\quad \text{as } T\to\infty,$$

$A_kG_j=\varnothing$ and $B_kG_j=\varnothing$ if $|k|\geqslant\theta^{\varepsilon j}$, when j is sufficiently large. Hence we have

$$\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant c_T}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,u))|}{H_2(t,a_T,v,c_T)\left(2(\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)+\log\log\tilde{H}_2(t,a_T,v,c_T))\right)^{1/2}}$$

$$\leqslant\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_kB_iG_j}\ \sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant c_T}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,u))|}{H_2(t,a_T,v,c_T)\left(2(\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)+\log\log\tilde{H}_2(t,a_T,v,c_T))\right)^{1/2}}$$

$$\leqslant\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_kB_iG_j}\ \sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant\theta^{i+1}}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant\theta^{k+1}}\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(t,\theta^{k+1},v,\theta^{i+1})(2(\log\theta^j+\log\log\tilde{H}_2(t,\theta^{k+1},v,\theta^{i+1})))^{1/2}}.$$

(2.6.75)

By (2.6.74) and Proposition 2.6.2, we derive

$$P\left\{\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_kB_iG_j}\ \sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant\theta^{i+1}}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant\theta^{k+1}}\frac{|X(R(t,s,v,u))|}{H_2(t,\theta^{k+1},v,\theta^{i+1})(2(\log\theta^j+\log\log\tilde{H}_2(t,\theta^{k+1},v,\theta^{i+1})))^{1/2}}\right.$$

$$\left.\geqslant(1+\varepsilon)^4\right\}$$

$$\leqslant C(\varepsilon)\sum_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_kB_iG_j}\left(\frac{d_T}{\theta^{i+1}}+1\right)\left(\frac{b_T}{\theta^{k+1}}+1\right)$$

$$\times\exp(-(1+\varepsilon^2)\log\theta^j)$$

$$\leqslant C(\varepsilon)\sum_{|k|,|i|\leqslant\theta^{\varepsilon j}}\theta^j\exp(-(1+\varepsilon)^2\log\theta^j)$$

$$\leqslant C(\varepsilon)\theta^{-\varepsilon^2 j}. \quad (2.6.76)$$

It follows from (2.6.75), (2.6.76) and the Borel-Cantelli lemma that

$$\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\ \sup_{0\leqslant u\leqslant c_T}\ \sup_{|t|\leqslant b_T}\ \sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,u))|}{H_2(t,a_T,v,c_T)\left(2(\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)+\log\log\tilde{H}_2(t,a_T,v,c_T))\right)^{1/2}}$$

$$\leqslant(1+\varepsilon)^5\quad \text{a.s.}$$

This proves (2.6.69) by the arbitrariness of ε.

We now consider (2.6.72). Note that

$$\liminf_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\ \sup_{0<t\leqslant b_T}\frac{|X(R(t,a_T,v,c_T))|}{H_2(t,a_T,v,c_T)\left(2\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\right)^{1/2}}$$

$$\geqslant\liminf_{j\to\infty}\inf_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \inf_{T\in A_kB_iG_j}\ \sup_{\substack{0\leqslant v\leqslant d_T\\0<t\leqslant b_T}}\frac{|X(R(t,a_T,v,c_T))|}{H_2(t,a_T,v,c_T)\left(2\log(\frac{b_T}{a_T}+1)(\frac{d_T}{c_T}+1)\right)^{1/2}}$$

$$\geqslant\liminf_{j\to\infty}\inf_{|k|,|i|\leqslant\theta^{\varepsilon j}}\ \inf_{T\in A_kB_iG_j}\ \sup_{\substack{0\leqslant v\leqslant d_T\\0<t\leqslant b_T}}$$

$$\frac{|X(R(t,\theta^{k+1},v,\theta^{i+1}))|}{(1+\varepsilon)H_2(t,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_kB_iC_j}\sup_{\theta^j\leqslant v\leqslant\theta^j+d_T}\sup_{0\leqslant u\leqslant(\theta-1)\theta^j}\sup_{0<t\leqslant b_T}\sup_{0\leqslant s\leqslant\theta^j}$$

$$\frac{|X(R(t,s,v,u))|}{H_2(t,\theta^{k+1},v-\theta^i,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_kB_iC_j}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant\theta^{j+1}}\sup_{\theta^k\leqslant t\leqslant\theta^k+b_T}\sup_{0\leqslant s\leqslant(\theta-1)\theta^k}$$

$$\frac{|X(R(t,s,v,u))|}{H_2(t-\theta^k,\theta^{k+1},v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$\geqslant\liminf_{j\to\infty}\inf_{|k|,|i|\leqslant\theta^{\varepsilon j}}\min_{m,n\geqslant\theta^j}\max_{\substack{0\leqslant l\leqslant m\\0\leqslant p\leqslant n}}$$

$$\frac{|X(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1}))|}{(1+\varepsilon)H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$-\varepsilon\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_kB_iC_j}\sup_{0\leqslant v\leqslant\theta^j+d_T}\sup_{0\leqslant u\leqslant(\theta-1)\theta^j}\sup_{0<t\leqslant b_T}\sup_{0\leqslant s\leqslant\theta^k}$$

$$\frac{|X(R(t,s,v,u))|}{H_2(t,\theta^{k+1},v,(\theta-1)\theta^j)(2\log\theta^j)^{1/2}}$$

$$-\varepsilon\limsup_{j\to\infty}\sup_{|k|,|i|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_kB_iC_j}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant\theta^{j+1}}\sup_{0<t\leqslant\theta^k+b_T}\sup_{0\leqslant s\leqslant(\theta-1)\theta^k}$$

$$\frac{|X(R(t,s,v,u))|}{H_2(t,(\theta-1)\theta^k,v,\theta^{i+1})(2\log\theta^j)^{1/2}}$$

$$=: I_1-I_2-I_3. \tag{2.6.77}$$

Along the lines of the proof of (2.6.69) and by (2.6.70) one can obtain

$$I_2+I_3\leqslant 2\varepsilon \quad \text{a.s.} \tag{2.6.78}$$

For I_1, in terms of (2.6.71), we can apply Slepian's inequality and get

$$P\left\{\min_{mn\geqslant\theta^j}\max_{0\leqslant l\leqslant m}\max_{0\leqslant p\leqslant n}\frac{X(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1}))}{H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\sum_{mn\geqslant\theta^j}\prod_{0\leqslant l\leqslant m}\prod_{0\leqslant p\leqslant n}P\left\{\frac{X(R(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1}))}{H_2(l\theta^{k+1},\theta^{k+1},p\theta^{i+1},\theta^{i+1})(2\log\theta^j)^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leqslant\sum_{m,n:\,mn\geqslant\theta^j}\left(1-\exp\left(-\frac{\log\theta^j}{(1+\varepsilon)^2}\right)\right)^{(m+1)(n+1)}$$

$$\leqslant\sum_{m,n:\,mn\geqslant\theta^j}\exp\left(-(m+1)(n+1)\theta^{-j/(1+\varepsilon)^2}\right)$$

$$\leqslant\theta^{2j}\exp(-\theta^{j\varepsilon})\leqslant\theta^{-j}$$

for every sufficiently large j. This implies

$$I_1\geqslant\frac{1}{(1+\varepsilon)^3} \quad \text{a.s.} \tag{2.6.79}$$

by the Borel-Cantelli lemma. From the above inequalities we finally conclude

$$\liminf_{T\to\infty}\sup_{\substack{0\leqslant v\leqslant d_T\\0<t\leqslant b_T}}\frac{|X(R(t,a_T,v,c_T))|}{H_2(t,a_T,v,c_T)\left(2\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}$$

$$\geqslant\frac{1}{(1+\varepsilon)^3}-2\varepsilon \quad \text{a.s.} \tag{2.6.80}$$

Now (2.6.72) follows from (2.6.80), (2.6.69) and (2.6.70), and so does (2.6.73). This completes the proof of Theorem 2.6.2.

The following corollaries deal with the examples given earlier in this section.

Corollary 2.6.1 *Let* $\{W(x,y);-\infty<x,y<\infty\}$ *be a standard two-parameter Wiener process. Assume that*

$$\log\log\left(Ta_T+\frac{1}{Ta_T}+\frac{1}{b_T}\right)=o\left(\log\frac{b_T}{a_T}\right)\quad \textit{as } T\to\infty.$$

Then

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{|W(t+a_T,T)-W(t,T)|}{\left(2Ta_T\log\frac{b_T}{a_T}\right)^{1/2}}=1 \quad \text{a.s},$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|W(t+s,T)-W(t,T)|}{\left(2Ta_T\log\frac{b_T}{a_T}\right)^{1/2}}=1 \quad \text{a.s.}$$

Corollary 2.6.2 *Let* $\{W(x,y);-\infty<x,y<\infty\}$ *be a standard two-parameter Wiener process. Assume that*

$$\log\log\left(a_T+c_T+a_Tc_T+\frac{1}{a_Tc_T}+\frac{1}{b_T}+\frac{1}{d_T}\right)$$
$$=o\left(\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right) \quad \text{as } T\to\infty.$$

Then

$$\lim_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant t\leqslant b_T}\frac{|W(R(t,a_T,v,c_T))|}{\left(2a_Tc_T\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}=1 \quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant u\leqslant c_T}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|W(R(t,s,v,u))|}{\left(2a_Tc_T\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}$$
$$=1 \quad \text{a.s.}$$

Corollary 2.6.3 *Let* $\{K(x,y);0\leqslant x\leqslant 1,0\leqslant y<\infty\}$ *be a Kiefer process,* a_T *and* b_T *be continuous functions with* $0\leqslant a_T+b_T\leqslant 1$. *Assume that*

$$\log\log\left(\frac{1}{b_T}+Ta_T+\frac{1}{Ta_T}\right)=o\left(\log\frac{b_T}{a_T}\right) \quad \text{as} \quad T\to\infty.$$

Then

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{|K(t+a_T,T)-K(t,T)|}{\left(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\right)^{1/2}}=1 \quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|K(t+s,T)-K(t,T)|}{\left(2Ta_T(1-a_T)\log\frac{b_T}{a_T}\right)^{1/2}}=1 \quad \text{a.s.}$$

Corollary 2.6.4 *Let* $\{K(x,y);0\leqslant x\leqslant 1,0\leqslant y<\infty\}$ *be a Kiefer process,* a_T, b_T, c_T, d_T *be continuous functions with* $0\leqslant a_T+b_T\leqslant 1$ *and* $0\leqslant a_T\leqslant 1/2$. *Assume that*

$$\log\log\left(\frac{1}{b_T}+\frac{1}{d_T}+c_T+\frac{1}{a_Tc_T}\right)=o\left(\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)$$

as $T\to\infty$. *Then*

$$\lim_{T\to\infty}\sup_{0\leqslant v\leqslant d_T}\sup_{0\leqslant t\leqslant b_T}\frac{|K(R(t,a_T,v,c_T))|}{\left(2a_T(1-a_T)c_T\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}$$
$$=1 \quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{\substack{0\leqslant v\leqslant d_T\\0\leqslant u\leqslant c_T}}\sup_{\substack{0\leqslant t\leqslant b_T\\0\leqslant s\leqslant a_T}}\frac{|K(R(t,s,v,u))|}{\left(2a_T(1-a_T)c_T\log\left(\frac{b_T}{a_T}+1\right)\left(\frac{d_T}{c_T}+1\right)\right)^{1/2}}=1 \quad \text{a.s.}$$

The proofs of Corollaries 2.6.1—2.6.4 are easy. The details are omitted here.

Corollary 2.6.5 *Let* $\{X(t,v);-\infty<t,v<\infty\}$ *be a two-parameter Gaussian process as in Example* 2.6.3. *Put*

$$H^2(a_T,T):=H_1^2(t,a_T,T)$$
$$=2\int_0^T\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)a_T))\,dx.$$

Assume that there exists $c_0>0$ *such that*

$$\int_{0<x\leqslant T,\lambda(x)\geqslant 1/s}\frac{\gamma(x)}{\lambda(x)}dx\leqslant c_0s\int_{0<x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)dx \qquad (2.6.81)$$

for each $0<s\leqslant a_T$,

$$\log\log\left(a_T+\frac{1}{b_T}+\tilde{H}(a_T,T)\right)=o\left(\log\frac{b_T}{a_T}\right) \quad \text{as } T\to\infty.$$

Then

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{|X(t+a_T,T)-X(t,T)|}{H(a_T,T)\left(2\log\frac{b_T}{a_T}\right)^{1/2}}=1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty}\sup_{|t|\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H(a_T,T)\left(2\log\frac{b_T}{a_T}\right)^{1/2}}=1\quad\text{a. s.}$$

Proof Noting that

$$EX(t,v)X(s,u)=\int_0^{v\wedge u}\exp(-\lambda(y)|t-s|)\frac{\gamma(y)}{\lambda(y)}\mathrm{d}y$$

for each $v,u>0$, we can verify that (2.6.45), (2.6.49), (2.6.28) are satisfied. We show below that $H^2(s,T)/s^\alpha$ is increasing in s on $(0,a_T)$, where $\alpha=1/(6(c_0+1))$. Let

$$f(s)=H^2(s,T)/s^\alpha=2\int_0^T\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))\mathrm{d}x/s^\alpha.$$

Then, by (2.6.81)

$$\begin{aligned}
f'(s)&=2s^{-\alpha-1}\left(-\alpha\int_0^T\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))\mathrm{d}x\right.\\
&\qquad\left.+\int_0^T s\gamma(x)\exp(-\lambda(x)s)\mathrm{d}x\right)\\
&\geqslant 2s^{-\alpha-1}\left(-\alpha\int_{0\leqslant x\leqslant T,\lambda(x)\geqslant 1/s}\frac{\gamma(x)}{\lambda(x)}\mathrm{d}x\right.\\
&\qquad\left.-\alpha s\int_{0\leqslant x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)\mathrm{d}x+\frac{s}{3}\int_{0\leqslant x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)\mathrm{d}x\right)\\
&\geqslant 2s^{-\alpha-1}\left(-\alpha(c_0+1)s\int_{0\leqslant x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)\mathrm{d}x\right.\\
&\qquad\left.+\frac{s}{3}\int_{0\leqslant x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)\mathrm{d}x\right)\\
&>0,
\end{aligned}$$

provided $0<\alpha<1/(3(c_0+1))$, as desired. Therefore, (2.6.46) is satisfied. Corollary 2.6.5 now follows from Theorem 2.6.1.

Corollary 2.6.6 *Let $d>0$, $b>0$, $\{X(t,v);-\infty<t,v<\infty\}$ be a two-parameter Gaussian process as in Example 2.6.3. Assume that $a_T\to 0$ and $c_T\to\infty$ as $T\to\infty$ and that*

$$\sup_{0<x<d+b}\lambda(x)<\infty,\tag{2.6.82}$$

$$x^{1-\alpha}\gamma(x)\leqslant c_0y^{1-\alpha}\gamma(y)\quad\textit{and}\ \gamma(y)y^{-1/\alpha}\leqslant c_0\gamma(x)x^{-1/\alpha}\tag{2.6.83}$$

for each $0<x<y\leqslant 1$, where $0<\alpha<1$, $c_0>0$. Then

$$\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d}\sup_{0\leqslant t\leqslant b}\frac{|X(R(t,a_T,v,c_T))|}{H(a_T,v,c_T)(2\log(c_Ta_T)^{-1})^{1/2}}=1\quad\text{a. s.},$$

$$\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d}\sup_{0\leqslant u\leqslant c_T}\sup_{|t|\leqslant b}\sup_{0\leqslant s\leqslant a_T}\frac{|X(R(t,s,v,u))|}{H(a_T,v,c_T)(2\log(c_Ta_T)^{-1})^{1/2}}=1\ \text{a. s.},$$

where $H^2(a_T,v,c_T)=2\int_v^{v+c_T}\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)a_T))\mathrm{d}x.$

Proof Put $M=\sup_{0<x\leqslant d+b}\lambda(x)$. Then $M<\infty$ by (2.6.82). It follows from (2.6.83) that

$$\int_0^{d+b}\gamma(x)\mathrm{d}x<\infty.\tag{2.6.84}$$

Clearly, for $0<s\leqslant 1/M$, $0\leqslant v+c\leqslant d+b$,

$$\begin{aligned}
\frac{s}{3}\int_v^{v+c}\gamma(x)\mathrm{d}x&\leqslant\int_v^{v+c}\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))\mathrm{d}x\\
&=\frac{1}{2}H^2(s,v,c)\\
&\leqslant s\int_v^{v+c}\gamma(x)\mathrm{d}x,
\end{aligned}$$

which implies that (2.6.29) is satisfied. We show next that (2.6.67) is also satisfied. It suffices to prove that

$$\lim_{\delta\to\infty}\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d+\frac{b}{2}-\delta c_T\leqslant u\leqslant c_T}\sup\frac{H^2(a_T,v+u,\delta c_T)}{H^2(a_T,v,c_T)}=0.\tag{2.6.85}$$

By (2.6.82) again, we see that

$$\frac{H^2(a_T,v,c)}{2a_T\int_v^{v+c}\gamma(x)\mathrm{d}x}\to 1$$

as $T\to\infty$, uniformly in $0<v\leqslant d+b$, $0<c<1$. Hence, equivalently, it is enough to show that

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{0\leqslant v\leqslant d+b/2}\sup_{-\delta c_T\leqslant u\leqslant c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)\mathrm{d}x}{\int_v^{v+c_T}\gamma(x)\mathrm{d}x}=0. \quad (2.6.86)$$

Noting that $\gamma(x)$ is a positive continuous function on $(0,\infty)$, we have

$$0<\inf_{1/3\leqslant x\leqslant d+b}\gamma(x)\leqslant\sup_{1/3\leqslant x\leqslant d+b}\gamma(x)<\infty.$$

Hence

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{1/2\leqslant v\leqslant d+b/2}\sup_{-\delta c_T\leqslant u\leqslant c_T}\frac{\int_{v+u}^{v+u+\delta c_T}\gamma(x)\mathrm{d}x}{\int_v^{v+c_T}\gamma(x)\mathrm{d}x}$$

$$\leqslant\lim_{\delta\to 0}\delta\frac{\sup_{1/3\leqslant x\leqslant d+b}\gamma(x)}{\inf_{1/3\leqslant x\leqslant d+b}\gamma(x)}=0. \quad (2.6.87)$$

If $0<v\leqslant 1/2, -\delta c_T\leqslant u\leqslant c_T/2$, then

$$\int_v^{v+c_T}\gamma(x)\mathrm{d}x\geqslant\int_{v+2c_T/3}^{v+c_T}\gamma(x)\mathrm{d}x\geqslant\frac{c_T}{3}\gamma(x_0), \quad (2.6.88)$$

where $v+c_T/3\leqslant x_0\leqslant v+c_T$. By (2.6.83) we find

$$\int_{v+u}^{v+u+\delta c_T}\gamma(x)\mathrm{d}x\leqslant c_0x_0^{1-\alpha}\gamma(x_0)\int_{v+u}^{v+u+\delta c_T}\frac{1}{x^{1-\alpha}}\mathrm{d}x$$

$$=\frac{c_0}{\alpha}x_0^{1-\alpha}\gamma(x_0)((v+u+\delta c_T)^\alpha-(v+u)^\alpha)$$

$$\leqslant\frac{c_0}{\alpha}(v+c_T)^{1-\alpha}\gamma(x_0)((v+u+\delta c_T)^\alpha-(v+u)^\alpha)$$

$$\leqslant\frac{6c_0(\delta^\alpha+\delta)c_T\gamma(x_0)}{\alpha}. \quad (2.6.89)$$

If $0<v\leqslant\frac{1}{2}$, $\frac{c_T}{2}\leqslant u\leqslant c_T$, then

$$\int_{v+u}^{v+u+\delta c_T}\gamma(x)\mathrm{d}x=\gamma(y_0)\delta c_T, \quad (2.6.90)$$

where $v+u\leqslant y_0\leqslant v+u+\delta c_T$. From (2.6.83) again, we obtain

$$\int_v^{v+c_T}\gamma(x)\mathrm{d}x\geqslant\int_v^{v+c_T/2}\gamma(x)\mathrm{d}x\geqslant\frac{\gamma(y_0)}{c_0y_0^{1/\alpha}}\int_v^{v+c_T/2}x^{\frac{1}{\alpha}}\mathrm{d}x$$

$$\geqslant\frac{\alpha\gamma(y_0)}{2c_0y_0^{1/\alpha}}\left(\left(v+\frac{c_T}{2}\right)^{1+1/\alpha}-v^{1+1/\alpha}\right)$$

$$\geqslant\frac{\alpha\gamma(y_0)}{2c_02^{1/\alpha}(v+c_T)^{1/\alpha}}\left(\left(v+\frac{c_T}{2}\right)^{1+1/\alpha}-v^{1+1/\alpha}\right)$$

$$\geqslant\frac{\alpha\gamma(y_0)c_T}{c_0(12)^{1+1/\alpha}}. \quad (2.6.91)$$

Combining (2.6.87)—(2.6.91) yields (2.6.86). This proves (2.6.67). Corollary 2.6.6 follows from Theorem 2.6.2.

Corollary 2.6.7 *Let $\{X(t,v);-\infty<t<\infty,0\leqslant v<\infty\}$ be a two-parameter Gaussian process as in Example 2.6.4. Assume that $\varphi_k(v)$ is non-decreasing in v for each k and that there exists $c_0>0$ such that*

$$\sum_{\lambda_k\geqslant 1/s}\varphi_k^2(v)\frac{\gamma_k}{\lambda_k}\leqslant c_0s\sum_{\lambda_k\leqslant 1/s}\varphi_k^2(v)\cdot\gamma_k\quad\textit{for } 0<s\leqslant 1, v>0,$$

and

$$\log\log\left(\sum_{k=0}^\infty\varphi_k^2(T)\frac{\gamma_k}{\lambda_k}\right)=o\left(\log\frac{1}{a_T}\right)\quad\textit{as } T\to\infty.$$

Then

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant 1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)\left(2\log\frac{1}{a_T}\right)^{1/2}}=1\quad\text{a.s.},$$

$$\limsup_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)\left(2\log\frac{1}{a_T}\right)^{1/2}}=1\quad\text{a.s.},$$

where $H_1^2(a_T,T)=2\sum_{k=1}^\infty\varphi_k^2(T)(1-\mathrm{e}^{-\lambda_ka_T})\frac{\gamma_k}{\lambda_k}$.

The proof is trivial. The details are omitted here.

2.7 Moduli of Continuity for Local Times of Gaussian Processes

Let $\{X(t);t\geqslant 0\}$ be a real-valued stochastic process. For any Borel set A of the real line let

$$H(A,t)=\lambda\{s;0\leqslant s\leqslant t,X(s)\in A\},\quad t\geqslant 0 \qquad (2.7.1)$$

be the occupation time of X, where λ is the Lebesgue measure. If, for each fixed t, $H(\cdot,t)$ is absolutely continuous with respect to Lebesgue measure, then its Radon-Nikodym derivative is called the local time of $X(\cdot)$ at t, denoted by $L(\cdot,t)$. In this case we say that the local time (occupation density) of $X(\cdot)$ exists. It follows from the definition of $L(x,t)$ that

$$L(x,0)=0,\quad L(x,s)\leqslant L(x,t),\quad \text{for } t\geqslant s\geqslant 0,\ x\in\mathbf{R}, \qquad (2.7.2)$$

$$H(A,t)=\int_A L(x,t)\mathrm{d}x \qquad (2.7.3)$$

and

$$H(A,t+h)-H(A,t)=\int_A (L(x,t+h)-L(x,t))\mathrm{d}x,\quad t,h\geqslant 0. \qquad (2.7.4)$$

Concerning the local time of a standard Wiener process there has been a great amount of elegant work. Hawkes (1971) showed the moduli of continuity in t: let $l(x,t)$ be the local time of W. Then

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1-h}\frac{l(0,t+h)-l(0,t)}{\{h\log h^{-1}\}^{1/2}}=1\quad \text{a.s.}, \qquad (2.7.5)$$

while Perkins (1981) obtained

$$\lim_{h\to 0}\sup\ \sup_{0\leqslant t\leqslant 1-h}\ \sup_{-\infty<x<\infty}\frac{l(x,t+h)-l(x,t)}{\left\{2h\log\frac{1}{h}\right\}^{1/2}}=1\quad \text{a.s.} \qquad (2.7.6)$$

Csáki, Csörgő, Földes and Révész (1983) showed that in (2.7.6) one can replace $\limsup_{h\to 0}$ by $\lim_{h\to 0}$ and proved also the following results:

Theorem 2.7.1 *Let $0<a_T\leqslant T$ be a non-decreasing function of $T\geqslant 0$, and assume that a_T/T is non-increasing. Then*

$$\lim_{T\to\infty}\sup\ \sup_{0\leqslant t\leqslant T-a_T}\frac{l(x,t+a_T)-l(x,t)}{\left\{a_T\left(\log\frac{T}{a_T}+2\log\log T\right)\right\}^{1/2}}=1\quad \text{a.s.} \qquad (2.7.7)$$

for each $x\in\mathbf{R}$, and

$$\lim_{T\to\infty}\sup\ \sup_{0\leqslant t\leqslant T-a_T}\ \sup_{-\infty<x<\infty}\frac{l(x,t+a_T)-l(x,t)}{\left\{2a_T\left(\log\frac{T}{a_T}+\log\log T\right)\right\}^{1/2}}=1\quad \text{a.s.} \qquad (2.7.8)$$

Moreover, if we also assume that $\lim_{T\to\infty}\left(\log\frac{T}{a_T}\right)/\log\log T=\infty$, *then* $\lim\sup_{T\to\infty}$ *can be raplaced by* $\lim_{T\to\infty}$ *in* (2.7.7) *and* (2.7.8).

Taking $a_T=T$ in both (2.7.7) and (2.7.8), we obtain the law of the iterated logarithm, proved by Kesten (1965), for the Brownian local time $l(\cdot,\cdot)$:

$$\lim_{T\to\infty}\sup\frac{l(x,T)}{(2T\log\log T)^{1/2}}=\lim_{T\to\infty}\sup\frac{\sup_{-\infty<x<\infty}l(x,T)}{(2T\log\log T)^{1/2}}=1\quad \text{a.s.} \qquad (2.7.9)$$

In this section we present and discuss results, which are analogues to (2.7.5), (2.7.6) and (2.7.7) for local time $L(x,t)$ of a Gaussian process $\{X(t);t\geqslant 0\}$. These results are due to Csörgő, Lin and Shao (1995).

2.7.1 Moments of increments of local times of Gaussian processes

Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero and stationary increments. Put

$$\sigma^2(h)=E(X(t+h)-X(t))^2,\quad t,h\geqslant 0.$$

It is known (cf. Berman 1969 and Geman 1976) that if

$$\int_0^t \frac{\mathrm{d}s}{\sigma(s)}<\infty \quad \text{for each } t>0, \tag{2.7.10}$$

then the local time $L(x,t)$ of X exists. It is also known that if $\sigma^2(h)$ is continuous and concave for $0\leqslant h\leqslant 1$, then the local time $L(x,t)$ of X exists and is jointly continuous almost surely (cf. Berman, 1972).

The following result gives estimates for moments of increments of $L(x,t)$ in t:

Proposition 2.7.1 *Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero, stationary increments and $X(0)=0$. Put $\sigma^2(h)=E(X(t+h)-X(t))^2$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,h_0)$, satisfying*

$$\sigma(ah)\geqslant c_0a^\alpha\sigma(h) \ \text{for}\ 0<a<1,\ 0\leqslant h\leqslant h_0 \tag{2.7.11}$$

for some $0<\alpha\leqslant 1/2$, $c_0>0$, $h_0>0$. Then

$$E(L(x,t+h)-L(x,t))^m$$

$$\leqslant\left(\frac{16h}{c_0\sigma(h)}\right)^m(m!)^\alpha\exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right) \tag{2.7.12}$$

for each integer $m\geqslant 1$, $0<h\leqslant h_0$, $x\in\mathbf{R}$.

To prove this result, we need some lemmas, the first three are on matrices and covariance matrices.

Lemma 2.7.1 *Let $\xi_1,\cdots,\xi_n$ be random variables with finite second moments and $\boldsymbol{A}_n$ be their covariance matrix. Then*

$$|\boldsymbol{A}_n|\leqslant|\boldsymbol{A}_{n-1}|\operatorname{Var}\xi_n, \tag{2.7.13}$$

where $|\boldsymbol{A}_n|$ denotes the determinant of $\boldsymbol{A}_n$.

Proof Write $\boldsymbol{A}_n=(a_{ij})$, $1\leqslant i,j\leqslant n$, where $a_{ij}=\operatorname{Cov}(\xi_i,\xi_j)$. Noting that

$$(a_{1n},\cdots,a_{n-1,n})'(a_{1n},\cdots,a_{n-1,n})$$

is positive semi-definte, we obtain

$$\begin{aligned}
|\boldsymbol{A}_n| &= \left|\left(a_{ij}-\frac{a_{ni}a_{nj}}{a_{nn}};1\leqslant i,j\leqslant n-1\right)\right|\operatorname{Var}\xi_n\\
&=\left|\left(\operatorname{Cov}\left(\xi_i-\frac{a_{1i}\xi_n}{a_{nn}},\xi_j-\frac{a_{1j}\xi_n}{a_{nn}}\right),1\leqslant i,j\leqslant n-1\right)\right|\operatorname{Var}\xi_n\\
&=|(a_{ij};1\leqslant i,j\leqslant n-1)\\
&\quad-(a_{1n},\cdots\cdots,a_{n-1,n})'(a_{1n},\cdots\cdots,a_{n-1,n})|\operatorname{Var}\xi_n\\
&\leqslant|(a_{ij};1\leqslant i,j\leqslant n-1)|\operatorname{Var}\xi_n\\
&=|\boldsymbol{A}_{n-1}|\operatorname{Var}\xi_n,
\end{aligned}$$

which is (2.7.13), as claimed.

Lemma 2.7.2 *Let $\boldsymbol{B}_n=(b_{ij};1\leqslant i,j\leqslant n)$ and $\widetilde{\boldsymbol{B}}_{n-1}=(b_{ij};2\leqslant i,j\leqslant n)$ be real valued matrices. Assume that for each $1\leqslant i\leqslant n$, $|b_{ii}|\geqslant\sum_{j\neq i}|b_{ij}|$. Then*

$$\begin{aligned}
|\boldsymbol{B}_n| &\geqslant\left(|b_{11}|-\sum_{i=2}^n|b_{1i}|\right)|\widetilde{\boldsymbol{B}}_{n-1}|\\
&\geqslant|b_{nn}|\prod_{i=1}^{n-1}\left(|b_{ii}|-\sum_{j=i+1}^n|b_{ij}|\right).
\end{aligned}$$

This lemma is due to Price (1951).

Lemma 2.7.3 *Let $\{\xi(t);t\geqslant 0\}$ be a Gaussian process with mean zero, stationary increments and $\xi(0)=0$. Put $\sigma^2(h)=E(\xi(t+h)-\xi(t))^2$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,h_0)$. For $t\leqslant t_1<t_2<\cdots<t_n\leqslant t+h_0$, let A_n be the covariance matrix of $\xi(t_1),\cdots,\xi(t_n)$. Then*

$$|A_n| \geqslant (1/2)^n\sigma^2(t_1)\prod_{i=2}^{n}\sigma^2(t_i-t_{i-1}).$$

Proof By the assumption that $\sigma^2(h)$ is concave, we have

$$\sigma^2(d-a)+\sigma^2(c-b)\leqslant\sigma^2(d-b)+\sigma^2(c-a) \quad (2.7.14)$$

for each $0\leqslant a\leqslant b\leqslant c\leqslant d\leqslant h_0+t$.

Let $\widetilde{A}_n$ be the covariance matrix of $\xi(t_1),\xi(t_2)-\xi(t_1),\cdots,\xi(t_n)-\xi(t_{n-1})$. It is easy to see that $|A_n|=|\widetilde{A}_n|$.

Put $\widetilde{A}_n=(a_{ij};1\leqslant i,j\leqslant n)$, where

$a_{11}=E\xi^2(t_1)$,

$a_{1i}=E\xi(t_1)(\xi(t_i)-\xi(t_{i-1}))$, for all $2\leqslant i\leqslant n$,

$a_{ij}=E(\xi(t_i)-\xi(t_{i-1}))(\xi(t_j)-\xi(t_{j-1}))$, for all $2\leqslant i,j\leqslant n$.

When $1<i<j\leqslant n$, we can write

$$\begin{aligned}a_{ij}&=\frac{1}{2}\left(\sigma^2(t_j-t_{i-1})+\sigma^2(t_{j-1}-t_i)-\sigma^2(t_j-t_i)-\sigma^2(t_{j-1}-t_{i-1})\right)\\&\leqslant 0\end{aligned}$$

by (2.7.14). When $1<i\leqslant n$, we have

$$a_{1i}=(\sigma^2(t_{i-1}-t_1)+\sigma^2(t_i)-\sigma^2(t_i-t_1)-\sigma^2(t_{i-1}))/2\leqslant 0$$

by (2.7.14) again. Therefore, we obtain for $1\leqslant i<n$

$$\begin{aligned}\sum_{j=i+1}^{n}|a_{ij}| &= -\sum_{j=i+1}^{n}a_{ij}\\&=-E(\xi(t_i)-\xi(t_{i-1}))(\xi(t_n)-\xi(t_i))\\&=\frac{1}{2}\left(\sigma^2(t_n-t_i)+\sigma^2(t_i-t_{i-1})-\sigma^2(t_n-t_{i-1})\right)\\&\leqslant\frac{1}{2}\sigma^2(t_i-t_{i-1})=\frac{1}{2}a_{ii},\end{aligned} \quad (2.7.15)$$

where $t_0=0$. For $2\leqslant i\leqslant n$, we have

$$\begin{aligned}\sum_{j=1}^{i-1}|a_{ij}| &= -\sum_{j=1}^{i-1}a_{ij}\\&=-E(\xi(t_i)-\xi(t_{i-1}))\xi(t_{i-1})\\&=\frac{1}{2}\left(\sigma^2(t_i-t_{i-1})-E\xi^2(t_i)+E\xi^2(t_{i-1})\right)\\&\leqslant\frac{1}{2}\sigma^2(t_i-t_{i-1}).\end{aligned}$$

It follows that

$$|a_{ii}|\geqslant\sum_{j\neq i}|a_{ij}|$$

for each $1\leqslant i\leqslant n$. Now the result follows from Lemma 2.7.2 and (2.7.15).

We now utilize the Fourier analytic approach to local times due to Berman (1969, 1974). Let

$$f(u,t)=\int_{-\infty}^{\infty}e^{iux}d_xH([0,x],t)=\int_0^t e^{iuX(s)}ds, \quad -\infty<u<\infty,$$

the Fourier transform of the occupation time $H([0,x],t)$. We can express $L(x,t)$ as the inverse Fourier transform of $f(u,t)$, namely

$$\begin{aligned}L(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iux}f(u,t)du\\&=\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}e^{-iux}e^{iuX(s)}du\,ds.\end{aligned} \quad (2.7.16)$$

Lemma 2.7.4 *Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero, stationary increments with incremental variance function $\sigma^2(h)$. Let $m\geqslant 1$ be an integer and $\boldsymbol{R}(s_1,\cdots,s_n)$ be the covariance matrix of $X(s_1)$, $X(s_2)-X(s_1)$, $\cdots$, $X(s_m)-X(s_{m-1})$. Then*

$E(L(x,t+h)-L(x,t))^m$

$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} m! \int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h} \exp\left(-\frac{x^2}{2EX^2(s_1)}\right)$$

$$\times |\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2} \mathrm{d}s_1\cdots\mathrm{d}s_m, \tag{2.7.17}$$

for each $x\in\mathbf{R}$, $t\geqslant 0$ *and* $h>0$.

Proof Put $v_j = \sum_{i=j}^{m} u_i, 1\leqslant j\leqslant m, V=(v_1,\cdots,v_m)$. Using (2.7.16), we can write

$$E(L(x,t+h)-L(x,t))^m$$

$$= \left(\frac{1}{2\pi}\right)^m \int_t^{t+h}\cdots\int_t^{t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}$$

$$\exp\left(-\mathrm{i}x\sum_{j=1}^m u_j\right) E\exp\left(\mathrm{i}\sum_{j=1}^m u_j X(s_j)\right) \mathrm{d}u_1\cdots\mathrm{d}u_m \mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$= \left(\frac{1}{2\pi}\right)^m m! \int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp(-\mathrm{i}xv_1)$$

$$\times E\exp\left[\mathrm{i}v_1X(s_1)+\mathrm{i}\sum_{j=2}^m v_j(X(s_j)-X(s_{j-1}))\right]\mathrm{d}v_1\cdots\mathrm{d}v_m\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$= \left(\frac{1}{2\pi}\right)^m m! \int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp(-\mathrm{i}xv_1)$$

$$\times \exp\left[-\mathrm{i}xv_1 - \frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right]\mathrm{d}v_1\cdots\mathrm{d}v_m\cdots\mathrm{d}s_1\cdots\mathrm{d}s_m.$$

To prove (2.7.17), it is suffices to show that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\mathrm{i}xv_1-\frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right]\mathrm{d}v_1\cdots\mathrm{d}v_m$$

$$\leqslant (2\pi)^{m/2}\exp\left(-\frac{x^2}{2EX^2(s_1)}\right)|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}. \tag{2.7.18}$$

If $|\boldsymbol{R}(s_1,\cdots,s_m)|=0$, then (2.7.18) is trivial. So, we assume $|\boldsymbol{R}(s_1,\cdots,s_m)|>0$. That is, $\boldsymbol{R}(s_1,\cdots,s_m)$ is positive definite. Hence $\boldsymbol{R}^{-1}(s_1,\cdots,s_m)$ is also positive definite. Let $(Y_1,\cdots,Y_m)$ be distributed according to the multivariate normal distribution with mean zero and the covariance matrix $\boldsymbol{R}^{-1}(s_1,\cdots,s_m)$. Then, the

density of $(Y_1,\cdots,Y_m)$ is given by

$$\left(\frac{1}{2\pi}\right)^{-m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{1/2}\exp\left(-\frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right).$$

Therefore

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\mathrm{i}xv_1-\frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right]\mathrm{d}v_1\cdots\mathrm{d}v_m$$

$$= (2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}E\mathrm{e}^{-\mathrm{i}xY_1}$$

$$= (2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\mathrm{e}^{-(x^2/2)EY_1^2}. \tag{2.7.19}$$

Put $\boldsymbol{R}(s_1,\cdots,s_n)=(\gamma_{ij};1\leqslant i,j\leqslant n)$. Then

$$EY_1^2=\frac{|(\gamma_{ij};2\leqslant i,j\leqslant n)|}{|(\gamma_{ij};1\leqslant i,j\leqslant n)|}\geqslant\frac{1}{\gamma_{11}}=\frac{1}{EX^2(s_1)} \tag{2.7.20}$$

by Lemma 2.7.1. This proves (2.7.18) by (2.7.19) and (2.7.20) and the proof of (2.7.17) is now completed.

Proof of Proposition 2.7.1 It follows from Lemmas 2.7.3 and 2.7.4 that

$$E(L(x,t+h)-L(x,t))^m$$

$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} m! \int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h} \frac{\exp(-x^2/(2EX^2(s_1)))2^m}{\sigma(s_1)\prod_{j=2}^m\sigma(s_j-s_{j-1})}\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$\leqslant \left(\frac{2}{\pi}\right)^{m/2} m!\exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$

$$\times \int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h} \frac{1}{\sigma(s_1)\prod_{j=2}^m\sigma(s_j-s_{j-1})}\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$= \left(\frac{2}{\pi}\right)^{m/2} m!h^m\exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$

$$\times \int\cdots\int_{0\leqslant s_1<s_2<\cdots<s_m\leqslant 1} \frac{1}{\sigma(s_1h+t)\prod_{j=2}^m\sigma((s_j-s_{j-1})h)}\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$\leqslant \left(\frac{2}{\pi}\right)^{m/2} \frac{m!h^m\exp(-x^2/(2\sigma^2(t+h)))}{\sigma(h+t)\sigma^{m-1}(h)c_0^m}$$

$$\times \int\cdots\int_{0\leqslant s_1<s_2<\cdots<s_m\leqslant 1} \frac{1}{s_1^\alpha\prod_{j=2}^m(s_j-s_{j-1})^\alpha}\mathrm{d}s_1\cdots\mathrm{d}s_m \tag{2.7.21}$$

by (2.7.11). By elementary calculations (cf. Ehm, 1981)

$$\int\cdots\int_{0\leqslant s_1<s_2<\cdots<s_m\leqslant 1}\prod_{j=1}^{m}(s_j-s_{j-1})^{-b_j}\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$=\Big(\prod_{j=1}^{m}\Gamma(1-b_j)\Big)\Big/\Gamma\Big(1+m-\sum_{j=1}^{m}b_j\Big)$$

for $b_j<1$, $j=1,\cdots,m$, where $\Gamma(\cdot)$ is the gamma function. Hence, we have

$$E(L(x,t+h)-L(x,t))^m$$

$$\leqslant\left(\frac{2}{\pi c_0^2}\right)^{m/2}m!\ \left(\frac{h}{\sigma(h)}\right)^m\exp\left(\frac{-x^2}{2\sigma^2(t+h)}\right)\frac{\Gamma^m(1-\alpha)}{\Gamma(1+m(1-\alpha))}. \tag{2.7.22}$$

It is easy to see that

$$\Gamma(1-\alpha)\leqslant\frac{2}{1-\alpha}\leqslant 4$$

for $0<\alpha\leqslant 1/2$. Noting that $\Gamma(y)$ is non-decreasing on $(3/2,\infty)$, we have

$$\Gamma(1+m(1-\alpha))\geqslant\Gamma(1+[m(1-\alpha)])=[m(1-\alpha)]!.$$

Using Stirling's formula, we obtain

$$\frac{m!}{[m(1-\alpha)]!}\leqslant 2^m\left(\frac{1}{1-\alpha}\right)^m(m!)^\alpha\leqslant 4^m(m!)^\alpha.$$

Therefore we conclude

$$E(L(x,t+h)-L(x,t))^m$$

$$\leqslant\left(\frac{16h}{c_0\sigma(h)}\right)^m(m!)^\alpha\exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right).$$

This proves (2.7.12).

For stationary Gaussian processes, we have

Proposition 2.7.2 *Let $\{X(t);t\geqslant 0\}$ be a stationary Gaussian process with mean zero and $EX^2(0)=1$. Put $\sigma^2(h)=E(X(t+h)-X(t))^2$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,$*

$h_0)$ with $\sigma^2(h_0)\leqslant 2$ and condition (2.7.11) is satisfied. Then

$$E(L(x,t+h)-L(x,t))^m$$

$$\leqslant\sigma(h)\left(\frac{16h}{c_0\sigma(h)}\right)^m(m!)^\alpha\exp(-x^2/2) \tag{2.7.23}$$

for each integer $m\geqslant 2$, $0<h\leqslant h_0$, $x\in\mathbf{R}$.

Proof The proof is similar to that of Proposition 2.7.1, using the following lemma instead of Lemma 2.7.3, and hence, omitted.

Lemma 2.7.5 *Let $\{\xi(t);t\geqslant 0\}$ be a stationary Gaussian Process with mean zero and $E\xi^2(0)=1$. Put $\sigma^2(h)=E(\xi(t+h)-\xi(t))^2$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,h_0)$ with $\sigma^2(h_0)\leqslant 2$. For $t\leqslant t_1<t_2<\cdots<t_n\leqslant t+h_0$, let $\mathbf{A}_n$ be the covariance matrix of $\xi(t_1),\cdots,\xi(t_n)$. Then*

$$|\mathbf{A}_n|\geqslant(1/2)^n\prod_{i=2}^{n}\sigma^2(t_i-t_{i-1}). \tag{2.7.24}$$

Proof Let $\widetilde{\mathbf{A}}_n$ be the covariance matrix of $\xi(t_n),\xi(t_{n-1})-\xi(t_n),\cdots,\xi(t_1)-\xi(t_2)$. Then $|\mathbf{A}_n|=|\widetilde{\mathbf{A}}_n|$. The rest of the proof is completely similar to that of Lemma 2.7.3 and so is omitted.

The next result is about the estimates of maxinum increments of local time of a Gaussian process with stationary increments.

Proposition 2.7.3 *Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero, stationary increments and $X(0)=0$. Put $\sigma^2(h)=E(X(t+h)-X(t))^2$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,1)$, satisfying*

$$\sigma(ah)\geqslant c_0a^\alpha\sigma(h)\quad \text{for } 0<a,h\leqslant 1, \tag{2.7.25}$$

for some $0<\alpha\leqslant 1/2$ and $c_0>0$. Then

$$E\sup_x (L(x,t+h)-L(x,t))^m \leqslant C_1 h^{-(4/3)\alpha}\left(\frac{524h}{c_0\sigma(h)}\right)^m (m!)^{1+\alpha} \tag{2.7.26}$$

for each even integer $m\geqslant 4, 0<h\leqslant 1, 0\leqslant t\leqslant 1$, *where*

$$C_1 = 5000(1+\sigma(2))c_0^{-8/3}\sigma^{-4/3}(1).$$

To prove Proposition 2.7.3, we need some more lemmas.

Lemma 2.7.6 *Let* $\{X(t);t\geqslant 0\}$ *be a Gaussian process with mean zero, stationary increments and incremental variance function* $\sigma^2(h)$. *Let* $m\geqslant 4$ *be an even integer and* $\boldsymbol{R}(s_1,\cdots,s_m)$ *be the covariance matrix of* $X(s_1), X(s_2)-X(s_1),\cdots,X(s_m)-X(s_{m-1})$. *Then*

$$E(L(x+y,t+h)-L(x+y,t)-L(x,t+h)+L(x,t))^m$$
$$\leqslant 3\left(\frac{1}{2\pi}\right)^{m/2} m!\,|y|^{2\delta}\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\exp\left(-\frac{x^2}{4EX^2(s_1)}\right)$$
$$\times|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2})\mathrm{d}s_1\cdots\mathrm{d}s_m \tag{2.7.27}$$

for each $0<\delta\leqslant 1$, $xy\geqslant 0$, $t\geqslant 0, h>0$, *where*

$$\rho_m = \frac{|(\gamma_{ij};1\leqslant i,j\leqslant m-1)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|},$$
$$\rho_1 = \frac{|(\gamma_{ij};2\leqslant i,j\leqslant m)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|},$$
$$\rho_2 = \frac{|(\gamma_{ij};1\leqslant i,j\leqslant m, i,j\neq 2)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|},$$
$$\boldsymbol{R}(s_1,\cdots,s_m) = (\gamma_{ij};1\leqslant i,j\leqslant m).$$

Proof Using (2.7.16), we have

$$L(x+y,t+h)-L(x+y,t)-L(x,t+h)+L(x,t)$$
$$=\frac{1}{2\pi}\int_t^{t+h}\int_{-\infty}^{\infty}(\mathrm{e}^{-\mathrm{i}u(x+y)}-\mathrm{e}^{-\mathrm{i}ux})\mathrm{e}^{\mathrm{i}uX(s)}\mathrm{d}u\mathrm{d}s. \tag{2.7.28}$$

Put $v_j=\sum_{l=j}^m u_l, 1\leqslant j\leqslant m$, $V=(v_1,\cdots,v_m)$, $v_{m+1}=0$. By (2.7.28), we can write

$$E(L(x+y,t+h)-L(x+y,t)-L(x,t+h)+L(x,t))^m$$

$$=\left(\frac{1}{2\pi}\right)^m\int_t^{t+h}\cdots\int_t^{t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(\mathrm{e}^{-\mathrm{i}(x+y)u_j}-\mathrm{e}^{-\mathrm{i}xu_j})$$
$$\times E\exp\left(\mathrm{i}\sum_{j=1}^m u_jX(s_j)\right)\mathrm{d}u_1\cdots\mathrm{d}u_m\mathrm{d}s_1\cdots\mathrm{d}s_m$$
$$=\left(\frac{1}{2\pi}\right)^m m!\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(\mathrm{e}^{-\mathrm{i}(x+y)u_j}-\mathrm{e}^{-\mathrm{i}xu_j})$$
$$\times E\exp[\mathrm{i}u_1X(s_1)+\mathrm{i}\sum_{j=2}^m u_j(X(s_j)-X(s_{j-1}))]\mathrm{d}u_1\cdots\mathrm{d}u_m\mathrm{d}s_1\cdots\mathrm{d}s_m$$
$$=\left(\frac{1}{2\pi}\right)^m m!$$
$$\times\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(\mathrm{e}^{-\mathrm{i}(x+y)(v_j-v_{j+1})}-\mathrm{e}^{-\mathrm{i}x(v_j-v_{j+1})})$$
$$\times\exp\left(-\frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right)\mathrm{d}v_1\cdots\mathrm{d}v_m\mathrm{d}s_1\cdots\mathrm{d}s_m. \tag{2.7.29}$$

To prove (2.7.27), it suffices to show that

$$I:=\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^m(\mathrm{e}^{-\mathrm{i}(x+y)(v_j-v_{j+1})}-\mathrm{e}^{-\mathrm{i}x(v_j-v_{j+1})})\right.$$
$$\left.\times\exp\left(-\frac{1}{2}V\boldsymbol{R}(s_1,\cdots,s_m)V'\right)\mathrm{d}v_1\cdots\mathrm{d}v_m\right|$$
$$\leqslant 3(2\pi)^{m/2}\exp\left(-\frac{x^2}{4EX^2(s_1)}\right)\cdot|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}$$
$$\times|y|^{2\delta}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2}). \tag{2.7.30}$$

If $|\boldsymbol{R}(s_1,\cdots,s_m)|=0$, then (2.7.30) is trivial. Hence we can assume $|\boldsymbol{R}(s_1,\cdots,s_m)|>0$. Then $\boldsymbol{R}(s_1,\cdots,s_m)$ is positive definite and so is $\boldsymbol{R}^{-1}(s_1,\cdots,s_m)$. Let $(Y_1,\cdots,Y_m)$ be distributed according to the multivariate normal distribution with mean zero and the covariance matrix $\boldsymbol{R}^{-1}(s_1,\cdots,s_m)$, $Y_{m+1}=0$. Then

$$I=(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}$$
$$\times\left|E\prod_{j=1}^m(\mathrm{e}^{-\mathrm{i}(x+y)(Y_j-Y_{j+1})}-\mathrm{e}^{-\mathrm{i}x(Y_j-Y_{j+1})})\right|$$

$$\leqslant (2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}$$
$$\times \left|E\prod_{j=2}^{m}(e^{i(x+y)(Y_j-Y_{j+1})}-e^{ix(Y_j-Y_{j+1})})\right.$$
$$\times E((e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)})|Y_2,\cdots,Y_m)\Big|$$
$$\leqslant (2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}E\Big\{|e^{-iyY_m}-1|$$
$$\times|E((e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)})|Y_2,\cdots,Y_m)|\Big\}. \tag{2.7.31}$$

Note that the conditional distribution of Y_1 given $Y_2,\cdots,Y_m$ is a unvariate Gaussian distribution with conditional mean $E(Y_1|Y_2,\cdots,Y_m)$ and conditional variance

$$E(Y_1-E(Y_1|Y_2,\cdots,Y_m))^2=\frac{1}{\gamma_{11}},\ \text{where } \gamma_{11}=EX^2(s_1).$$

Therefore, we have

$$\left|E\left((e^{-i(x+y)(Y_1-Y_2)}-e^{-ix(Y_1-Y_2)})|Y_2,\cdots,Y_m\right)\right|$$
$$=\left|\exp\left(i(x+y)Y_2-i(x+y)E(Y_1|Y_2,\cdots,Y_m)-\frac{(x+y)^2}{2\gamma_{11}}\right)\right.$$
$$\left.-\exp\left(ixY_2-ixE(Y_1|Y_2,\cdots,Y_m)-\frac{x^2}{2\gamma_{11}}\right)\right|$$
$$\leqslant\left|\exp\left(iyY_2-iyE(Y_1|Y_2,\cdots,Y_m)-\frac{(x+y)^2}{2\gamma_{11}}\right)-e^{-x^2/(2\gamma_{11})}\right|$$
$$\leqslant\exp\left(-\frac{(x+y)^2}{2\gamma_{11}}\right)|\exp(iyY_2-iyE(Y_1|Y_2,\cdots,Y_m))-1|$$
$$+\left|\exp\left(-\frac{(x+y)^2}{2\gamma_{11}}\right)-\exp\left(-\frac{x^2}{2\gamma_{11}}\right)\right|$$
$$\leqslant\exp\left(-\frac{x^2}{2\gamma_{11}}\right)|y|^\delta(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta)$$
$$+2\exp\left(-\frac{x^2}{4\gamma_{11}}\right)\left|\frac{y}{\sqrt{\gamma_{11}}}\right|^\delta$$

$$\leqslant\exp\left(-\frac{x^2}{2\gamma_{11}}\right)|y|^\delta\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right). \tag{2.7.32}$$

Here we used the following elementary conclusion:

$$e^{-b^2}-e^{-a^2}\leqslant 2e^{-b^2/2}((a-b)\wedge 1)$$

for each $a\geqslant b\geqslant 0$.

From (2.7.32) and (2.7.31) it follows that

$$I\leqslant(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^\delta E\Big\{|e^{-iyY_m}-1|$$
$$\times\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right)\Big\}$$
$$\leqslant(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^{2\delta}$$
$$\times E\left\{|Y_m|^\delta\left(|Y_2|^\delta+|E(Y_1|Y_2,\cdots,Y_m)|^\delta+\frac{2}{\gamma_{11}^{\delta/2}}\right)\right\}$$
$$\leqslant(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^{2\delta}$$
$$\times(EY_m^2)^{\delta/2}\left((EY_2^2)^{\delta/2}+(EY_1^2)^{\delta/2}+\frac{2}{\gamma_{11}^{\delta/2}}\right)$$
$$=(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^{2\delta}$$
$$\times\left(\frac{|(\gamma_{ij};1\leqslant i,j\leqslant m-1)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}$$
$$\times\left(\left(\frac{|(\gamma_{ij};1\leqslant i,j\leqslant m,i,j\neq 2)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}\right.$$
$$\left.+\left(\frac{|(\gamma_{ij};2\leqslant i,j\leqslant m)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}+\frac{2}{\gamma_{11}^{\delta/2}}\right)$$
$$\leqslant(2\pi)^{m/2}|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\exp\left(-\frac{x^2}{4\gamma_{11}}\right)|y|^{2\delta}$$
$$\times\left(\frac{|(\gamma_{ij};1\leqslant i,j\leqslant m-1)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}$$

$$\times\left(\left(\frac{|(\gamma_{ij};1\leqslant i,j\leqslant m,i,j\neq 2)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}+\left(\frac{|(\gamma_{ij};2\leqslant i,j\leqslant m)|}{|(\gamma_{ij};1\leqslant i,j\leqslant m)|}\right)^{\delta/2}\right) \tag{2.7.33}$$

by Lemma 2.7.1. This proves (2.7.29). The proof of Lemma 2.7.5 is now completed.

Lemma 2.7.7 *Under the assumptions and notations of Lemma 2.7.6, we have*

$$E\sup_x(L(x,t+h)-L(x,t))^m$$
$$\leqslant 4\left(1-\left(\frac{1}{2}\right)^{(2\delta-1)/m}\right)^{-m}m!\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\left(1+\sqrt{EX^2(s_1)}\right)$$
$$\times|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2})\mathrm{d}s_1\cdots\mathrm{d}s_m \tag{2.7.34}$$

for each $1/2<\delta\leqslant 1$, *even number* $m\geqslant 4$, $t\geqslant 0$, $h>0$.

Proof It is clear that

$$E\sup_x(L(x,t+h)-L(x,t))^m$$
$$\leqslant 2^m\sum_{k=-\infty}^{\infty}E(L(k,t+h)-L(k,t))^m$$
$$+2^m\sum_{k=-\infty}^{\infty}E\sup_{0\leqslant y\leqslant 1}(L(k+y,t+h)$$
$$-L(k+y,t)-L(k,t+h)+L(k,t))^m. \tag{2.7.35}$$

By Lemma 2.7.4, we have

$$E(L(k,t+h)-L(k,t))^m$$
$$\leqslant\left(\frac{1}{2\pi}\right)^{m/2}m!\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\exp\left(-\frac{k^2}{2EX^2(s_1)}\right)$$
$$\times|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\mathrm{d}s_1\cdots\mathrm{d}s_m. \tag{2.7.36}$$

A combination of Lemma 2.7.6 and a theorem of Móricz (1982) yields

$$E\sup_{0\leqslant y\leqslant 1}(L(k+y,t+h)-L(k+y,t)-L(k,t+h)+L(k,t))^m$$

$$\leqslant 3\times 2^{m+1}\left(1-\left(\frac{1}{2}\right)^{(2\delta-1)/m}\right)^{-m}\left(\frac{1}{2\pi}\right)^{m/2}m!$$
$$\times\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\exp\left(-\frac{k^2}{4EX^2(s_1)}\right)$$
$$\times|\boldsymbol{R}(s_1,\cdots,s_m)|^{-1/2}\rho_m^{\delta/2}(\rho_1^{\delta/2}+\rho_2^{\delta/2})\mathrm{d}s_1\cdots\mathrm{d}s_m. \tag{2.7.37}$$

Now (2.7.34) follows from (2.7.35), (2.7.36) and (2.7.37) via elementary calculations.

Proof of Proposition 2.7.3 In terms of Lemmas 2.7.1 and 2.7.3, we get

$$\rho_m\leqslant 2\sigma^{-2}(s_m-s_{m-1}),\quad \rho_1\leqslant 2(EX^2(s_1))^{-1},\quad \rho_2\leqslant 2\sigma^{-2}(s_2-s_1)$$

and

$$|\boldsymbol{R}(s,\cdots,s_m)|^{-1/2}\leqslant 2^m(EX^2(s_1)\sigma^2(s_2-s_1)\cdots\sigma^2(s_m-s_{m-1}))^{-1/2}.$$

Taking $\delta=2/3$ in Lemma 2.7.7, along the lines of the proof of Proposition 2.7.1, we obtain

$$E\sup_x(L(x,t+h)-L(x,t))^m$$
$$\leqslant 8\times 2^m\left(1-\left(\frac{1}{2}\right)^{1/3m}\right)^{-m}m!\ (1+\sigma(2))$$
$$\times\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\frac{1}{\sigma(s_1)\sigma(s_2-s_1)\cdots\sigma^{5/3}(s_m-s_{m-1})}$$
$$\times(\sigma^{-2/3}(s_1)+\sigma^{-2/3}(s_2-s_1))\mathrm{d}s_1\cdots\mathrm{d}s_m$$
$$\leqslant 16\times 2^m(6m)^m m!(1+\sigma(2))h^m$$
$$\times\int\cdots\int_{t\leqslant s_1<s_2<\cdots<s_m\leqslant t+h}\frac{1}{\sigma(s_1h)\sigma((s_2-s_1)h)\cdots\sigma^{5/3}((s_m-s_{m-1})h)}$$
$$\times((\sigma^{-2/3}(s_1h)+\sigma^{-2/3}((s_2-s_1)h))\mathrm{d}s_1\cdots\mathrm{d}s_m$$
$$\leqslant 16\times 2^m(6m)^m m!(1+\sigma(2))h^m\sigma(h)^{-m-4/3}c_0^{-m-4/3}$$
$$\times\int\cdots\int_{0\leqslant s_1<s_2<\cdots<s_m\leqslant 1}s_1^{-\alpha}(s_2-s_1)^{-\alpha}\cdots(s_m-s_{m-1})^{-5\alpha/3}$$
$$\times(s_1^{-2\alpha/3}+(s_2-s_1)^{-2\alpha/3})\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$\leqslant 32 \times 12^m m^m m!(h/\sigma(h))^m c_0^{-m-4/3}\sigma^{-4/3}(h)(1+\sigma(2))$$

$$\times \frac{\Gamma^{m-2}(1-\alpha)\Gamma^2(1-\frac{5}{3}\alpha)}{\Gamma(1+m(1-\alpha)-\frac{4}{3}\alpha)}$$

$$\leqslant 5000 \times (1+\sigma(2))c_0^{-8/3}\sigma^{-4/3}(1)h^{-4\alpha/3}\left(\frac{524h}{c_0\sigma(h)}\right)^m (m!)^{1+\alpha},$$

where, in the second inequality, we have used the fact that $1-2^{-1/(3m)} \geqslant 1/(6m)$ for $m \geqslant 4$. This proves (2.7.26).

For the moments of maximum increments of local time of a stationary Gaussian process, we have

Proposition 2.7.4 *Let $\{X(t);t\geqslant 0\}$ be a stationary Gaussian process with mean zero and $EX^2(0)=1$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,1)$, satisfying (2.7.25). Then*

$$E\sup_x (L(x,t+h)-L(x,t))^m \leqslant C_2 h^{-(1/3)\alpha}\left(\frac{262h}{c_0\sigma(h)}\right)^m (m!)^{1+\alpha} \tag{2.7.38}$$

for each even integer $m\geqslant 4$, $0<h\leqslant 1$, $0\leqslant t\leqslant 1$, where $C_2 = 10^4 c_0^{-8/3}\sigma^{-1/3}(1)$.

The proof is similar to that of Proposition 2.7.3 and is omitted.

2.7.2 Increments of local times of Gaussian processes

Now we present an analogous result to (2.7.5) and (2.7.7) for Gaussian processes with stationary increments.

Theorem 2.7.2 *Let a_T and b_T be non-negative functions of $T\geqslant 0$. Put $a^* = \sup_{T\geqslant 0} a_T$. Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero and stationary increments. Assume $X(0)=0$ and that*

$\sigma^2(h)$ is non-decreasing, continuous and concave on $(0,a^*)$. Suppose also that there exist constants $0<\alpha\leqslant 1/2$ and $c_0>0$ such that

$$\sigma(ah) \geqslant c_0 a^\alpha \sigma(h) \quad \text{for all } 0\leqslant a\leqslant 1, 0<h\leqslant a^* \tag{2.7.39}$$

and assume

$$\frac{1+b_T}{a_T} \to \infty \quad \text{as } T\to\infty. \tag{2.7.40}$$

Then

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant b_T} \frac{L(x,t+a_T)-L(x,t)}{a_T\left(\log\frac{b_T}{a_T}+\log\log(a_T+\frac{1}{a_T})\right)^\alpha/\sigma(a_T)} \leqslant \frac{160}{c_0} \quad \text{a.s.} \tag{2.7.41}$$

Before we prove Theorem 2.7.2, here are some immediate consequences:

Corollary 2.7.1 *Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero and stationary increments. Assume $X(0)=0$ and that $\sigma^2(h)$ is non-decreasing, continuous and concave on $(0,1)$, satisfying*

$$\sigma(ah) \geqslant c_0 a^\alpha \sigma(h) \quad \text{for all } 0\leqslant a, h\leqslant 1 \tag{2.7.42}$$

for some $0<\alpha\leqslant 1/2$, $c_0>0$. Then

$$\limsup_{h\to 0} \frac{L(0,h)}{h(2\log\log(1/h))^\alpha/\sigma(h)} \leqslant \frac{160}{c_0} \quad \text{a.s.}, \tag{2.7.43}$$

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \frac{L(0,t+h)-L(0,t)}{h(2\log(1/h))^\alpha/\sigma(h)} \leqslant \frac{160}{c_0} \quad \text{a.s.} \tag{2.7.44}$$

Corollary 2.7.2 *Let $\{Z(t);t\geqslant 0\}$ be a fractional Wiener process of order α, $0<\alpha\leqslant 1/2$, i.e., a centered Gaussian process with stationary increments and $\sigma^2(h)=h^{2\alpha}$. Then*

$$\limsup_{h\to 0} \frac{L(0,h)}{h^{1-\alpha}(2\log\log(1/h))^\alpha} \leqslant 200 \quad \text{a.s.}, \tag{2.7.45}$$

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \frac{L(0,t+h)-L(0,t)}{h^{1-\alpha}(2\log(1/h))^\alpha} \leqslant 200 \quad \text{a.s.} \tag{2.7.46}$$

Remark 2.7.1 Consider the case of $\alpha=1/2$, i. e. $Z(\cdot)$ is a standard Wiener process with the local time $l(x,t)$. According to the law of the iterated logarithm due to Kesten (1965)(cf. (2.7.9)), we have

$$\limsup_{T\to\infty}\frac{l(0,T)}{(2T\log\log T)^{1/2}}=1 \quad \text{a.s.} \tag{2.7.47}$$

Applying the method of Kesten (1965), one can also prove that

$$\limsup_{h\to 0}\frac{l(0,h)}{(2h\log\log(1/h))^{1/2}}=1 \quad \text{a.s.} \tag{2.7.48}$$

This means that the bound we get in (2.7.45) is of the precise order.

Now we turn to prove Theorem 2.7.2. The proof needs the following lemma.

Lemma 2.7.8 *Under the assumptions of Theorem 2.7.2, we have*

$$P\left\{L(x,t+h)-L(x,t)\geqslant\frac{16h}{c_0\sigma(h)}y\right\}$$
$$\leqslant\frac{K_\alpha\exp\{-x^2/(2\sigma^2(t+h))\}}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}} \tag{2.7.49}$$

for each $0<h\leqslant a^*$, $y>0$, $x\in\mathbf{R}$, *where* K_α *is a positive constant depending only on* α.

Proof Lemma 2.7.8 follows immediately from Proposition 2.7.1 and the following lemma.

Lemma 2.7.9 *Let* ξ *be a non-negative random variable. Assume that*

$$E\xi^m\leqslant C(m!)^\alpha$$

for some $C>0$, $\alpha>0$ *and each* $m\geqslant 2$. *Then*

$$P\{\xi>y\}\leqslant\frac{K_\alpha C}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}} \tag{2.7.50}$$

for each $y>0$, *where* K_α *is a positive constant depending only on* α.

Proof When $0<y\leqslant 2^\alpha$, by Chebyshev's inequality

$$P\{\xi>y\}\leqslant\frac{E\xi^2}{y^2}\leqslant\frac{K_\alpha C}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}}.$$

For $y>2^\alpha$, let $m=[y^{1/\alpha}]$. By Stirling's formula and Chebyshev's inequality again,

$$\begin{aligned}P\{\xi>h\}&\leqslant\frac{E\xi^m}{y^m}\leqslant\frac{C(m!)^\alpha}{y^m}\\&\leqslant\frac{C(3m(m/e)^m)^\alpha}{y^m}\leqslant C(3y^{1/\alpha})^\alpha e^{-m\alpha}\\&\leqslant C3^\alpha y\exp\{-(y^{1/\alpha}-1)\alpha\}\leqslant K_\alpha C\exp(-y^{1/\alpha}/2)\\&\leqslant K_\alpha C(\exp(y^{1/\alpha}/4)-1)^{-2\alpha}\end{aligned}$$

as desired.

Proof of Theorem 2.7.2 Let $1<\theta<\frac{5}{4}$. Define

$$A_k=\{T;\theta^k<a_T\leqslant\theta^{k+1}\},\quad -\infty<k<\infty,$$

$$A_{kj}=\{T;\theta^j\leqslant\frac{b_T+a_T}{\theta^k}<\theta^{j+1},T\in A_k\},\quad j=0,1,\cdots,$$

$$\beta_T=\log\frac{b_T}{a_T}+\log\log\left(a_T+\frac{1}{a_T}\right).$$

It is easy to see that

$$\beta_T\geqslant\beta_{k,j}:=\log\theta^j+\log\log\theta^{|k|}$$

for each $T\in A_{k,j}$.

Noting that $L(x,t)$ is non-decreasing in t for each fixed x, we have

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{L(x,t+a_T)-L(x,t)}{a_T\beta_T^\alpha/\sigma(a_T)}$$
$$\leqslant\limsup_{|k|+j\to\infty}\sup_{l\geqslant j}\sup_{T\in A_{k,l}}\sup_{0\leqslant t\leqslant b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^\alpha}$$

$$\leqslant \limsup_{|k|+j\to\infty} \sup_{l\geqslant j} \sup_{T\in A_{k,l}} \sup_{0\leqslant t\leqslant(\theta^{l+1}-1)\theta^k} \frac{\sigma(\theta^{k+1})(L(x,t+\theta^{k+1})-L(x,t))}{\theta^k\beta_{k,l}^{\alpha}}$$

$$\leqslant 2\limsup_{|k|+j\to\infty} \sup_{l\geqslant j} \max_{0\leqslant m\leqslant\theta^l} \frac{\sigma(\theta^{k+1})(L(x,(m+1)\theta^{k+1})-L(x,m\theta^{k+1}))}{\theta^k\beta_{k,l}^{\alpha}}. \tag{2.7.51}$$

Applying Lemma 2.7.8, we obtain

$$P\left\{\sup_{l\geqslant j}\max_{0\leqslant m\leqslant\theta^l} \frac{\sigma(\theta^{k+1})(L(x,(m+1)\theta^{k+1})-L(x,m\theta^{k+1}))}{\theta^k\beta_{k,l}^{\alpha}} \geqslant \theta\left(\frac{2\theta}{\alpha}\right)^{\alpha}\frac{16}{c_0}\right\}$$

$$\leqslant \sum_{l=j}^{\infty}\sum_{m=0}^{[\theta^l]} K_{\alpha}e^{-\theta\beta_{k,l}} \leqslant C\sum_{l=j}^{\infty}\theta^{l(\theta-1)}\log^{-\theta}\theta^{|k|}$$

$$\leqslant C\theta^{j(\theta-1)}(|k|+1)^{-\theta}, \tag{2.7.52}$$

where C is a constant, depending only on θ and α. From (2.7.51), (2.7.52) and the Borel-Cantelli lemma it follows that

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^{\alpha}}$$

$$\leqslant\frac{32}{c_0}\theta(2\theta/\alpha)^{\alpha}\leqslant\frac{160}{c_0}\theta^2 \quad \text{a.s.}$$

Therefore

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\frac{\sigma(a_T)(L(x,t+a_T)-L(x,t))}{a_T\beta_T^{\alpha}}\leqslant\frac{160}{c_0} \quad \text{a.s.}$$

as claimed in (2.7.41).

Similar to Lemma 2.7.8, a combination of Lemma 2.7.9 and Proposition 2.7.2 yields

Lemma 2.7.10 *Under the assumptions of Proposition 2.7.2, we have*

$$P\left\{L(x,t+h)-L(x,t)\geqslant\frac{16h}{c_0\sigma(h)}y\right\}\leqslant\frac{K_{\alpha}\exp(-x^2/2)\cdot\sigma(h)}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}}$$

for each $0<h\leqslant h_0, t\geqslant 0, y>0$, where K_{α} is a positive constant depending only on α.

Using Lemma 2.7.10 instead of Lemma 2.7.8, one can proceed along the same lines as the proof of Theorem 2.7.2 and get the next theorem on path properties of the local time for a stationary Gaussian process.

Theorem 2.7.3 *Let b_h be a non-negative function of h on $(0,1)$. Let $\{X(t);t\geqslant 0\}$ be a stationary Gaussian process with mean zero and $EX^2(0)=1$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,1)$, satisfying (2.7.42) for some $0<\alpha\leqslant 1/2, c_0>0$. Then*

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant b_h}\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))\beta_h^{\alpha}}\leqslant\frac{160}{c_0} \quad \text{a.s.} \tag{2.7.53}$$

for each $x\in\mathbf{R}$, where

$$\beta_h=\log\left(1+\left(\frac{h+b_h}{h}\right)\sigma(h)\log^{3/2}\frac{1}{h}\right). \tag{2.7.54}$$

Here are some immediate consequences:

Corollary 2.7.3 *Under the assumptions of Theorem 2.7.3, we have*

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant h^{\theta}}\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))\log^{\alpha}(1+h^{\theta-1}\sigma(h)\log^{3/2}(1/h))}$$

$$\leqslant\frac{160}{c_0} \quad \text{a.s.} \tag{2.7.55}$$

and

$$\limsup_{h\to 0}\frac{L(x,h)}{h\sigma^{\alpha-1}(h)\log^{2\alpha}(1/h)}=0 \quad \text{a.s.} \tag{2.7.56}$$

for each $x\in\mathbf{R}$, $0\leqslant\theta\leqslant 1$.

Corollary 2.7.4 *Let $\{X(t);t\geqslant 0\}$ be a stationary Gaussian process with mean zero and $EX^2(0)=1$. Assume that $\sigma^2(h)$ is non-decreasing, continuous and concave on $(0,1)$, satisfying $c_1h^{\alpha}\leqslant$*

$\sigma(h)\leqslant c_0h^\alpha$ *for some* $0<\alpha\leqslant\frac{1}{2}$, c_1, $c_0>0$ *and all* $0<h\leqslant 1$. *Then*

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant h^\theta}\frac{L(x,t+h)-L(x,t)}{h^{1+\alpha(\theta-2+\alpha)}\log^{2\alpha}(1/h)}=0\quad \text{a.s.} \tag{2.7.57}$$

for each $x\in\mathbf{R}$, *and* $0\leqslant\theta\leqslant 1$.

Corollary 2.7.5 *Let* $\{X(t);t\geqslant 0\}=\{\sum_{k=1}^\infty X_k(t);t\geqslant 0\}$, *where* $\{X_k(t);t\geqslant 0\}$ *are independent Ornstein-Uhlenbeck processes with coefficients* $\gamma_k\geqslant 0$ *and* $\lambda_k>0$. *Assume* $\Gamma_0=\sum_{k=1}^\infty\gamma_k/\lambda_k=1$ *and* (2.7.42) *for some* $0<\alpha\leqslant 1/2$, $c_0>0$. *Then* (2.7.55) *and* (2.7.56) *hold. In particular, we have*

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant h^\theta}\frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))\log^{1/2}(1+h^{\theta-1}\sigma(h)\log^{3/2}(1/h))}$$
$$\leqslant\frac{160}{c_0}\quad \text{a.s.}, \tag{2.7.58}$$

$$\limsup_{h\to 0}\frac{L(x,h)}{h\sigma^{-1/2}(h)\log(1/h)}=0\quad \text{a.s.} \tag{2.7.59}$$

for each $x\in\mathbf{R}$, $0\leqslant\theta\leqslant 1$.

Remark 2.7.2 Comparing (2.7.47) with (2.7.59), we find that the limit behavior of the local time of Gaussian processes with stationary increments and $X(0)=0$ is quite different from that of stationary Gaussian processes.

2.7.3 Maximum moduli of continuity of local times of Gaussian processes

Let $\{X(t);t\geqslant 0\}$ be a Gaussian process with mean zero, stationary increments. We study the supremum in $x\in\mathbf{R}$ of moduli of continuity of $L(x,t)$ in $t\geqslant 0$. The following are two analogous results to (2.7.6).

Theorem 2.7.4 *Let* $\{X(t);t\geqslant 0\}$ *be a Gaussian process with mean zero, stationary increments and* $X(0)=0$, $L(x,t)$ *be the local time of* $X(\cdot)$. *Put* $\sigma^2(h)=E(X(t+h)-X(t))^2$. *Assume that* $\sigma^2(h)$ *is non-decreasing, continuous and concave on* $(0,1)$ *and that there exist constants* $0<\alpha\leqslant 1/2$ *and* $c_0>0$ *such that*

$$\sigma(ah)\geqslant c_0a^\alpha\sigma(h)\qquad \text{for all } 0<a,h\leqslant 1. \tag{2.7.60}$$

Then

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{-\infty<x<\infty}\frac{L(x,t+h)-L(x,t)}{h(\log(1/h))^{\alpha+1}/\sigma(h)}\leqslant\frac{524(3+4\alpha)^{\alpha+1}}{c_0}\ \text{a.s.} \tag{2.7.61}$$

Theorem 2.7.5 *Let* $\{X(t);t\geqslant 0\}$ *be a stationary Gaussian process with mean zero and* $EX^2(0)=1$. *Assume that* $\sigma^2(h)$ *is non-decreasing, continuous and concave on* $(0,1)$, *satisfying* (2.7.60) *for some* $0<\alpha\leqslant 1/2$ *and* $c_0>0$. *Then* (2.7.61) *holds.*

The proof of Theorems 2.7.4 and 2.7.5 is based on the following two lemmas, whose proof is similar to that of Lemma 2.7.8 by using Proposition 2.7.3 (resp. Proposition 2.7.4) instead of Proposition 2.7.1.

Lemma 2.7.11 *Under the assumptions of Theorem* 2.7.4, *we have*

$$P\left\{\sup_x(L(x,t+h)-L(x,t))>y\,\frac{524h}{c_0\sigma(h)}\right\}$$
$$\leqslant K_\alpha C_1h^{-4\alpha/3}\exp(-y^{1/(1+\alpha)}/2) \tag{2.7.62}$$

for each $y>1$, $0\leqslant t,h\leqslant 1$, *where* C_1 *is defined as in Proposition* 2.7.3 *and* K_α *is a positive number depending only on* α.

Lemma 2.7.12 *Under the assumptions of Theorem* 2.7.5, *we have*

$$P\left\{\sup_x(L(x,t+h)-L(x,t))>y\,\frac{524h}{c_0\sigma(h)}\right\}$$

$$\leqslant K_\alpha C_2 h^{-\alpha/3}\exp(-y^{1/(1+\alpha)}/2) \tag{2.7.63}$$

for each $y>1, 0\leqslant t, h\leqslant 1$, *where* C_2 *is defined as in Proposition* 2.7.4 *and* K_α *is a positive number depending only on* α.

Proof of Theorem 2.7.4 Let $1<\theta<5/4$. Then

$$\begin{aligned}
&\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_x \frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha+1}} \\
&\leqslant \limsup_{k\to\infty}\sup_{\theta^{-k-1}\leqslant h\leqslant \theta^{-k}}\sup_{0\leqslant t\leqslant 1}\sup_x \frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha+1}} \\
&\leqslant \limsup_{k\to\infty}\sup_{0\leqslant t\leqslant 1}\sup_x \frac{L(x,t+\theta^{-k})-L(x,t)}{(\theta^{-k-1}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}} \\
&\leqslant 2\theta\limsup_{k\to\infty}\max_{0\leqslant j\leqslant \theta^k}\sup_x \frac{L(x,(j+1)\theta^{-k})-L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}}.
\end{aligned} \tag{2.7.64}$$

From (2.7.62) it follows that

$$\begin{aligned}
&P\left\{\max_{0\leqslant j\leqslant\theta^k}\sup_x \frac{L(x,(j+1)\theta^{-k})-L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}} \geqslant \frac{524}{c_0}((3+4\alpha)\theta)^{1+\alpha}\right\} \\
&\leqslant K_\alpha(\theta^k+1)C_1\cdot\theta^{4\alpha/3}\exp(-(3+4\alpha)\log\theta^k/2) \\
&\leqslant K\theta^{-k(\theta-1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{k\to\infty}\sup_{0\leqslant j\leqslant\theta^k}\sup_x \frac{L(x,(j+1)\theta^{-k})-L(x,j\theta^{-k})}{(\theta^{-k}/\sigma(\theta^{-k}))(\log\theta^k)^{\alpha+1}} \\
&\geqslant \frac{524}{c_0}((3+4\alpha)\theta)^{1+\alpha} \quad \text{a.s.}
\end{aligned} \tag{2.7.65}$$

By the Borel-Cantelli lemma. This proves (2.7.61) by (2.7.64), (2.7.65) and by taking θ arbitrarily near to 1.

Proof of Theorem 2.7.5 The proof is similar to that of Theorem 2.7.4 and therefore is omitted.

Remark 2.7.3 Let $\{W(t);t\geqslant 0\}$ be a standard Wiener process with local time $l(x,t)$. From (2.7.61) we obtain

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_x \frac{l(x,t+h)-l(x,t)}{h^{1/2}(\log(1/h))^{3/2}} \leqslant 262(48)^{3/2} \quad \text{a.s.}$$

Comparing this result to that of (2.7.6) we conclude that the result (2.7.61) is not the best possibly. We believe that (2.7.61) can be replaced by

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_x \frac{L(x,t+h)-L(x,t)}{(h/\sigma(h))(\log(1/h))^{\alpha}} \leqslant c_0^* \quad \text{a.s.}$$

for some constant c_0^*, under the assumptions of Theorem 2.7.4 or Theorem 2.7.5.

Chapter 3

Moduli of Continuity and Large Increments for Infinite Dimensional Gaussian Processes

The study of path properties of an infinite dimensional Gaussian process was beginning from the study of an infinite dimensional Ornstein-Uhlenbeck process. As we have mentioned in Section 2.1.5, the infinite dimensional Ornstein-Uhlenbeck process was first studied by Dawson (1972) as a stationary solution of a kind of the infinite array of stochastic differential equations. Such processes have been extensively studied in the literature since the appearance of Dawson (1972). As the extension of the study of this kind of processes, furthermore, some authors have investigated more infintie dimensional Gaussian processes. In this chapter, we shall introduce their path behavior, including the continuity, the moduli of continuity and large increments.

3.1 Continuity of l^p-valued Gaussian Processes

Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent stationary Gaussian processes with $EX_k(t)=0$ and $\sigma_k^2(h)=E(X_k(t+h)-X_k(t))^2$, where $\sigma_k(h)$ is a non-decreasing continuous function for every $k\geqslant 1$. First, we assume that $X_k(\cdot)$ is a Ornstein-Uhlenbeck (O-U) process with coefficients $\gamma_k\geqslant 0$ and $\lambda_k>0$, i.e. $X_k(\cdot)$ is a stationary, mean zero Gaussian process with $EX_k(s)X_k(t)=(\gamma_k/\lambda_k)\exp\{-\lambda_k|t-s|\}$. Obviously,

$$\sigma_k^2(h):=E(X_k(t+h)-X_k(t))^2=\frac{2\gamma_k}{\lambda_k}(1-e^{\lambda_k h}),\ h>0. \tag{3.1.1}$$

(3.1.1) implies

$$(2/e)\gamma_k(\lambda_k^{-1}\wedge h)\leqslant\sigma_k^2(h)\leqslant 2\gamma_k(\lambda_k^{-1}\wedge h). \tag{3.1.2}$$

In this section, we study continuity of $Y(\cdot)$ in l^p, $p\geqslant 1$.

3.1.1 l^2-valued O-U process

We first quote the following inequality (cf. Iscoe et al. 1990) which follows from Theorem 2.1.6 and Corollary 2.1.2: There exist $0<c_p\leqslant C_p<\infty$ such that

$$c_p\left\{E\|Y(0)\|_{l^p}+\sup_{\|(y_k)\|_{l^q}\leqslant 1}\int_0^{1/2}\frac{(\sum_{k=1}^{\infty}y_k^2\sigma_k^2(u))^{1/2}}{u(\log(1/u))^{1/2}}du\right\}$$
$$\leqslant E\sup_{t\in[0,1]}\|Y(t)\|_{l^p}$$

$$\leqslant C_p\left\{E\,\|\,Y(0)\,\|_{l^p}+\sup_{\|(y_k)\|_{l^q}\leqslant 1}\int_0^{1/2}\frac{(\sum_{k=1}^{\infty}y_k^2\sigma_k^2(u))^{1/2}}{u(\log(1/u))^{1/2}}du\right\},\tag{3.1.3}$$

where $1/p+1/q=1$.

We consider the case of $p=2$. For $x\geqslant 0$, let $K(x)=\{k\in\mathbf{N};\ \gamma_k>\lambda_k x\}$, $\lambda(x)=\sup\{\lambda_k;k\in K(x)\}$, $\lambda(x)=0$ when $K(x)$ is an empty set. The following theorem is due to Fernique (1989).

Theorem 3.1.1 *$Y(\cdot)$ is continuous almost surely in l^2 if and only if $\sum_{k=1}^{\infty}\gamma_k/\lambda_k<\infty$ and $\int_0^{\infty}\log^+(\lambda(x))dx<\infty$.*

Proof We first prove the sufficiency. For any $\delta>0$, $\rho>0$ and integer $N>0$ write

$$\begin{aligned}&P\{\sup_{|t-s|\leqslant\delta,s,t\in[0,1]}\|\,Y(t)-Y(s)\,\|_{l^2}\geqslant 3\rho\}\\&\leqslant P\Big\{\sup_{|t-s|\leqslant\delta,s,t\in[0,1]}\Big(\sum_{k\leqslant N}|X_k(t)-X_k(s)|^2\Big)^{1/2}\geqslant\rho\Big\}\\&\quad+2P\Big\{\sup_{t\in[0,1]}\sum_{k>N}X_k(t)^2\geqslant\rho\Big\}.\end{aligned}\tag{3.1.4}$$

The first term on the right hand side of (3.1.4) goes to zero as $\delta\to 0$ because the O-U processes are continuous. We employ (3.1.3) to estimate the second probability on the right hand side of (3.1.4), but we consider $k\geqslant N$ instead of $k\geqslant 1$. Corresponding to $E\,\|\,Y(0)\,\|_{l^2}$,

$$\sum_{k\geqslant N}EX_k(0)^2=\sum_{k\geqslant N}\gamma_k/\lambda_k<\varepsilon\tag{3.1.5}$$

for any given $\varepsilon>0$ provided $N=N(\varepsilon)$ is large enough. Letting $x=(\log u^{-1})^{1/2}$ in the integral in (3.1.3) and noting (3.1.2), we may replace this integral by

$$M(y):=\int_{\log 2}^{\infty}\Big(\sum_{k\geqslant N}y_k^2\gamma_k(\lambda_k^{-1}\wedge e^{-x^2})\Big)^{1/2}dx$$

with $y=(y_1,y_2,\cdots)$ by adjusting the constants c_p and C_p. Obviously,

$$M(y)=M_1(y)+M_2(y),$$

where

$$M_1(y)=\int_{\log 2}^{\infty}\Big(\sum_{k\geqslant N}y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx,$$

$$M_2(y)=\int_{\log 2}^{\infty}\Big(\sum_{k\geqslant N}y_k^2\gamma_k e^{-x^2}I(\lambda_k\leqslant e^{x^2})\Big)^{1/2}dx.$$

Let B be an unit ball in l^2. It is clear that

$$\sup_{y\in B}M_2(y)\leqslant\sum_{k\geqslant N}\gamma_k/\lambda_k<\varepsilon.\tag{3.1.6}$$

Hence it suffices to estimate $M_1(y)$. For $j=\cdots,-1,0,1,\cdots$, put

$$K_j=\{k;2^{-j-1}<\gamma_k/\lambda_k\leqslant 2^{-j}\},\ \lambda_j^*=\sup_{k\in K_j}\lambda_k,$$

$$J:=J(N)=\sup\{j;\sup_{k\geqslant N}\gamma_k/\lambda_k\leqslant 2^{-j}\}.$$

Then

$$\begin{aligned}\sup_{y\in B}M_1(y)&\leqslant\sup_{y\in B}\int_{\log 2}^{\infty}\Big(\sum_{j\geqslant J}2^{-j}\sum_{k\in K_j}y_k^2I(\lambda_j^*>e^{x^2})\Big)dx\\&\leqslant\sup_{z\in B}\int_{\log 2}^{\infty}\sum_{j\geqslant J}2^{-j/2}|z_j|I(\lambda_j^*>e^{x^2})dx\\&\leqslant\sup_{z\in B}\sum_{j\geqslant J}2^{-j/2}(\log^+\lambda_j^*)^{1/2}|z_j|\\&\leqslant\Big(\sum_{j\geqslant J}2^{-j}\log^+\lambda_j^*\Big)^{1/2}<\varepsilon\end{aligned}\tag{3.1.7}$$

provided N is large enough (hence, J is large enough) by the condition $\int_{-\infty}^{\infty}\log^+(\lambda(x))dx<\infty$. Combining (3.1.5)—(3.1.7) implies that the last term in (3.1.4) can be made arbitrarily small by choosing N large enough. Therefore we conclude that $Y(\cdot)$ is continuous a.s. in l^2.

Next we prove the necessity. Suppose that $Y(\cdot)$ is continuous almost surely in l^2. Then it is locally bounded almost sure-

ly. By Theorem 2.1.1, we have $E\sup_{t\in[0,1]}\|Y(t)\|_{l^2}<\infty$. Hence (3.1.3) implies that

$$E\|Y(0)\|_{l^2}=\sum_{k=1}^{\infty}\gamma_k/\lambda_k<\infty,\quad L:=\sup_{y\in B}L(y)<\infty, \tag{3.1.8}$$

where

$$L(y)=\int_0^\infty\Big(\sum_{k=1}^\infty y_k^2\gamma_k(\lambda_k^{-1}\wedge e^{-x^2})\Big)^{1/2}dx. \tag{3.1.9}$$

Put $H_n=\{k;\exp(n^2)\leqslant\lambda_k<\exp((n+1)^2)\}$, $s_n=\sup_{k\in H_n}\gamma_k/\lambda_k$, $z_n^2=\sum_{k\in H_n}y_k^2$. Then

$$\begin{aligned}L_1(y)&:=\int_0^\infty\Big(\sum_{k=1}^\infty y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx\\&=\sum_{n=1}^\infty\int_{n-1}^n\Big(\sum_{k=1}^\infty y_k^2\gamma_k\lambda_k^{-1}I(\lambda_k>e^{x^2})\Big)^{1/2}dx\\&\geqslant\sum_{n=1}^\infty\Big(\sum_{m\geqslant n}\sum_{k\in H_m}y_k^2\gamma_k\lambda_k^{-1}\Big)^{1/2}\\&=:L_1'(y).\end{aligned}$$

Obviously,

$$L_1'(y)\leqslant\sum_{n=1}^\infty\Big(\sum_{m\geqslant n}z_m^2s_m\Big)^{1/2}.$$

On the other hand, for any $z'\in B$, if we define $k_m=\sup\{k;\gamma_k/\lambda_k=s_m,k\in H_m\}$, and for $k\in H_m$, let $y'_{k_m}=z'_m$, $y'_k=0$, $k\neq k_m$, then $y'=(y'_1,y'_2,\cdots)\in B$. Therefore we have

$$\sup_{y\in B}L_1'(y)=\sup_{z\in B}\sum_{n=1}^\infty\Big(\sum_{m\geqslant n}z_m^2s_m\Big)^{1/2}=:L_2(s).$$

We may assume that $\{s_m\}$ is decreasing, otherwise there exists a decreasing sequence $\{s'_m\}$ with $s_m\leqslant s'_m$ such that $L_2(s)=L_2(s')$. Let $z_m^2=ms_m\Big/\sum_{k=1}^\infty ks_k$. Then

$$\begin{aligned}L_2(s)^2&\geqslant\Big(\sum_{n=1}^\infty\Big(\sum_{m\geqslant n}ms_m^2\Big)^{1/2}\Big)\Big(\sum_{k=1}^\infty ks_k\Big)^{-1/2}\\&\geqslant\Big(\sum_{n=1}^\infty\Big(n\sum_{m=n}^{2n-1}s_m^2\Big)^{1/2}\Big)\Big(\sum_{k=1}^\infty ks_k\Big)^{-1/2}\\&\geqslant\Big(\sum_{n=1}^\infty ns_{2n-1}\Big)\Big(\sum_{k=1}^\infty ks_k\Big)^{-1/2}\\&\geqslant\frac12\Big(\sum_{n=1}^\infty ns_n\Big)^{1/2}.\end{aligned} \tag{3.1.10}$$

By noting $L_2(s)<\infty$ from (3.1.9) we obtain

$$\sum_{n=1}^\infty ns_n<\infty. \tag{3.1.11}$$

Now we show that

$$\int_0^\infty\log^+(\lambda(x))dx<\infty. \tag{3.1.12}$$

Note that $\lambda(x)$ is a non-increasing function. Obviously (3.1.8) implies $\sup_{k\geqslant1}\gamma_k/\lambda_k<\infty$. Hence $\lambda(x)=0$ for $x>\sup_{k\geqslant1}\gamma_k/\lambda_k$. Let $x_0=\sup\{x;\lambda(x)>e\}$. Then $x_0<\infty$. So it suffices for (3.1.12) to show that

$$\int_0^{x_0}\log(\lambda(x))dx<\infty. \tag{3.1.13}$$

If $\sup_{k\geqslant1}\lambda_k<\infty$, which implies that $\lambda(x)$ is bounded, it is easy to see that (3.1.13) is true. If $\sup_{k\geqslant1}\lambda_k=\infty$, then $\lambda(x)\to\infty$ as $x\to0$. Let $y=\log\lambda(x)$. Then $x=\lambda^{-1}(e^y)$. We have

$$\begin{aligned}\int_0^{x_0}\log(\lambda(x))dx&=x\log(\lambda(x))\Big|_0^{x_0}-\int_0^{x_0}x\,d\log\lambda(x)\\&=\lambda^{-1}(e^y)y\Big|_\infty^1+\int_1^\infty\lambda^{-1}(e^y)dy.\end{aligned}$$

In this case we have

$$y\lambda^{-1}(e^y)\to0\quad\text{as}\quad y\to\infty$$

and

$$\int_1^\infty \lambda^{-1}(e^y)dy = \sum_{n=1}^\infty \int_{n^2}^{(n+1)^2} \lambda^{-1}(e^y)dy$$
$$\leqslant \sum_{n=1}^\infty ((n+1)^2 - n^2) \max_{k\in H_n} \gamma_k/\lambda_k$$
$$\leqslant 2\sum_{n=1}^\infty (n+1)s_n < \infty$$

by (3. 1. 11). Combining these results we obtain (3. 1. 13). Necessity is proved. This completes the proof of Theorem 3. 1. 1.

From this theorem, we may give some sufficient conditions for continuity of $Y(\cdot)$ in l^2. One of them is as follows:

Corollary 3. 1. 1 *$Y(\cdot)$ is continuous a. s. in l^2 if*

$$\sum_{k=1}^\infty (\gamma_k/\lambda_k)(1 + \log^+ \lambda_k) < \infty.$$

Remark 3. 1. 1 Fernique (1990a) extended Theorem 3. 1. 1 to the case of $p\geqslant 2$; $Y(\cdot)$ is continuous a. s. in $l^p(p\geqslant 2)$ if and only if

$$\sum_{k=1}^\infty (\gamma_k/\lambda_k)^{p/2} < \infty$$

and

$$\int_0^\infty x^{p/2-1}(\log^+ \lambda(x))^{p/2}dx < \infty.$$

3. 1. 2 l^p-valued O-U process

In this subsection we will use theory of Dirichlet forms to obtain the condition under which the process $Y(\cdot)$ has continuous sample paths in $l^p(p\geqslant 1)$. The process $Y(\cdot)$ lives on the state space $\mathbf{R}^\infty$ and has invariant measure

$$m = \prod_{k=1}^\infty N(0,\gamma_k/\lambda_k).$$

A generic point in $\mathbf{R}^\infty$ will be denoted by $\boldsymbol{x}=(x_1,x_2,\cdots)$. The Dirichlet's form $\mathscr{E}$ associated with the process $Y(\cdot)$ is given as the closure of the following bilinear form defined over $L^2(Y,P)$:

$$\mathscr{E}(u,v) = \frac{1}{2}\int \sum_{k=1}^\infty \gamma_k\left(\frac{\partial u}{\partial x_k}\right)\left(\frac{\partial v}{\partial x_k}\right)dm,$$

where

$$u,v\in \mathscr{D}(\mathscr{E}) := \{u\in L^2; u=\varphi(x_1,\cdots,x_k), \varphi\in C_0^\infty(\mathbf{R}^k), k\geqslant 1\},$$

$C_0^\infty(\mathbf{R}^k)$ is a space of all continuous functions on $\mathbf{R}^k$ with compact support and continuous derivatives of any order. In order to show continuity we need the following lemma of Fukushima (1980). Define the form $\mathscr{E}_1$ on $\mathscr{D}(\mathscr{E})$ by

$$\mathscr{E}_1(u,v)=\mathscr{E}(u,v)+(u,v)_{L^2}.$$

Lemma 3. 1. 1 *Every $u\in\mathscr{D}(\mathscr{E})$ has a quasi-continuous version $\tilde{u}$ so that*

$$P\{t\to\tilde{u}(Y(t)) \text{ is continuous}\}=1.$$

Furthermore, if $u_n\to u$ in $\mathscr{E}_1$-norm, then for every $T>0$ there exists a subsequence $\{u_{n_k}\}$ so that

$$P\{\tilde{u}_{n_k}(Y(t)) \text{convergences to } \tilde{u}(Y(t)) \text{ uniformly on } [0,T]\}=1.$$

The following result is due to Schmuland (1990). Define

$$\delta_k=4^{p/2}(\gamma_k/\lambda_k)^{p/2}\log^{p/2}(\gamma_k(\gamma_k/\lambda_k)^{p/2-1}\vee e).$$

Theorem 3. 1. 2 *If $\sum_{k=1}^\infty \delta_k < \infty$, then $Y(\cdot)$ is continuous a. s. in l^p.*

Proof By the assumption, $\sum_{k=1}^\infty(\gamma_k/\lambda_k)^{p/2} < \infty$, which means that $Y(\cdot)$ is in l^p a. s. Define

$$u_n = \sum_{k=1}^n |x_k(t)|^p \vee \delta_k.$$

The function u_n belongs to the domain of the closed form $\mathscr{E}$ with

$$\frac{\partial u_n(t)}{\partial x_k(t)}=p\,|x_k(t)|^{p-1}I\{|x_k(t)|^p>\delta_k\}I\{k\leqslant n\}.$$

Therefore, for $n>m$,

$$\begin{aligned}
&\mathscr{E}(u_n-u_m,u_n-u_m)\\
&\quad=\frac{p^2}{2}\int\sum_{k=m+1}^{n}\gamma_k|x_k|^{2(p-1)}I\{|x_k|^p>\delta_k\}\mathrm{d}m(\mathrm{d}x)\\
&\quad\leqslant c\sum_{k=m+1}^{n}\gamma_k\left(\frac{\gamma_k}{\lambda_k}\right)^{p-1}\exp\left\{-\frac{1}{4}\delta_k^{2/p}/\left(\frac{\gamma_k}{\lambda_k}\right)\right\}\\
&\quad\leqslant c\sum_{k=m+1}^{n}\left(\frac{\gamma_k}{\lambda_k}\right)^{p/2}.
\end{aligned}\tag{3.1.14}$$

Here for the inequality we used the fact

$$\int_x^\infty y^p\,\mathrm{e}^{-y^2/2}\mathrm{d}y\leqslant\mathrm{e}^{-x^2/2}P(x)=O(\mathrm{e}^{-x^2/4})$$

as $x\to\infty$, where $P(\cdot)$ is some polynomial. Now $u_n\uparrow u=\sum_{k=1}^\infty|x_k|^p\vee\delta_k$ and $u\leqslant\|x\|_{l^p}^p+\sum_{k=1}^\infty\delta_k$ which belongs to L^2, so $u_n\to u$ in L^2. (3.1.14) implies that $\{u_n\}$ is $\mathscr{E}$-Cauchy so $u-u_n\to 0$ in $\mathscr{E}_1$-norm.

Applying Lemma 3.1.1 we find that for any $T>0$

$$\begin{aligned}
&P\left\{\sup_{0\leqslant t\leqslant T}\sum_{k=n+1}^\infty|X_k(t)|^p\xrightarrow{n\to\infty}0\right\}\\
&\geqslant P\left\{\sup_{0\leqslant t\leqslant T}(u-u_n)Y(t)\xrightarrow{n\to\infty}0\right\}=1,
\end{aligned}$$

which gives the required conclusion.

Corollary 3.1.2 *If* $\sum_{k=1}^\infty(\gamma_k/\lambda_k)^{p/2}\log(\lambda_k\vee\mathrm{e})^{p/2}<\infty$ *for* $1\leqslant p<2$ *or* $\sum_{k=1}^\infty(\gamma_k/\lambda_k)^{p/2}\log(\gamma_k\vee\mathrm{e})^{p/2}<\infty$ *for* $2\leqslant p<\infty$, *then* $Y(\cdot)$ *is continuous a.s. in* l^p.

Remark 3.1.2 By the previous proof, the independence between components of $Y(\cdot)$ is not needed for the conclusion of Theorem 3.1.2.

3.1.3 l^p-valued Gaussian process

Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^\infty$ be a sequence of Gaussian processes defined as in the beginning of this section. Put

$$\begin{aligned}
&\sigma(p,h)=\left(\sum_{k=1}^\infty\sigma_k^p(h)\right)^{1/p},\quad p\geqslant1,\\
&\sigma^*(h)=\max_{k\geqslant1}\sigma_k(h),\\
&\tilde\sigma(p,h)=\begin{cases}\sigma\left(\dfrac{2p}{2-p},h\right) & \text{if } 1\leqslant p<2,\\ \sigma^*(h) & \text{if } p\geqslant2,\end{cases}\\
&\delta_p^p=E|N(0,1)|^p,\quad p\geqslant1.
\end{aligned}$$

Since $E\|Y(t+h)-Y(t)\|_{l^p}^p=\delta_p^p\sum_{k=1}^\infty\sigma_k^p(h)$, $Y(t+h)-Y(t)\in l^p$, $p\geqslant1$, almost surely for fixed t and h if and only if

$$\sigma(p,h)<\infty,\tag{3.1.15}$$

and $Y(t)\in l^p$ almost surely for every t if and only if (3.1.15) and also

$$\sum_{k=1}^\infty E|X_k(0)|^p<\infty.\tag{3.1.16}$$

Lemma 3.1.2 *With* $p\geqslant1$, *we have*

$$P\{\|Y(t+h)-Y(t)\|_{l^p}\geqslant\delta_p\sigma(p,h)+x\tilde\sigma(p,h)\}\leqslant2\exp(-x^2/2)\tag{3.1.17}$$

for any $t,x,h\geqslant0$.

Proof Let $\{\xi_n;\ n\geqslant1\}$ be independent normal random

variables with $E\xi_n=0$ and $\sum_{k=1}^{\infty}(E\xi_k^2)^{p/2}<\infty$. It is well known that

$$\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}=\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}\xi_k a_k,$$

where $q=p/(p-1)$, $a=(a_1,a_2,\cdots)\in l^q$. Using Borell's inequality (Theorem 1.1.1), we have that for any $x>0$

$$\begin{aligned}
&P\Big\{\Big|\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}-E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}\Big|\ge x\Big\}\\
&=P\Big\{\Big|\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}a_k\xi_k-E\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}a_k\xi_k\Big|\ge x\Big\}\\
&\le 2\exp\Big\{-\frac{x^2}{2\sup_{\|a\|_q\le 1}E(\sum_{k=1}^{\infty}a_k\xi_k)^2}\Big\}\\
&=2\exp\Big\{-\frac{x^2}{2\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}a_k^2E\xi_k^2}\Big\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}a_k^2E\xi_k^2\\
&\le\begin{cases}\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{\frac{q}{q-2}}\Big)^{\frac{q-2}{q}}\sup_{\|a\|_q\le 1}\Big(\sum_{k=1}^{\infty}|a_k|^q\Big)^{\frac{2}{q}} & \text{if } 1\le p<2,\\ \max_{k\ge 1}E\xi_k^2\sup_{\|a\|_q\le 1}\sum_{k=1}^{\infty}a_k^2 & \text{if } p\ge 2\end{cases}\\
&=\begin{cases}\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/(2-p)}\Big)^{(2-p)/p} & \text{if } 1\le p<2,\\ \max_{k\ge 1}E\xi_k^2 & \text{if } p\ge 2,\end{cases}
\end{aligned}$$

we obtain

$$\begin{aligned}
&P\Big\{\Big|\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}-E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}\Big|\ge x\Big\}\\
&=\begin{cases}2\exp\Big\{-\frac{x^2}{2(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/(2-p)})^{(2-p)/p}}\Big\} & \text{if } 1\le p<2,\\ 2\exp\Big\{-\frac{x^2}{2\max_{k\ge 1}E\xi_k^2}\Big\} & \text{if } p\ge 2.\end{cases}
\end{aligned}\tag{3.1.18}$$

Since

$$E\Big(\sum_{k=1}^{\infty}|\xi_k|^p\Big)^{1/p}\le\Big(\sum_{k=1}^{\infty}E|\xi_k|^p\Big)^{1/p}=\delta_p\Big(\sum_{k=1}^{\infty}(E\xi_k^2)^{p/2}\Big)^{1/p}$$

by Hölder's inequality, (3.1.18) implies (3.1.17). The lemma is proved.

Next, we eastablish Fernique's type inequality from (3.1.17). We consider more general process, which is not necessarily Gaussian. The following lemma is due to Csáki, Csörgő and Shao (1992).

Lemma 3.1.3 *Let B be a separable Banach space with norm $\|\cdot\|$ and let $\{\Gamma(t);-\infty<t<\infty\}$ be a stochastic process with values in B. Let P be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is P-almost surely separable with respect to $\|\cdot\|$ and that for $|t|\le t_0, 0<x^*\le x$ and $0<h\le h_0$ there exist non-negative monotonically non-decreasing functions $\sigma_1(h)$ and $\sigma_2(h)$ such that*

$$P\{\|\Gamma(t+h)-\Gamma(t)\|\ge x\sigma_1(h)+\sigma_2(h)\}\le K\exp(-\gamma x^\beta)\tag{3.1.19}$$

with some $K,\gamma,\beta>0$. Then

$$\begin{aligned}
&P\{\sup_{0\le t\le T}\sup_{0\le s\le a}\|\Gamma(t+s)-\Gamma(t)\|\ge x(\sigma_1(a)+\sigma_1(a,k))\\
&\quad+\sigma_1^*(a,k)+\sigma_2(a)+\sigma_2(a,k)\}\\
&\le 4\Big(\frac{T}{a}+1\Big)K2^{2^{k+1}}\exp(-\gamma x^\beta)
\end{aligned}\tag{3.1.20}$$

for any $0\leqslant T\leqslant t_0, 0<a\leqslant h_0$, $x\geqslant x^*$ *and* $k\geqslant 3$, *where*

$$\sigma_1(a,k) = 2^{3+1/\beta}\int_{2^{k-3}}^{\infty}\frac{\sigma_2(ae^{-z})}{z}\mathrm{d}z,$$

$$\sigma_2(a,k) = 6\int_{2^{k-3}}^{\infty}\frac{\sigma_2(ae^{-z})}{z}\mathrm{d}z,$$

$$\sigma_1^*(a,k) = 4\left(\frac{14}{\gamma}\right)^{1/\beta}\beta\int_{2^{\frac{k-2}{\beta}}}^{\infty}\sigma_1(ae^{-z^\beta})\mathrm{d}z.$$

Proof For any positive real number t and $k\geqslant 3$ put $t_j = a[t\frac{2^{2^j}}{a}]/2^{2^j}, R=2^{2^k}$. Denote by $\mathscr{T}=\{t_j; t\geqslant 0, j\geqslant 1\}$. We have for any $t,s\in\mathscr{T}$,

$$\begin{aligned}
\|\Gamma(t+s)-\Gamma(t)\| &\leqslant \|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\quad + \|\Gamma(t+s)-\Gamma((t+s)_k)\| + \|\Gamma(t)-\Gamma(t_k)\| \\
&\leqslant \|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\quad + \sum_{j=0}^{\infty}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| \\
&\quad + \sum_{j=0}^{\infty}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|,
\end{aligned}$$

where the second inequality is since the two infinite series are both sums of finite many numbers. Since $\Gamma(\cdot)$ is almost surely separable and $\mathscr{T}$ is dense in $\mathbf{R}^+$, it follows that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\| \\
&= \sup_{\substack{t\in\mathscr{T}\\0\leqslant t\leqslant T}}\sup_{\substack{s\in\mathscr{T}\\0\leqslant s\leqslant a}}\|\Gamma(t+s)-\Gamma(t)\| \\
&\leqslant \sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\quad + \sum_{j=0}^{\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| \\
&\quad + \sum_{j=0}^{\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|.
\end{aligned}$$

Since

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-\frac{1}{R})}|(t+s)_k-t_k|\leqslant a,$$

$$\sup_{0\leqslant t\leqslant T}\sup_{a(1-1/R)\leqslant s\leqslant a}|(t+s)_k-(t+a(1-1/R))_k|\leqslant 2a2^{-2^k},$$

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}|(t+s)_{k+j+1}-(t+s)_{k+j}|\leqslant a2^{-2^{k+j}},$$

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\leqslant \sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-\frac{1}{R})}\|\Gamma((t+s)_k)-\Gamma(t_k)\| \\
&\quad + \sup_{0\leqslant t\leqslant T}\sup_{a(1-\frac{1}{R})\leqslant s\leqslant a}\|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)\|,
\end{aligned}$$

we get from (3.1.19) for each $x\geqslant x^*$ and $x_j\geqslant x^*$

$$\begin{aligned}
&P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a(1-\frac{1}{R})}\|\Gamma((t+s)_k)-\Gamma(t_k)\|\geqslant x\sigma_1(a)+\sigma_2(a)\} \\
&\leqslant 2KR^2(T/a+1)\exp(-\gamma x^\beta), \\
&P\{\sup_{0\leqslant t\leqslant T}\sup_{a(1-1/R)\leqslant s\leqslant a}\|\Gamma((t+s)_k)-\Gamma((t+a(1-1/R))_k)\| \\
&\qquad \geqslant x\sigma_1(2a/R)+\sigma_2(2a/R)\} \\
&\leqslant 2KR(T/a+1)\exp(-\gamma x^\beta), \\
&\quad P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\| \\
&\qquad \geqslant x_j\sigma_1(a/2^{2^{k+j}})+\sigma_2(a/2^{2^{k+j}})\} \\
&\quad \leqslant 2K2^{2^{k+j+1}}(T/a+1)\exp(-\gamma x_j^\beta),
\end{aligned}$$

as well as

$$\begin{aligned}
&P\{\sup_{0\leqslant t\leqslant T}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|\geqslant x_j\sigma_1(a/2^{2^{k+j}})+\sigma_2(a/2^{2^{k+j}})\} \\
&\leqslant K2^{2^{k+j+1}}(T/a+1)\exp(-\gamma x_j^\beta).
\end{aligned}$$

Now we put $\gamma x_j^\beta=\gamma x^\beta+2^{k+j+1}$. Then

$$\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(\gamma x_j^\beta) = \sum_{j=0}^{\infty}2^{2^{k+j+1}}e^{-2^{k+j+1}}e^{-\gamma x^\beta}\leqslant\exp(-\gamma x^\beta).$$

From the definition of x_j, we see that

$$x_j\leqslant 2^{\frac{1}{\beta}}x+(2/\gamma)^{\frac{1}{\beta}}2^{\frac{k+j+1}{\beta}},$$

$$x\sigma_1\left(\frac{2a}{R}\right)+\sigma_2\left(\frac{2a}{R}\right)+2\sum_{j=0}^{\infty}x_j\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)+2\sum_{j=0}^{\infty}\sigma_2\left(\frac{a}{2^{2^{k+j}}}\right)$$

$$\leqslant x\left(\sigma_1\left(\frac{2a}{2^{2^k}}\right)+2^{1+\frac{1}{\beta}}\sum_{j=0}^{\infty}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)\right)$$

$$+2\left(\frac{2}{\gamma}\right)^{\frac{1}{\beta}}\sum_{j=0}^{\infty}2^{\frac{k+j+1}{\beta}}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)+\sigma_2\left(\frac{2a}{R}\right)+2\sum_{j=0}^{\infty}\sigma_2\left(\frac{a}{2^{2^{k+j}}}\right),$$

$$\sigma_1\left(\frac{2a}{2^{2^k}}\right)+2^{1+\frac{1}{\beta}}\sum_{j=0}^{\infty}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)$$

$$\leqslant\left(1+2^{\frac{1}{\beta}+1}\right)\sum_{j=0}^{\infty}\sigma_1\left(\frac{2a}{2^{2^{k+j}}}\right)$$

$$\leqslant\left(1+2^{\frac{1}{\beta}+1}\right)\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}}\frac{\sigma_1(2a/2^z)}{z}\mathrm{d}z/\ln 2$$

$$\leqslant 2^{3+\frac{1}{\beta}}\int_{2^{k-1}}^{\infty}\frac{\sigma_1(2a/2^z)}{z}\mathrm{d}z$$

$$\leqslant 2^{3+\frac{1}{\beta}}\int_{2^{k-3}}^{\infty}\frac{\sigma_1(ae^{-z})}{z}\mathrm{d}z$$

$$=\sigma_1(a,k)$$

and

$$2\left(\frac{2}{\gamma}\right)^{\frac{1}{\beta}}\sum_{j=0}^{\infty}2^{\frac{k+j+1}{\beta}}\sigma_1\left(\frac{a}{2^{2^{k+j}}}\right)$$

$$\leqslant\frac{2^{\frac{2}{\beta}+1}\left(\frac{2}{\gamma}\right)^{\frac{1}{\beta}}}{\beta(2^{\frac{1}{\beta}}-1)}\sum_{j=0}^{\infty}\int_{2^{k+j-1}}^{2^{k+j}}z^{\frac{1}{\beta}-1}\sigma_1(a2^{-z})\mathrm{d}z$$

$$\leqslant\frac{2^{\frac{1}{\beta}+1}\left(\frac{4}{\gamma}\right)^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}}-1}\int_{2^{\frac{k-1}{\beta}}}^{\infty}\sigma_1\left(a2^{-z^{\beta}}\right)\mathrm{d}z$$

$$\leqslant 4\left(\frac{14}{\gamma}\right)^{\frac{1}{\beta}}\beta\int_{2^{\frac{k-2}{\beta}}}^{\infty}\sigma_1\left(ae^{-z^{\beta}}\right)\mathrm{d}z$$

$$=\sigma_1^*(a,k),$$

as well as

$$2\sum_{j=0}^{\infty}\sigma_2\left(\frac{a}{2^{2^{k+j}}}\right)+\sigma_2\left(\frac{2a}{2^{2^k}}\right)\leqslant 6\int_{2^{k-3}}^{\infty}\frac{\sigma_2(ae^{-z})}{z}\mathrm{d}z$$

$$=\sigma_2(a,k).$$

Combining all the above inequalities shows (3. 1. 20) is true. This completes the proof of Lemma 3. 1. 3.

Using Lemmas 3. 1. 2 and 3. 1. 3 we give the following sufficient conditions for the a. s. continuity of $Y(\cdot)\in l^p, p\geqslant 1$.

Theorem 3. 1. 3 *Assume that* (3. 1. 16),

$$\int_1^{\infty}\frac{\sigma(p,e^{-z})}{z}\mathrm{d}z<\infty \qquad (3.1.21)$$

and

$$\int_1^{\infty}\widetilde{\sigma}(p,e^{-z^2})\mathrm{d}z<\infty \qquad (3.1.22)$$

are satisfied. Then $Y(\cdot)\in l^p$, $p\geqslant 1$, *has a. s. continuous sample paths.*

Proof It suffices to show that

$$\lim_{h\to 0}P\left\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}\sum_{k=1}^{\infty}|X_k(t+s)-X_k(t)|^p\geqslant\varepsilon^p\right\}=0$$

for any $\varepsilon>0$, $T>0$, namely

$$\lim_{h\to 0}\lim_{N\to\infty}P\left\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\geqslant\varepsilon^p\right\}=0. \qquad (3.1.23)$$

Clearly, (3. 1, 22) implies $\int_1^{\infty}\sigma_k(e^{-z^2})\mathrm{d}z<\infty$ for every $k\geqslant 1$. So by Theorem 2. 1. 6 $X_k(\cdot)$ is an a. s. continuous Gaussian process for every $k\geqslant 1$ and so is $Y_N(\cdot):=\{X_k(\cdot)\}_{k=1}^N$ in l^p for every $N\geqslant 1$. Using Lemmas 3. 1. 2 and 3. 1. 3 we have

$$P\left\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}\left(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\right)^{1/p}\geqslant x(\widetilde{\sigma}(p,h)\right.$$

$$+\tilde{\sigma}_1(p,h))+\tilde{\sigma}_2(p,h)+\delta_p(\sigma(p,h)+\sigma_1(p,h))\Big\}$$
$$\leqslant 8(T+1)h^{-32}\exp(-x^2/2)$$
for any $x>0$ and $N\geqslant 1$, where
$$\tilde{\sigma}_1(p,h)=16\int_{\log h^{-1}}^{\infty}\frac{\tilde{\sigma}(p,he^{-z})}{z}\mathrm{d}z,$$
$$\tilde{\sigma}_2(p,h)=11\int_{(\log h^{-1})^2}^{\infty}\tilde{\sigma}(p,he^{-z^2})\mathrm{d}z,$$
$$\sigma_1(p,h)=6\int_{\log h^{-1}}^{\infty}\frac{\sigma(p,he^{-z})}{z}\mathrm{d}z.$$
Since $\tilde{\sigma}(p,h)$ is non-decreasing in h, it follows from (3.1.22) that
$$\tilde{\sigma}(p,h)=o((\log h^{-1})^{-1/2})\quad\text{as}\quad h\to 0,$$
which, further, implies
$$\tilde{\sigma}_1(p,h)=o((\log h^{-1})^{-1/2})\quad\text{as}\quad h\to 0.$$
Also, in terms of (3.1.22) and (3.1.21), we have
$$\tilde{\sigma}_2(p,h)+\delta_p(\sigma(p,h)+\sigma_1(p,h))\to 0\quad\text{as}\quad h\to 0.$$
Consequently, we obtain
$$\lim_{h\to 0}\lim_{N\to\infty}P\Big\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}\Big(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\Big)^{1/p}\geqslant\varepsilon\Big\}$$
$$\leqslant\lim_{h\to 0}\lim_{N\to\infty}P\Big\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}\Big(\sum_{k=1}^{N}|X_k(t+s)-X_k(t)|^p\Big)^{1/p}\geqslant\varepsilon/2$$
$$+\tilde{\sigma}_2(p,h)+\delta_p(\sigma(p,h)+\sigma_1(p,h))\Big\}$$
$$\leqslant 8(T+1)\lim_{h\to 0}h^{-32}\exp\Big\{-\frac{\varepsilon^2}{8(\tilde{\sigma}(p,h)+\tilde{\sigma}_1(p,h))^2}\Big\}$$
$$\leqslant 8(T+1)\lim_{h\to 0}h^{-32}\exp(-34\log h^{-1})=0.$$
This proves (3.1.23) and completes the proof of Theorem 3.1.3.

3.2 The Increments for B-valued Stochastic Processes

In this section we shall establish general theorems on increments, both large and small, for stochastic processes due to Csáki, Csörgő and Shao (1992). Let B be a separable Banach space with norm $\|\cdot\|$ and let $\{\Gamma(t);-\infty<t<\infty\}$ be a stochastic process with values in Banach space B. Let P be the probability measure generated by $\Gamma(\cdot)$.

Theorem 3.2.1 *Let a_T, b_T, C_T, $\sigma_1(T)$, $\sigma_2(T)$ be non-negative continuous functions. Assume that both a_T and b_T are either non-decreasing or non-increasing functions of T and that*
$$C_T+\sigma_1(T)+\sigma_1^{-1}(T)\to\infty\quad as\quad T\to\infty,\tag{3.2.1}$$
$$P\{\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(T)+\sigma_2(T)\}$$
$$\leqslant C_T\exp(-\gamma x^{\beta})\tag{3.2.2}$$
for any
$$\left(\frac{1}{\gamma}\left(\log C_T+\log\log\left(\sigma_1(T)+\frac{1}{\sigma_1(T)}\right)\right)\right)^{\frac{1}{\beta}}\leqslant x$$
$$\leqslant(1+\delta)\left(\frac{1}{\gamma}\left(\log C_T+\log\log\left(\sigma_1(T)+\frac{1}{\sigma_1(T)}\right)\right)\right)^{\frac{1}{\beta}}+\delta\frac{\sigma_2(T)}{\sigma_1(T)}\tag{3.2.3}$$
with some $\gamma,\beta,\delta>0$. Then, we have
$$\lim_{T\to\infty}\sup\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|\leqslant 1\quad\text{a.s.},\tag{3.2.4}$$
where
$$\alpha_T^{-1}=\sigma_1(T)\left(\frac{1}{\gamma}\left(\log C_T+\log\log\left(\sigma_1(T)+\frac{1}{\sigma_1(T)}\right)\right)\right)^{\frac{1}{\beta}}+\sigma_2(T).$$

Proof Without loss of generality, assume $0<\delta<1/2$ and both a_T and b_T are non-decreasing. Let $1<\theta<1+\delta/2$. Define

$$A_k=\{T;\theta^k<\sigma_1(T)\leqslant\theta^{k+1}\},\quad -\infty<k<\infty,$$

$$A_{k,j}=\{T;2^j\leqslant C_T<2^{j+1},T\in A_k\},\quad j\geqslant 0,$$

$$A_{k,j,i}=\{T;\theta^i<\sigma_2(T)\leqslant\theta^{i+1},T\in A_{k,j}\},\quad -\infty<i<\infty,$$

$$T_{k,j,i}=\sup\{T;T\in A_{k,j,i}\}.$$

Write $a(T)=a_T$ and $b(T)=b_T$. Noting that (3.2.1) is satisfied and using the continuity of $a_T,b_T,C_T,\sigma_1(T)$ and $\sigma_2(T)$, we have

$$\begin{aligned}
&\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|\\
&\leqslant\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{T\in A_{k,j}}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\alpha_T\|\Gamma(t+s)-\Gamma(t)\|\\
&\leqslant\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{T\in A_{k,j}}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\\
&\qquad\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\sigma_2(T)}\\
&\leqslant\max\Bigg\{\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{i\leqslant k}\sup_{T\in A_{k,j}}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\\
&\qquad\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log(2^j\log\theta^{|k|})))^{1/\beta}},\\
&\quad\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{i\geqslant k}\sup_{0\leqslant t\leqslant b(T_{k,j,i})}\sup_{0\leqslant s\leqslant a(T_{k,j,i})}\\
&\qquad\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\theta^i}\Bigg\}.
\end{aligned}\tag{3.2.5}$$

We show first that

$$\begin{aligned}
&\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{i\geqslant k}\sup_{0\leqslant t\leqslant b(T_{k,j,i})}\sup_{0\leqslant s\leqslant a(T_{k,j,i})}\\
&\qquad\frac{\|\Gamma(t+s)-\Gamma(t)\|}{\theta^k(\frac{1}{\gamma}(\log 2^j+\log\log\theta^{|k|}))^{1/\beta}+\theta^i}
\end{aligned}$$

$$\begin{aligned}
&\leqslant\theta\limsup_{|k|+l\to\infty}\sup_{j\geqslant l}\sup_{i\geqslant k}\sup_{0\leqslant t\leqslant b(T_{k,j,i})}\sup_{0\leqslant s\leqslant a(T_{k,j,i})}\alpha_{T_{k,j,i}}\|\Gamma(t+s)-\Gamma(t)\|\\
&\leqslant\theta^2\quad\text{a.s.}
\end{aligned}\tag{3.2.6}$$

By (3.2.2), we find

$$\begin{aligned}
&P\Big\{\sup_{j\geqslant l}\sup_{i\geqslant k}\sup_{0\leqslant t\leqslant b(T_{k,j,i})}\sup_{0\leqslant s\leqslant a(T_{k,j,i})}\alpha_{T_{k,j,i}}\|\Gamma(t+s)-\Gamma(t)\|\geqslant\theta\Big\}\\
&\leqslant\sum_{j\geqslant l}\sum_{i\geqslant k}C_{T_{k,j,i}}\exp\Bigg\{-\gamma\Bigg((\theta-1)\frac{\sigma_2(T_{k,j,i})}{\sigma_1(T_{k,j,i})}\\
&\qquad+\theta\Big(\frac{1}{\gamma}(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\Big)^{\frac{1}{\beta}}\Bigg)^{\beta}\Bigg\}\\
&\leqslant\sum_{j\geqslant l}\sum_{i\geqslant k}C_{T_{k,j,i}}\exp\Bigg(-\gamma\Bigg((\theta-1)\theta^{i-k-1}\\
&\qquad+\theta\Big(\frac{1}{\gamma}(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\Big)^{\frac{1}{\beta}}\Bigg)^{\beta}\Bigg)\\
&\leqslant\sum_{j\geqslant l}\sum_{\substack{i\geqslant k\\ \theta^{i-k}\leqslant j^{2/\beta}+k^2}}C_{T_{k,j,i}}\exp\big(-\theta^\beta(\log C_{T_{k,j,i}}+\log\log\theta^{|k|})\big)\\
&\qquad+\sum_{j\geqslant l}\sum_{\theta^{i-k}>j^{2/\beta}+k^2}C_{T_{k,j,i}}\exp(-\gamma(\theta-1)^\beta\theta^{(i-k-1)\beta})\\
&\leqslant c\Big(\sum_{j\geqslant l}(j^{2/\beta}+\log|k|)2^{-j(\theta^\beta-1)}(|k|+1)^{-\theta^\beta}\\
&\qquad+\sum_{j\geqslant l}2^{-j}\exp(-\gamma(1-\frac{1}{\theta})^\beta(j^{2/\beta}+k^2)^\beta)\Big)\\
&\leqslant c2^{-\frac{l(\theta^\beta-1)}{2}}(1+|k|)^{-\frac{\theta^\beta+1}{2}},
\end{aligned}\tag{3.2.7}$$

where c is a constant, depending only on θ,β and γ. Therefore

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{|k|=0}^{\infty}P\Big\{\sup_{j\geqslant l}\sup_{i\geqslant k}\sup_{0\leqslant t\leqslant b(T_{k,j,i})}\sup_{0\leqslant s\leqslant a(T_{k,j,i})}\alpha_{T_{k,j,i}}\|\Gamma(t+s)-\Gamma(t)\|\\
&\qquad>\theta\Big\}<\infty.
\end{aligned}\tag{3.2.8}$$

Now (3.2.6) follows from (3.2.8) and the Borel-Cantelli lemma.

Next we prove that

$$\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{T\in A_{k,j}}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\frac{\| \Gamma(t+s)-\Gamma(t) \|}{\theta^k(\frac{1}{\gamma}\log(2^j\log\theta^{|k|}))^{1/\beta}}$$

$$\leq\theta^2 \quad \text{a. s.} \tag{3.2.9}$$

Put

$$T_{k,j}=\sup\{T;T\in A_{k,j},\sigma_2(T)\leq\theta^{k+1}\}.$$

Then

$$\sigma_2(T_{k,j})\leq\theta^{k+1}$$

and

$$\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{T\in A_{k,j}}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\frac{\| \Gamma(t+s)-\Gamma(t) \|}{\theta^k(\frac{1}{\gamma}\log(2^j\log\theta^{|k|}))^{1/\beta}}$$

$$\leq\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{0\leq t\leq b(T_{k,j})}\sup_{0\leq s\leq a(T_{k,j})}\frac{\| \Gamma(t+s)-\Gamma(t) \|}{\theta^k(\frac{1}{\gamma}\log(2^j\log\theta^{|k|}))^{1/\beta}+\sigma_2(T_{k,j})}$$

$$\leq\theta\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{0\leq t\leq b(T_{k,j})}\sup_{0\leq s\leq a(T_{k,j})}\alpha_{T_{k,j}}\| \Gamma(t+s)-\Gamma(t) \|.$$

The rest of the proof of (3. 2. 9) follows along the lines of the proof of (3. 2. 6), and (3. 2. 4) now follows from (3. 2. 5), (3. 2. 6), (3. 2. 9) and the arbitrariness of δ. This completes the proof of Theorem 3. 2. 1.

The next theorem indicates that the assumption that both a_T and b_T are either non-decreasing or non-increasing functions of T can removed under some additional conditions.

Theorem 3. 2. 2 *Let $a_T,b_T,\sigma_1(T)$ and $\sigma_2(T)$ be non-negative continuous functions. Assume that*

$$\frac{b_T}{a_T}+\sigma_1(T)+\frac{1}{\sigma_1(T)}\to\infty \quad \text{as} \quad T\to\infty, \tag{3.2.10}$$

$$P\{\sup_{0\leq t\leq b}\sup_{0\leq s\leq a_T}\| \Gamma(t+s)-\Gamma(t) \|\geq x\sigma_1(T)+\sigma_2(T)\}$$

$$\leq A(1+\frac{b}{a_T})\exp(-\gamma x^\beta) \tag{3.2.11}$$

for any $b\geq b_T$, and

$$\left(\frac{1}{\gamma}\left(\log(\frac{b}{a_T}+1)+\log\log(\sigma_1(T)+\frac{1}{\sigma_1(T)})\right)\right)^{\frac{1}{\beta}}\leq x$$

$$\leq(1+\delta)\left(\frac{1}{\gamma}\left(\log(\frac{b}{a_T}+1)+\log\log(\sigma_1(T)+\frac{1}{\sigma_1(T)})\right)\right)^{\frac{1}{\beta}}$$

$$+\delta\frac{\sigma_2(T)}{\sigma_1(T)}$$

with some γ,β,δ and $A>0$. Then we have

$$\lim_{T\to\infty}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\alpha_T^*\| \Gamma(t+s)-\Gamma(t) \|\leq 1 \quad \text{a. s.}, \tag{3.2.12}$$

where $1/\alpha_T^*=\sigma_1(T)(\frac{1}{\gamma}(\log(b_T/a_T+1)+\log\log(\sigma_1(T)+\sigma_1^{-1}(T))))^{1/\beta}+\sigma_2(T)$.

Proof Let $C_T=1+b_T/a_T$. Assume, without loss of generality, $0<\delta<1/2$. Let $\theta,A_k,A_{k,j},A_{k,j,i}$ be as in the proof of Theorem 3. 2. 1. Put

$$b(T_{k,j,i})=\sup\{b_T;T\in A_{k,j,i}\}$$

and

$$a(T_{k,j,i}^*)=\sup\{a_T;T\in A_{k,j,i}\}.$$

It is easy to see that $b(T_{k,j,i})\geq b(T_{k,j,i}^*)$ and

$$2^j\leq\frac{b(T_{k,j,i}^*)}{a(T_{k,j,i}^*)}+1\leq\frac{b(T_{k,j,i})}{a(T_{k,j,i}^*)}+1\leq\frac{b(T_{k,j,i})}{a(T_{k,j,i})}+1\leq 2^{j+1}.$$

Hence, along the way of the proof of Theorem 3. 2. 1 we find

$$\lim_{T\to\infty}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\alpha_T^*\| \Gamma(t+s)-\Gamma(t) \|$$

$$\leq\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{T\in A_{k,j}}\sup_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\alpha_T^*\| \Gamma(t+s)-\Gamma(t) \|$$

$$\leq\lim_{|k|+l\to\infty}\sup_{j\geq l}\sup_{i}\sup_{T\in A_{k,j,i}}\sup_{0\leq t\leq b(T_{k,j,i})}\sup_{0\leq s\leq a(T_{k,j,i}^*)}\alpha_T^*$$

$$\times\| \Gamma(t+s)-\Gamma(t) \|$$

$$\leq\theta^2 \quad \text{a. s.}$$

This, by the arbitrariness of δ, proves (3. 2. 12).

Using Lemma 3.1.3, we can get some corollaries of the previous theorem.

Lemma 3.2.1 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ *and* $\sigma_2(h)$ *be as in Lemma* 3.1.3 *and assume that* $\sigma_1(x)/x^\alpha$ *and* $\sigma_2(x)/x^\alpha$ *are quasi-increasing on* $(0,h_0)$ *for some* $\alpha>0$. *Then for any* $0<\varepsilon<1$ *there exists* $C=C(\varepsilon,\beta,\gamma,\alpha)$ *such that*

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(h)+(1+\varepsilon)\sigma_2(h)\}$$

$$\leqslant CK\left(\frac{T}{h}+1\right)\exp\left(-\frac{\gamma x^\beta}{1+\varepsilon}\right) \tag{3.2.13}$$

for every $x\geqslant\max\left(1,\dfrac{x^*}{1-\varepsilon}\right)$, $0\leqslant T\leqslant t_0$ *and* $0<h\leqslant h_0$.

Proof Since $\sigma_1(x)/x^\alpha$ and $\sigma_2(x)/x^\alpha$ are quasi-increasing on $(0,h_0)$, there is a positive c_0 such that

$$\sigma_i(ht)\leqslant c_0t^\alpha\sigma_i(h),\quad i=1,2 \tag{3.2.14}$$

for all $0<t\leqslant 1$. From (3.2.14) it is easy to find that

$$\sigma_i(h,k)\leqslant 2^{3+\frac{1}{\beta}}c_0\mathrm{e}^{-\alpha(k-3)}\alpha^{-1}\sigma_i(h),$$

$$\sigma_i^*(h,k)\leqslant 4\left(\frac{14}{\gamma}\right)^{\frac{1}{\beta}}\beta c_0\mathrm{e}^{-\alpha(k-2)}\alpha^{-1}\sigma_1(h),\quad i=1,2.$$

Hence for $\delta=\min(\varepsilon,1-(1+\varepsilon)^{-1/\beta})$, we can take k such that

$$\sigma_2(h,k)+\sigma_1(h,k)+\sigma_1^*(h,k)\leqslant\frac{\delta}{2}(\sigma_1(h)+\sigma_2(h)).$$

By Lemma 3.1.3 we have

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(h)+(1+\varepsilon)\sigma_2(h)\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x(1-\delta)(\sigma_1(h)$$

$$+\sigma_1(h,k))+\sigma_1^*(h,k)+\sigma_2(h)+\sigma_2(h,k)\}$$

$$\leqslant 4K\left(\frac{T}{h}+1\right)2^{2^{k+1}}\exp(-\gamma(x(1-\delta))^\beta)$$

$$\leqslant 4K\left(\frac{T}{h}+1\right)2^{2^{k+1}}\exp\left(-\frac{\gamma x^\beta}{1+\varepsilon}\right).$$

Now put $C=C(\varepsilon,\beta,\gamma,\alpha)=4\cdot 2^{2^{k+1}}$, as desired, and the proof is completed.

Remark 3.2.1 Strictly speaking, the constant C in (3.2.13) depends not only on ε, α, γ, β but also on c_0 in (3.2.14). But, for the sake of convenience, we shall continue writing C for the constant $C=C(\varepsilon,\alpha,\gamma,\beta)$ in the sequel.

Lemma 3.2.2 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ *and* $\sigma_2(h)$ *be as in Lemma* 3.1.3 *and assume that* $\sigma_1(x)(\log x^{-1})^\alpha$ *and* $\sigma_2(x)(\log x^{-1})^\alpha$ *are quasi-increasing on* $(0,1/2)$ *for some* $\alpha>1/\beta$, *namely, there is a* $c_0>0$ *such that*

$$\sigma_i(ht)\leqslant c_0\sigma_i(h)\left(\log\frac{1}{h}\right)^\alpha\Big/\left(\log\frac{1}{h}+\log\frac{1}{t}\right)^\alpha,\quad i=1,2 \tag{3.2.15}$$

for each $0<t\leqslant 1$, $0<h\leqslant\frac{1}{2}$. *Then, for any* $\varepsilon>0$, *we have*

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(h)(1+c_1c_0)$$

$$+\sigma_2(h)(1+c_1c_0)+c_2c_0\sigma_1(h)(\log h^{-1})^{1/\beta}\}$$

$$\leqslant 8K\left(\frac{T}{h}+1\right)\frac{1}{h^{2\varepsilon}}\exp(-\gamma x^\beta) \tag{3.2.16}$$

for every $x\geqslant\max(x^*,1)$, $0\leqslant T\leqslant t_0$ *and* $0<h\leqslant\min(\mathrm{e}^{-8/\varepsilon},h_0,1/2)$, *where*

$$c_1=2^{3+1/\beta}\left(1+\frac{\varepsilon}{8}\right)^{-\alpha}\left(\frac{1}{\alpha}+\log(1+\frac{8}{\varepsilon})\right),$$

$$c_2=4\left(\frac{14}{\gamma}\right)^{1/\beta}\beta\frac{\beta\alpha}{\beta\alpha-1}\left(1+\frac{\varepsilon}{4}\right)^{-(\beta\alpha-1)/\beta}.$$

Proof Put $2^k=\varepsilon\log h^{-1}$ in Lemma 3.1.3. By (3.2.15) we have for $i=1,2$

$$\sigma_i(h,k)\leqslant c_02^{3+1/\beta}\int_{2^{k-3}}^\infty\frac{\sigma_i(h)(\log 1/h)^\alpha}{z(\log(h+z)^{-1})^\alpha}\mathrm{d}z$$

$$= c_0 2^{3+1/\beta}\sigma_i(h)\int_{\frac{\varepsilon}{8}\log\frac{1}{h}}^{\infty}\frac{(\log h^{-1})^{\alpha}}{z(\log(h^{-1}+z)^{-1})^{\alpha}}\mathrm{d}z$$

$$= c_0 2^{3+1/\beta}\sigma_i(h)\int_{\varepsilon/8}^{\infty}\frac{1}{z(z+1)^{\alpha}}\mathrm{d}z$$

$$\leqslant c_0 2^{3+1/\beta}\sigma_i(h)\left(\int_{\varepsilon/8}^{1+\varepsilon/8}\frac{1}{z(1+\varepsilon/8)^{\alpha}}\mathrm{d}z+\int_{1+\varepsilon/8}^{\infty}\frac{\mathrm{d}z}{z^{1+\alpha}}\right)$$

$$\leqslant c_0 2^{3+1/\beta}\sigma_i(h)(1+\varepsilon/8)^{-\alpha}\left(\frac{1}{\alpha}+\log\left(1+\frac{8}{\varepsilon}\right)\right)$$

$$= c_0 c_1 \sigma_i(h)$$

and

$$\sigma_1^*(h,k)\leqslant 4\left(\frac{14}{\gamma}\right)^{1/\beta}\beta c_0\int_{2^{\frac{k-2}{\beta}}}^{\infty}\frac{\sigma_1(h)\log^{\alpha}h^{-1}}{(z^{\beta}+\log h^{-1})^{\alpha}}\mathrm{d}z$$

$$\leqslant 4\left(\frac{14}{\gamma}\right)^{1/\beta}\beta c_0\sigma_1(h)\left(\log\frac{1}{h}\right)^{1/\beta}\int_{(\varepsilon/4)^{1/\beta}}^{\infty}\frac{1}{(1+z^{\beta})^{\alpha}}\mathrm{d}z$$

$$\leqslant 4\left(\frac{14}{\gamma}\right)^{1/\beta}\beta c_0\sigma_1(h)\left(\log\frac{1}{h}\right)^{\frac{1}{\beta}}\left(\int_{(\varepsilon/4)^{1/\beta}}^{(1+\varepsilon/4)^{1/\beta}}\frac{\mathrm{d}z}{(1+\varepsilon/4)^{\alpha}}+\int_{(1+\varepsilon/4)^{1/\beta}}^{\infty}\frac{\mathrm{d}z}{z^{\alpha\beta}}\right)$$

$$\leqslant 4\left(\frac{14}{\gamma}\right)^{1/\beta}c_0\sigma_1(h)(\log h^{-1})^{\frac{1}{\beta}}\frac{\beta^2\alpha}{\beta\alpha-1}\left(1+\frac{\varepsilon}{4}\right)^{-\alpha-1/\beta}$$

$$= c_0\sigma_1(h)(\log h^{-1})^{1/\beta}c_2.$$

Now (3. 2. 16) follows from Lemma 3. 1. 3.

Clearly, the inequalities (3. 2. 13) and (3. 2. 16) enable one to study the increments of $\Gamma(\cdot)$ for small h over the interval $(0,1)$ and as we will see, even to establish some surprising results. The next versions of (3. 2. 13) and (3. 2. 16) are for the sake of studying large increments of $\Gamma(\cdot)$.

Lemma 3. 2. 3 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ *and* $\sigma_2(h)$ *be as in Lemma* 3. 1. 3 *with* $t_0=h_0=\infty$. *Assume that* $\sigma_1(x)/x^{\alpha}$ *and* $\sigma_2(x)/x^{\alpha}$ *are quasi-increasing on* $(0,\infty)$ *for some* $\alpha>0$. *Then for any* $0<\varepsilon<1$ *there exists* $C=C(\varepsilon,\gamma,\alpha,\beta)$ *such that*

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(a)+(1+\varepsilon)\sigma_2(a)\}$$

$$\leqslant CK\left(\frac{T}{a}+1\right)\exp\left(-\frac{\gamma x^{\beta}}{1+\varepsilon}\right) \tag{3. 2. 17}$$

for every $x\geqslant\max(1,x^*/(1-\varepsilon))$ *and* T, $a>0$.

Lemma 3. 2. 4 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(x)$ *and* $\sigma_2(x)$ *be as in Lemma* 3. 1. 3 *with* $t_0=h_0=\infty$. *Assume that* $\sigma_1(x)/(\log x)^{\alpha}$ *and* $\sigma_2(x)/(\log x)^{\alpha}$ *are quasi-increasing on* $(2,\infty)$ *for some* $\alpha>1/\beta$ *and that* $\int_1^{\infty}\sigma_1(\mathrm{e}^{-z^{\beta}})\mathrm{d}z<\infty$, $\int_1^{\infty}\sigma_2(\mathrm{e}^{-z})\mathrm{d}z<\infty$. *Then, there exist positive numbers* c_1, c_2 *and* a_0 *such that*

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|\geqslant xc_1\sigma_1(a)+c_1\sigma_2(a)\}$$

$$\leqslant c_2KT\exp(-\gamma x^{\beta}) \tag{3. 2. 18}$$

for every $x\geqslant\max(1,x^*)$ *and* $T\geqslant a\geqslant a_0$.

The proofs of (3. 2. 17) and (3. 2. 18) are similar to those of Lemmas 3. 2. 1 and 3. 2. 2 and are therefore omitted.

Combining Theorem 3. 2. 2 with previous lemmas, we deduce

Corollary 3. 2. 1 *Assume that* $\Gamma(\cdot)$ *is P-almost surely continuous with respect to* $\|\cdot\|$ *and that there exist non-negative monotone non-decreasing continuous functions* $\sigma_1(h)$ *and* $\sigma_2(h)$ *such that*

$$P\{\|\Gamma(t+s)-\Gamma(t)\|\geqslant x\sigma_1(h)+\sigma_2(h)\}\leqslant K\exp(-\gamma x^{\beta})$$

for each $t\geqslant0$, $h>0$ *and* $x\geqslant x^*>0$ *with some* K, $\gamma,\beta>0$. *Moreover, assume that* $\sigma_1(x)/x^{\alpha}$ *and* $\sigma_2(x)/x^{\alpha}$ *are quasi-increasing on* $(0,\infty)$ *for some* $\alpha>0$ *and that* a_T *and* b_T *are continuous functions with*

$$\frac{b_T}{a_T}+\sigma_1(a_T)+\frac{1}{\sigma_1(a_T)}\to\infty \quad as \quad T\to\infty.$$

Then we have

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\beta_T \| \Gamma(t+s)-\Gamma(t)\| \le 1 \quad \text{a.s.}, \tag{3.2.19}$$

where $\beta_T^{-1}=\sigma_1(a_T)(\gamma^{-1}(\log(1+b_T/a_T)+\log\log(\sigma_1(a_T)+\sigma_1(a_T)^{-1})))^{1/\beta}+\sigma_2(a_T)$.

Proof By Lemma 3.2.3, for every $0<\varepsilon<1$, $b>0$ and $x\geq \max(1,x^*/(1-\varepsilon))$ we have

$$P\{\sup_{0\le t\le b}\sup_{0\le s\le a_T}\|\Gamma(t+s)-\Gamma(t)\|\geq x\sigma_1(a_T)+(1+\varepsilon)\sigma_2(a_T)\}$$

$$\leq K\left(1+\frac{b}{a_T}\right)\exp\left(-\frac{\gamma x^\beta}{1+\varepsilon}\right).$$

Consequently, by Theorem 3.2.2

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\beta_T(\varepsilon)\|\Gamma(t+s)-\Gamma(t)\|\le 1 \quad \text{a.s.}, \tag{3.2.20}$$

where $\beta_T^{-1}(\varepsilon)=\sigma_1(a_T)\left(\frac{1+\varepsilon}{\gamma}(\log(1+b_T/a_T)+\log\log(\sigma_1(a_T)+\sigma_1(a_T)^{-1}))\right)^{1/\beta}+(1+\varepsilon)\sigma_2(a_T)$. (3.2.19) now follows from (3.2.20) and the arbitrariness of ε.

Corollary 3.2.2 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ *and* $\sigma_2(h)$ *be as in Lemma* 3.2.1 *with* $t_0=1$. *Then we have*

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\theta_h\|\Gamma(t+s)-\Gamma(t)\|\le 1 \quad \text{a.s.}, \tag{3.2.21}$$

where $\theta_h^{-1}=\sigma_1(h)(\gamma^{-1}(\log h^{-1}+\log\log\sigma_1(h)^{-1}))^{1/\beta}+\sigma_2(h)$.

Corollary 3.2.3 *Let* $\{\Gamma(t);-\infty<t<\infty\}$, $\sigma_1(h)$ *and* $\sigma_2(h)$ *be as in Lemma* 3.2.2 *with* $t_0=1$. *Then, there exists a positive* C *such that*

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\theta_h\|\Gamma(t+s)-\Gamma(t)\|\le C \quad \text{a.s.} \tag{3.2.22}$$

The proofs of Corollaries 3.2.2 and 3.2.3 are similar to that of Corollary 3.2.1 and are therefore omitted here.

3.3 The Increments for l^p-valued Gaussian Processes

Let $\{Y(t);t\geq 0\}=\{X_k(t);t\geq 0\}_{k=1}^\infty$ be a sequence of independent Gaussian processes with $EX_k(t)=0$ and stationary increments $\sigma_k^2(h)=E(X_k(t+h)-X_k(t))^2$, where $\sigma_k(h)$ is assumed to be a non-decreasing continuous function for every $k\geq 1$. We shall continue employing the notations introduced in subsection 3.1.3.

3.3.1 The increments when $\sigma(p,h)/h^\alpha$ and/or $\tilde{\sigma}(p,h)/h^\alpha$ are quasi-increasing

We first consider the case when $\sigma(p,h)$ and/or $\tilde{\sigma}(p,h)$ are infinite. The following two propositions will be used in the proofs of main theorems.

Proposition 3.3.1 *Let* a_T, $T>0$, *be a positive continuous function and* b_T, $T>0$, *be a non-negative continuous function. Put* $a^*=\sup_{T>0}a_T$. *Assume that* $\sigma(p,h)/h^\alpha$ *and* $\tilde{\sigma}(p,h)/h^\alpha$ *are quasi-increasing on* $(0,a^*)$ *for some* $\alpha>0$ *and that*

$$\frac{1+b_T}{a_T}+a_T\to\infty \quad as \quad T\to\infty. \tag{3.3.1}$$

Then we have

$$\lim_{T\to\infty}\sup_{0\le t\le b_T}\sup_{0\le s\le a_T}\beta(p,T)\|Y(t+s)-Y(t)\|_{l^p}\le 1 \quad \text{a.s.}, \tag{3.3.2}$$

where $\beta(p,T)^{-1}=\delta_p\sigma(p,a_T)+\tilde{\sigma}(p,a_T)(2(\log(b_T/a_T)+$

$\log\log(a_T+1/a_T)))^{1/2}$.

Proof Recalling that

$$\widetilde{\sigma}(p,h) = \begin{cases} \delta^{-1}_{2p/(2-p)}\Big(E\Big(\sum_{k=1}^{\infty}|X_k(t+h)-X_k(t)|^{2p/(2-p)}\Big)\Big)^{(2-p)/2p} & \text{if } 1\leqslant p<2, \\ \max_{k\geqslant 1}(E(X_k(t+h)-X_k(t))^2)^{1/2} & \text{if } p\geqslant 2, \end{cases}$$

and using Minkowski's inequality, we obtain

$$\widetilde{\sigma}(p,2h)\leqslant 2\widetilde{\sigma}(p,h) \quad \text{for each } h>0 \text{ and } p\geqslant 1. \tag{3.3.3}$$

From (3.3.3) it follows easily that

$$\widetilde{\sigma}(p,h)+\frac{1}{\widetilde{\sigma}(p,h)} \leqslant 4\Big(h+\frac{1}{h}\Big)\Big(\widetilde{\sigma}(p,1)+\frac{1}{\widetilde{\sigma}(p,1)}\Big) \quad \text{for each } h>0. \tag{3.3.4}$$

By (3.3.4), Lemma 3.1.3 and Theorem 3.1.3, we conclude that (3.3.2) holds.

Remark 3.3.1 Let $\sigma_*(p,h)$ and $\widetilde{\sigma}_*(p,h)$ be non-decreasing functions such that $\sigma(p,h)\leqslant\sigma_*(p,h)$ and $\widetilde{\sigma}(p,h)\leqslant\widetilde{\sigma}_*(p,h)$ for each $h>0$. Assume that $\sigma_*(p,h)/h^\alpha$ and $\widetilde{\sigma}_*(p,h)/h^\alpha$ are quasi-increasing on $(0,a^*)$ for some $\alpha>0$. Clearly, (3.1.17) remains true if $\sigma(p,h)$ and $\widetilde{\sigma}(p,h)$ are replaced by $\sigma_*(p,h)$ and $\widetilde{\sigma}_*(p,h)$, respectively. Hence, (3.3.2) remains valid with $\sigma_*(p,a_T)$ and $\widetilde{\sigma}_*(p,a_T)$ instead of $\sigma(p,a_T)$ and $\widetilde{\sigma}(p,a_T)$, respectively.

Proposition 3.3.2 *Let a_T and b_T be positive continuous functions. Assume that* (3.1.15), *as well as*

$$\frac{\log(b_T/a_T)}{\log\log(a_T+(1/a_T))}\to\infty \quad \text{as} \quad T\to\infty \tag{3.3.5}$$

and

$$\limsup_{T\to\infty}\max_{(b_T/a_T)^\varepsilon\leqslant j\leqslant b_T/a_T}\max_{k\geqslant 1}\{\sigma_k^{-2}(a_T) \times E[(X_k(a_T)-X_k(0))(X_k(ja_T)-X_k((j-1)a_T))]\}\leqslant 0 \tag{3.3.6}$$

for each $\varepsilon>0$, are satisfied. Then we have

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\widetilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}}\geqslant 1 \quad \text{a.s.} \tag{3.3.7}$$

Proof Let $1<\theta<65/64$. Define

$$A_k=\Big\{T; 2^k\leqslant\frac{b_T}{a_T}\leqslant 2^{k+1}\Big\}, \quad k\geqslant 0,$$

$$A_{k,j}=\{T;\theta^{j-1}\leqslant\widetilde{\sigma}(p,a_T)\leqslant\theta^j, T\in A_k\}, \quad -\infty<j<\infty,$$

$$b(T_{k,j})=\inf\{b_T;T\in A_{k,j}\}, \quad a_{k,j}=a(T^*_{k,j})=\inf\{a_T;T\in A_{k,j}\}.$$

By (3.3.4) and (3.3.5), one finds that

$$A_{k,j}=\varnothing \quad \text{for every } |j|\geqslant e^k, \tag{3.3.8}$$

provided that k is sufficiently large. It is also easy to see that

$$2^k\leqslant\frac{b(T_{k,j})}{a(T_{k,j})}\leqslant\frac{b(T_{k,j})}{a(T^*_{k,j})}\leqslant\frac{b(T^*_{k,j})}{a(T^*_{k,j})}\leqslant 2^{k+1}. \tag{3.3.9}$$

Therefore

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\widetilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}} \\
&\geqslant\liminf_{k\to\infty}\inf_j\inf_{T\in A_{k,j}}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\widetilde{\sigma}(p,a_T)(2\log(b_T/a_T))^{1/2}} \\
&\geqslant\liminf_{k\to\infty}\min_{|j|\leqslant e^k}\sup_{0\leqslant t\leqslant b(T_{k,j})}\sup_{0\leqslant s\leqslant a_{k,j}}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\theta^j(2\log 2^{k+1})^{1/2}} \\
&\geqslant\liminf_{k\to\infty}\min_{|j|\leqslant e^k}\max_{0\leqslant i\leqslant 2^{k(2-\theta)}} \frac{\|Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\theta\widetilde{\sigma}(p,a_{k,j})(2\log 2^k)^{1/2}}.
\end{aligned} \tag{3.3.10}$$

We proceed with the proof by considering the two cases of $1\leqslant p<2$ and $2\leqslant p<\infty$ separately.

Case I $1\leqslant p<2$. In this case, by (3.1.17), we have

$$\| Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}$$
$$\geqslant\Big[\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)}\Big]^{-(p-1)/p}\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2(p-1)/(2-p)}$$
$$\times\ (X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-X_v(i2^{k(\theta-1)}a_{k,j})). \qquad (3.3.11)$$

Consider

$$\xi(k,j;i)$$
$$=\frac{\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2(p-1)/(2-p)}(X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-X_v(i2^{k(\theta-1)}a_{k,j}))}{\widetilde{\sigma}(p,a_{k,j})(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)})^{(p-1)/p}}$$
$$=\frac{\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2(p-1)/(2-p)}(X_v(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-X_v(i2^{k(\theta-1)}a_{k,j}))}{(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)})^{1/2}},$$

$k=1,2,\cdots,|j|\leqslant e^k$, $0\leqslant i\leqslant 2^{k(2-\theta)}$. For j, k fixed and $0\leqslant i<m\leqslant 2^{k(2-\theta)}$, we have

$$E\{\xi(k,j;i)\xi(k,j;m)\}$$
$$=\Big(\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)}\Big)^{-1}\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{4(p-1)/(2-p)}$$
$$\times E\{(X_v(a_{k,j})-X_v(0))\times(X_v((m-i)2^{k(\theta-1)}a_{k,j}+a_{k,j})$$
$$-X_v((m-i)2^{k(\theta-1)}a_{k,j}))\}, \qquad (3.3.12)$$

by the fact that $\{X_v(t);t\geqslant 0\}_{v=1}^{\infty}$ is a sequence of independent Gaussian processes with stationary increments. Noting that

$$\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{4(p-1)/(2-p)}\sigma_v^2(a_{k,j})=\sum_{v=1}^{\infty}\sigma_v(a_{k,j})^{2p/(2-p)},$$

and using the assumption (3.3.6), we deduce from (3.3.12) that

$$E\xi(k,j;i)\xi(k,j;m)\leqslant\theta-1 \qquad (3.3.13)$$

for every $|j|\leqslant e^k$, $0\leqslant i<m\leqslant 2^{k(2-\theta)}$, provided that k is sufficiently large. Also, clearly

$$E\xi^2(k,j;i)=1. \qquad (3.3.14)$$

Let $\{\eta_i;0\leqslant i\leqslant 2^{k(2-\theta)}\}$ and τ be independent normal random variables with mean zero and with $E\eta_i^2=2-\theta$, $0\leqslant i\leqslant 2^{k(2-\theta)}$ and $E\tau^2=\theta-1$. Define $\tau_i=\tau+\eta_i$, $0\leqslant i\leqslant 2^{k(2-\theta)}$. Note that

$$E\xi^2(k,j;i)=E\tau_i^2=1,\quad 0\leqslant i\leqslant 2^{k(2-\theta)},$$
$$E\{\xi(k,j;i)\xi(k,j;m)\}\leqslant E\{\tau_i\tau_m\},\quad 0\leqslant i\neq m\leqslant 2^{k(2-\theta)},$$

for k sufficiently large. Therefore, by Slepian's inequality,

$$P\{\max_{0\leqslant i\leqslant 2^{k(2-\theta)}}\xi(k,j;i)\leqslant((2-\theta)^2-2(\theta-1)^{1/2})(2\log 2^k)^{1/2}\}$$
$$\leqslant P\{\max_{0\leqslant i\leqslant 2^{k(2-\theta)}}\tau_i\leqslant((2-\theta)^2-2(\theta-1)^{1/2})(2\log 2^k)^{1/2}\}$$
$$\leqslant P\{\max_{0\leqslant i\leqslant 2^{k(2-\theta)}}\eta_i\leqslant(2-\theta)^2(2\log 2^k)^{1/2}\}$$
$$+P\{\tau\geqslant 2(\theta-1)^{1/2}(2\log 2^k)^{1/2}\}$$
$$\leqslant(\Phi((2-\theta)^{3/2}(2\log 2^k)^{1/2}))^{2^{k(2-\theta)}}+\exp(-4\log 2^k)$$
$$\leqslant 2^{-4k}+\left(1-\frac{\exp(-(2-\theta)^3\log 2^k)}{3(1+(2-\theta)^{3/2}(2\log 2^k)^{1/2})}\right)^{2^{k(2-\theta)}}$$
$$\leqslant 2^{-4k}+\exp\left(-\frac{2^{k(2-\theta)}2^{-k(2-\theta)^3}}{k}\right)$$
$$\leqslant 2^{-4k}+\exp\left(-\frac{2^{k(2-\theta)(\theta-1)}}{k}\right) \qquad (3.3.15)$$

for every k big enough.

Putting the above inequalities together and applying the Borel-Cantelli lemma, we conclude

$$\liminf_{k\to\infty}\min_{|j|\leqslant e^k}\max_{0\leqslant i\leqslant 2^{k(2-\theta)}}\frac{\| Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\theta\widetilde{\sigma}(p,a_{k,j})(2\log 2^k)^{1/2}}$$
$$\geqslant\frac{(2-\theta)^2-2(\theta-1)^{1/2}}{\theta}\quad \text{a.s.} \qquad (3.3.16)$$

which yields (3.3.7), by (3.3.10) and the arbitrariness of $1<\theta$

$<65/64$.

Case Ⅱ $p\geqslant 2$. Take $N_{k,j}$ such that $\sigma_{N_{k,j}}(a_{k,j})=\sigma^*(a_{k,j})$. Clearly

$$\frac{\| Y(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\tilde{\sigma}(p,a_{k,j})}$$
$$\geqslant\frac{X_{N_{k,j}}(i2^{k(\theta-1)}a_{k,j}+a_{k,j})-X_{N_{k,j}}(i2^{k(\theta-1)}a_{k,j})}{\sigma_{N_{k,j}}(a_{k,j})}.$$

Along the lines of the proof of Case Ⅰ, we conclude that (3.3.7) remains true in this case as well.

Remark 3.3.2 From the proof of Proposition 3.3.2, one can conclude also that (3.3.7) remains true if (3.3.5) is replaced by

$$\log\log\left(a_T+\frac{1}{a_T}\right)=O\left(\log\frac{b_T}{a_T}\right)\quad\text{and}\quad\frac{b_T}{a_T}\to\infty\quad\text{as}\quad T\to\infty.$$

Moreover, if the conditions (3.3.5) and (3.3.6) are replaced by

$$\frac{\log(b_T/a_T)}{\log\log\log(a_T+1/a_T)}\to\infty\quad\text{as}\quad T\to\infty$$

and

$$E\{(X_k(a)-X_k(0))(X_k(c)-X_k(b))\}\leqslant 0$$

for each $k\geqslant 1$, $0\leqslant a\leqslant b\leqslant c$, respectively, then (3.3.7) holds true.

Now we state the main results in this subsection. They are due to Csörgő and Shao (1993).

Theorem 3.3.1 *Let a_T, $T>0$, be a positive continuous function. Put $a^*=\sup_{T>0}a_T$. Assume that $\tilde{\sigma}(p,h)/h^\alpha$ and $\sigma(p,h)/h^\alpha$ are quasi-increasing on $(0,a^*)$ for some $\alpha>0$ and also that*

$$\lim_{T\to\infty}\frac{\log(T/a_T)}{\log\log T}=\infty,\tag{3.3.17}$$

$$\sigma(p,a_T)=o\left(\tilde{\sigma}(p,a_T)\left(\log\frac{T}{a_T}\right)^{1/2}\right)\quad\text{a.s.}\quad T\to\infty,$$
(3.3.18)

$$\limsup_{T\to\infty}\max_{(T/a_T)^\varepsilon\leqslant j\leqslant T/a_T}\max_{k\geqslant 1}\{\sigma_k^{-2}(a_T)$$
$$\times E[(X_k(a_T)-X_k(0))(X_k(ja_T)-X_k((j+1)a_T))]\}\leqslant 0$$
(3.3.19)

for each $\varepsilon>0$. Then we have

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,a_T)(2\log(T/a_T))^{1/2}}=1\quad\text{a.s.}$$
(3.3.20)

Theorem 3.3.2 *Let a_T, $T>0$, be a positive continuous function satisfying (3.3.17). Assume that $\sigma(p,h)/h^\alpha$ and $\tilde{\sigma}(p,h)/h^\alpha$ are quasi-increasing on $(0,a^*)$ for some $\alpha>0$ and*

$$\tilde{\sigma}(p,a_T)\left(\log\frac{T}{a_T}\right)^{1/2}=o(\sigma(p,a_T))\quad\text{as}\quad T\to\infty.\tag{3.3.21}$$

Then we have

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,a_T)}=1\quad\text{a.s.}\tag{3.3.22}$$

Theorem 3.3.3 *Assume that $\tilde{\sigma}(p,h)/h^\alpha$ is quasi-increasing on $(0,1)$ for some $\alpha>0$. Moreover, suppose that*

$$\sigma(p,h)=o\left(\tilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2}\right)\quad\text{as}\quad h\to 0,\tag{3.3.23}$$

$$\limsup_{h\to 0}\max_{h^{-\varepsilon}\leqslant j\leqslant h^{-1}}\max_{k\geqslant 1}\frac{E[(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))]}{\sigma_k^2(h)}$$
$$\leqslant 0\tag{3.3.24}$$

for each $\varepsilon>0$. Then we have

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}}=1\quad\text{a.s.}\tag{3.3.25}$$

Theorem 3.3.4 *Assume that $\sigma(p,h)/h^\alpha$ is quasi-increasing on $(0,1)$ for some $\alpha>0$ and that*

$$\tilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2}=o(\sigma(p,h))\quad\text{as}\quad h\to 0,\tag{3.3.26}$$

Then we have

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}=1 \quad \text{a. s.} \tag{3. 3. 27}$$

Remark 3. 3. 3 If

$$E\{(X_k(b)-X_k(a))(X_k(d)-X_k(c))\}\le 0 \tag{3. 3. 28}$$

for every $0\le a\le b\le c\le d<\infty$ and for every $k\ge 1$, then, obviously, (3. 3. 19) and (3. 3. 24) are satisfied. In particular, if $\sigma_k^2(h)$ is concave on $(0,\infty)$ for each $k\ge 1$, then (3. 3. 28) is true and hence (3. 3. 19) and (3. 3. 24) are satisfied. In fact, condition (3. 3. 19) or (3. 3. 24) is really much weaker than (3. 3. 28).

Remark 3. 3. 4 We call attention to the normalizing constants in Theorems 3. 3. 1 and 3. 3. 3 being completely different from those of Theorems 3. 3. 2 and 3. 3. 4. The conclusions of the latter two theorems may be somewhat surprising at first sight. We should, however, note that, under the conditions of Theorems 3. 3. 2 and 3. 3. 4, respectively, we have

$$\| Y(a_T)-Y(0)\|_{l^p}\sim \delta_p\sigma(p,a_T), \quad T\to\infty$$

and

$$\| Y(h)-Y(0)\|_{l^p}\sim \delta_p\sigma(p,h), \quad h\to 0$$

and, consequently, their conclusions are like laws of large numbers. On the other hand, the conclusions of Theorems 3. 3. 1 and 3. 3. 3, respectively, may be compared to large and small increments of a standard Wiener process (cf. Theorems 0. 1 and 0. 2 or Chapter 1 of Csörgő and Révész (1981)).

Proof of Theorem 3. 3. 1 This is an immediate consequence of Propositions 3. 3. 1 and 3. 3. 2.

Proof of Theorem 3. 3. 2 By Proposition 3. 3. 1, we have

$$\limsup_{T\to\infty}\sup_{0\le t\le T}\sup_{0\le s\le a_T}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,a_T)}\le 1 \quad \text{a. s.},$$

So it suffices to show that

$$\liminf_{T\to\infty}\sup_{0\le s\le a_T}\frac{\| Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\ge 1 \quad \text{a. s.} \tag{3. 3. 29}$$

Let $1<\theta<65/64$. Define

$$B_k=\left\{T;2^k\le\frac{T}{a_T}\le 2^{k+1}\right\}, \quad k\ge 0,$$

$$B_{k,j}=\{T;\theta^j\le\sigma(p,a_T)\le\theta^{j+1},T\in B_k\}, \quad -\infty<j<\infty,$$

$$a_{k,j}=a(T_{k,j})=\inf\{a_T;T\in B_{k,j}\}.$$

Similar to (3. 3. 4), we have

$$\sigma(p,a_T)\le 2(1+a_T)\sigma(p,1)$$

and hence

$$B_{k,j}=\varnothing, \quad \text{if } |j|>e^k$$

by (3. 3. 17), provided k is sufficiently large. Therefore

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le s\le a_T}\frac{\| Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\\
&\ge\liminf_{k\to\infty}\min_{|j|\le e^k}\inf_{T\in B_{k,j}}\sup_{0\le s\le a_T}\frac{\| Y(s)-Y(0)\|_{l^p}}{\delta_p\sigma(p,a_T)}\\
&\ge\liminf_{k\to\infty}\min_{|j|\le e^k}\frac{\| Y(a_{k,j})-Y(0)\|_{l^p}}{\theta\delta_p\sigma(p,a_{k,j})}.
\end{aligned} \tag{3. 3. 30}$$

Applying Hölder's inequality, we find that

$$\begin{aligned}
E\| Y(a_T)-Y(0)\|_{l^p}&\ge\frac{(E\| Y(a_T)-Y(0)\|_{l^p}^p)^{(2p-1)/p}}{(E\| Y(a_T)-Y(0)\|_{l^p}^{2p})^{(p-1)/p}}\\
&\ge\frac{(\delta_p^p\sum_{k=1}^{\infty}\sigma_k^p(a_T))^{(2p-1)/p}}{((\delta_p^p\sum_{k=1}^{\infty}\sigma_k^p(a_T))^2+\delta_{2p}^{2p}\sum_{k=1}^{\infty}\sigma_k^{2p}(a_T))^{(p-1)/p}}\\
&\ge\frac{(\delta_p^p\sum_{k=1}^{\infty}\sigma_k^p(a_T))^{(2p-1)/p}}{((\delta_p^p\sum_{k=1}^{\infty}\sigma_k^p(a_T))^2+\delta_{2p}^{2p}\sigma^{*p}(a_T)\sum_{k=1}^{\infty}\sigma_k^p(a_T))^{(p-1)/p}}\\
&=\frac{\delta_p\sigma(p,a_T)}{(1+\delta_{2p}^{2p}\delta_p^{-p}(\sigma^*(a_T)/\sigma(p,a_T))^p)^{(p-1)/p}}.
\end{aligned} \tag{3. 3. 31}$$

Therefore, by (3. 3. 21),

$$\liminf_{T\to\infty} \frac{E\,\| Y(a_T) - Y(0) \|_{l^p}}{\delta_p \sigma(p, a_T)} \geqslant 1. \tag{3.3.32}$$

From (3. 1. 18) and (3. 3. 32) it follows that for every k sufficiently large, $|j| \leqslant e^k$,

$$\begin{aligned} & P\left\{ \frac{\| Y(a_{k,j}) - Y(0) \|_{l^p}}{\delta_p \sigma(p, a_{k,j})} \leqslant 2 - \theta \right\} \\ & \leqslant P\Big\{ \| Y(a_{k,j}) - Y(0) \|_{l^p} - E\,\| Y(a_{k,j}) - Y(0) \|_{l^p} \\ & \qquad \leqslant -\frac{\theta - 1}{2} \delta_p \sigma(p, a_{k,j}) \Big\} \\ & \leqslant 2 \exp\left(-\frac{(\theta-1)^2 \delta_p^2 \sigma^2(p, a_{k,j})}{\delta \tilde{\sigma}^2(p, a_{k,j})} \right) \\ & \leqslant 2 \exp\left(-4 \log \frac{T_{k,j}}{a(T_{k,j})} \right) \\ & \leqslant 2 \times 2^{-4k}, \end{aligned} \tag{3.3.33}$$

which, together with the Borel-Cantelli lemma, implies

$$\liminf_{k\to\infty} \min_{|j| \leqslant e^k} \frac{\| Y(a_{k,j}) - Y(0) \|_{l^p}}{\theta \delta_p \sigma(p, a_{k,j})} \geqslant \frac{2-\theta}{\theta} \quad \text{a. s.} \tag{3.3.34}$$

This proves (3. 3. 29) by (3. 3. 30), (3. 3. 34) and the arbitrariness of $\theta > 1$.

Proof of Theorem 3. 3. 3 From Proposition 3. 3. 2, we have

$$\liminf_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} \geqslant 1 \quad \text{a. s.}$$

It suffices to prove that

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} \leqslant 1 \quad \text{a. s.} \tag{3.3.35}$$

For any fixed $\varepsilon > 0$, put

$$\sigma_*(p,h) = \varepsilon \sup\nolimits_{0\leqslant s\leqslant h} \tilde{\sigma}(p,h)(\log(1/s))^{1/2}, \quad 0 < h \leqslant 1.$$

Noting that $\tilde{\sigma}(p,h)/h^{\alpha}$ is quasi-increasing, one can see that there exists a constant c_0, independent of ε, such that

$$\begin{aligned} \varepsilon \sigma(p,h) \left(\log \frac{1}{h} \right)^{1/2} & \leqslant \sigma_*(p,h) = \varepsilon \sup_{0\leqslant s\leqslant h} \frac{\tilde{\sigma}(p,s)}{s^{\alpha}} s^{\alpha} \left(\log \frac{1}{s} \right)^{1/2} \\ & \leqslant \varepsilon c_0 \tilde{\sigma}(p,h) \left(\log \frac{1}{h} \right)^{1/2} \end{aligned} \tag{3.3.36}$$

for $0 < h < 1$. Moreover, $\sigma_*(p, h)$ is non-decreasing, $\sigma_*(p,h)/h^{\alpha/2}$ is quasi-increasing and $\sigma(p, h) \leqslant \sigma_*(p, h)$ by (3. 3. 23), provided that h is sufficiently small. Hence, using Remark 3. 3. 1, we obtain

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2} + \sigma_*(p,h)} \leqslant 1 \quad \text{a. s.}$$

Thus, by (3. 3. 36),

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} \leqslant 1 + \varepsilon c_0.$$

This proves (3. 3. 35) by the arbitrariness of ε.

Proof of Theorem 3. 3. 4 Similar to the proof of (3. 3. 35), by Proposition 3. 3. 1 we have

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\delta_p \sigma(p,h)} \leqslant 1 \quad \text{a. s.} \tag{3.3.37}$$

On the other hand, along the lines of the proof of (3. 3. 29), we can also obtain

$$\liminf_{h\to 0} \sup_{0\leqslant s\leqslant h} \frac{\| Y(s) - Y(0) \|_{l^p}}{\delta_p \sigma(p,h)} \geqslant 1 \quad \text{a. s.} \tag{3.3.38}$$

A combination of (3. 3. 37) with (3. 3. 38) yields (3. 3. 27), as desired.

As applications of the previous theorems, we give moduli of continuity and large increments for l^p-valued fractional Wiener process and moduli of continuity for l^p-valued fractional Ornstein-Uhlenbeck process.

Let $\{\xi(t);t\geqslant 0\}$ be a centered Gaussian process with stationary increments. $\xi(t)$ is called a fractional Wiener process (or self-similar Gaussian process) of order γ if $E\xi^2(t)=t^{2\gamma}$, where $0<\gamma<1$. When $\gamma=1/2$, $\xi(t)$ is a well-known Wiener process.

Let $p\geqslant 1$, $\{c_n;n\geqslant 1\}$ be non-negative numbers. Put

$$c(p) = \left(\sum_{k=1}^{\infty} c_k^p\right)^{1/p}, \tag{3.3.39}$$

$$\tilde{c}(p) = \begin{cases} c\left(\dfrac{2p}{2-p}\right) & \text{if } 1 \leqslant p < 2, \\ \max_{k\geqslant 1} c_k & \text{if } p \geqslant 2. \end{cases} \tag{3.3.40}$$

Let $\{Y(t);t\geqslant 0\}=\{c_k\xi_k(t);t\geqslant 0\}_{k=1}^{\infty}$, where $\xi_k(t)$ are independent fractional Wiener processes of order γ, $0<\gamma<1$. Set $\sigma_k^2(h)=c_k^2E\xi_k^2(h)=c_k^2h^{2\gamma}$. Define $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ as above. Clearly, we have

$$\sigma(p,h) = h^\gamma c(p), \quad \tilde{\sigma}(p,h) = h^\gamma \tilde{c}(p). \tag{3.3.41}$$

Assume

$$0 < \sum_{k=1}^{\infty} c_k^p < \infty. \tag{3.3.42}$$

Then, by Theorem 3.1.3, $Y(\cdot)\in l^p$ has a.s. continuous sample paths. Noting that for each $k\geqslant 1$, $a>0$, $j>2$,

$$\begin{aligned}&\frac{Ec_k\xi_k(a)(c_k\xi_k(ja)-c_k\xi_k((j-1)a))}{\sigma_k^2(a)}\\ &=\frac{E\xi_1(a)(\xi_1(ja)-\xi_1((j-1)a))}{E\xi_1^2(a)}\\ &=\frac{1}{2}((j+1)^{2\gamma}+(j-1)^{2\gamma}-2j^{2\gamma})\end{aligned}$$

and

$$\lim_{j\to\infty}((j+1)^{2\gamma}+(j-1)^{2\gamma}-2j^{2\gamma}) = 0,$$

we see that conditions (3.3.19) and (3.3.24) are satisfied.

Hence, from Theorems 3.3.1, 3.3.3, we obtain the following corollary.

Corollary 3.3.1 *Let $p\geqslant 1$, $\{\xi_k(t);t\geqslant 0\}$, $k=1,2,\cdots$, be independent fractional Wiener processes of order γ, $0<\gamma<1$. Let $\{Y(t);t\geqslant 0\}=\{c_k\xi_k(t);t\geqslant 0\}_{k=1}^{\infty}$. Assume that (3.3.42) is satisfied. Then we have*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{h^\gamma\tilde{c}(p)(2\log(1/h))^{1/2}}=1 \quad \text{a.s.}, \tag{3.3.43}$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{a_T^\gamma\tilde{c}(p)(2\log(T/a_T))^{1/2}}=1 \quad \text{a.s.} \tag{3.3.44}$$

for any positive continuous function a_T with

$$\lim_{T\to\infty}\log(T/a_T)/\log\log T=\infty.$$

Let $\{Y(t);t\geqslant 0\}=\{X_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k. It is easy to see that

$$\{X_k(t);t\geqslant 0\}_{k=1}^{\infty} \quad \text{and} \quad \left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}};t\geqslant 0\right\}_{k=1}^{\infty}$$

have the same distribution, where $\{W_k(t)\}_{k=1}^{\infty}$ are independent standard Wiener processes. Hence, without loss of generality, we can write

$$X_k(t) = \left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, \quad t\geqslant 0,\ k=1,2,\cdots$$

and keep the path property of $Y(\cdot)$ without change. This relationship and the notion of fractional Wiener processes lead in a natural way to introducing fractional Ornstein-Uhlenbeck processes and to studying their path behavior.

Let $\{\xi(t);t\geqslant 0\}$ be a fractional Wiener process of order γ, where $0<\gamma<1$. A stationary Gaussian process $\{X(t);t\geqslant 0\}$ is called a fractional Ornstein-Uhlenbeck process of order γ with

coefficients a and b if

$$\{X(t);t\geqslant 0\} \quad \text{and} \quad \left\{\left(\frac{a}{b}\right)^{1/2}\frac{\xi(\mathrm{e}^{2bt})}{\mathrm{e}^{2\gamma bt}};t\geqslant 0\right\}$$

have the same distribution, that is, $EX(t)=0$, and

$$E\{X(t)X(s)\}=\frac{a}{2b}(\mathrm{e}^{2\gamma b(t-s)}+\mathrm{e}^{2\gamma b(s-t)}-|\mathrm{e}^{b(t-s)}-\mathrm{e}^{b(s-t)}|^{2\gamma}) \tag{3.3.45}$$

for all t, $s\geqslant 0$, where $a\geqslant 0$, $b>0$.

Clearly, $\{X(t);t\geqslant 0\}$ is the usual Ornstein-Uhlenbeck process if $\gamma=1/2$.

In what follows, we will always let $\{Y(t);t\geqslant 0\}=\{X_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent fractional Ornstein-Uhlenbeck processes of order γ with coefficients γ_k and λ_k, where $0<\gamma<1$, $\gamma_k\geqslant 0$, $\lambda_k>0$. Put

$$\begin{aligned}\sigma_k^2(h)&=E(X_k(t+h)-X_k(t))^2\\&=\frac{\gamma_k}{\lambda_k}(2+(\mathrm{e}^{\lambda_k h}-\mathrm{e}^{-\lambda_k h})^{2\gamma}-\mathrm{e}^{2\gamma\lambda_k h}-\mathrm{e}^{-2\gamma\lambda_k h}),\end{aligned}$$

for $h\geqslant 0$, $k=1,2,\cdots$. Let $p\geqslant 1$. Define $\sigma(p,h)$, $\widetilde{\sigma}(p,h)$ and δ_p as above. From Theorem 3.3.1 we obtain

Corollary 3.3.2 *Assume that $\widetilde{\sigma}(p,h)/h^{\alpha}$ quasi-increasing on* $(0,1)$ *for some $\alpha>0$. If*

$$\sigma(p,h)=o\left(\widetilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2}\right) \quad as \quad h\to 0, \tag{3.3.46}$$

then

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\sigma(p,h)(2\log(1/h))^{1/2}}=1 \quad \text{a.s.} \tag{3.3.47}$$

Assume that $\sigma(p,h)/h^{\alpha}$ is quasi-increasing on $(0,1)$ for some $\alpha>0$. If

$$\widetilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2}=o(\sigma(p,h)) \quad \text{as } h\to 0, \tag{3.3.48}$$

then

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}=1 \quad \text{a.s.} \tag{3.3.49}$$

Proof By Theorems 3.3.3 and 3.3.4, it suffices to verify that

$$\limsup_{h\to 0}\max_{\log(1/h)\leqslant j\leqslant 1/h}\max_{k\geqslant 1}\frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))\}}{\sigma_k^2(h)}\leqslant 0. \tag{3.3.50}$$

We can verify that $\sigma_k^2(h)$ is concave on $(0,\infty)$ if $0<\gamma\leqslant 1/2$. Hence, (3.3.50) is satisfied in this case.

We consider below the case of $1/2<\gamma<1$. We have

$$\begin{aligned}&\frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h))\}}{\sigma_k^2(h)}\\&=\frac{f(j\lambda_k h)+f((j-2)\lambda_k h)-2f((j-1)\lambda_k h)}{2(2+f(\lambda_k h))}\\&=\frac{f''(\xi)(\lambda_k h)^2}{2(2+f(\lambda_k h))}\end{aligned}$$

for every $h>0$, $j\geqslant 6$, $k\geqslant 1$ and for some $(j-2)\lambda_k h\leqslant\xi\leqslant j\lambda_k h$, where

$$f(x)=(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma}-\mathrm{e}^{2\gamma x}-\mathrm{e}^{-2\gamma x}.$$

By an elementary computation we can show (3.3.50).

A consequence below gives specific meaning to Corollary 3.3.2 in terms of the coefficients γ_k, λ_k and of the order γ.

Corollary 3.3.3 *Assume that*

$$0<\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/2}\lambda_k^{\gamma p}<\infty.$$

Then we have

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\Gamma(p,\gamma)h^{\gamma}(2\log(1/h))^{1/2}}=1 \quad \text{a.s.},$$

where

$$\Gamma(p,\gamma)=\begin{cases}\left(\sum_{k=1}^{\infty}\left(\frac{\gamma_k}{\lambda_k}\right)^{p/(2-p)}(2\lambda_k)^{2\gamma p/(2-p)}\right)^{(2-p)/2p} & if\ 1\leqslant p<2,\\ \max_{k\geqslant 1}(\gamma_k/\lambda_k)^{1/2}\cdot(2\lambda_k)^{\gamma} & if\ p\geqslant 2.\end{cases}$$

In particular, taking $\gamma=1/2$, $p=2$, we obtain moduli of continuity for l^2-norm squared process generated by Ornstein-Uhlenbeck processes. Let

$$\chi^2(t)=\|Y(t)\|_{l^2}^2=\sum_{k=1}^{\infty}X_k^2(t).$$

Put

$$\Gamma_0=E\chi^2(t)=\sum_{k=1}^{\infty}(\gamma_k/\lambda_k),\ \Gamma_1=\sum_{k=1}^{\infty}\gamma_k,\ \Gamma_2=\sum_{k=1}^{\infty}\gamma_k^2/\lambda_k,$$

and

$$B=\max_{k\geqslant 1}\gamma_k,\ M=\max_{k\geqslant 1}\gamma_k^2/\lambda_k.$$

Then Corollary 3.3.3 implies that

Corollary 3.3.4 *Assume that $\Gamma_0<\infty$, $\Gamma_1<\infty$. Then*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{2(Bh\log(1/h))^{1/2}}=1\quad a.s.$$

Csörgő and Lin (1990) have studied the other form of moduli of continuity with the range of t tending to infinity as $h\to 0$. In this case, the normalized factor should be quite different.

Theorem 3.3.5 *Assume that $\Gamma_0<\infty$, $\Gamma_2<\infty$ and that $T_h\uparrow\infty$ continuously as $h\to 0$. Then*

$$\limsup_{h\to 0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8Mh)^{1/2}2\log(T_h/h)}\leqslant 1.$$

If, in addition, T_h satisfies

$$(\log T_h)/\log(1/h)\to\infty\quad as\quad h\to 0,$$

then we have

$$\lim_{h\to 0}\sup_{|t|\leqslant T_h}\frac{|\chi^2(t+h)-\chi^2(t)|}{(8Mh)^{1/2}2\log T_h}=1\quad a.s.,$$

$$\lim_{h\to 0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8Mh)^{1/2}2\log T_h}=1\quad a.s.$$

Remark 3.3.5 In this case, we can, for example, take

$$T_h=\exp\{\log(1/h)\log\log\cdots\log(1/h)\},$$

where for small enough $h>0$, $\log\log\cdots\log(1/h)$ stands for taking logarithm any given finite number of times, resulting in the modulus $(8Mh)^{1/2}2\log(1/h)\log\log\cdots\log(1/h)$ for the $\chi^2(\cdot)$ process.

The proof of Theorem 3.3.5 hinges on the following lemma.

Lemma 3.3.1 *Assume that $\Gamma_0<\infty$ and $\Gamma_2<\infty$. For any $\varepsilon>0$ there exist $h(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for any $T_h>h(\varepsilon)$, $h<h(\varepsilon)$ we have*

$$P\{|\chi^2(t+h)-\chi^2(t)|\geqslant x(8Mh)^{1/2}\}\geqslant\frac{1}{7x}\exp\left(-\frac{x}{1+\varepsilon}\right)$$

for any fixed t, and

$$P\{\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}|\chi^2(t+s)-\chi^2(t)|\geqslant x(8Mh)^{1/2}\}$$
$$\leqslant(CT_h/h)\exp\left(-\frac{x}{1+\varepsilon}\right)$$

for any $x\geqslant(8/\varepsilon^2)(\Gamma_2/M)^{1/2}$.

The proofs of Theorem 3.3.5 and Lemma 3.3.1 will not be presented here.

3.3.2 The large increments when $\sigma(p,h)$ is bounded

For some Gaussian processes, the conditions of the previous theorems are not satisfied. For example, $\sigma(p,h)$ is bounded as h

$\to\infty$ for Ornstein-Uhlenbeck processes. Now we establish large increment results for $Y(\cdot)$ with bounded $\sigma(p,h)$. To this end, we first give a finer version of Lemma 3.1.3.

Lemma 3.3.2 *Let B be a separable Banach space with norm $\|\cdot\|$ and let $\{\Gamma(t);t\geqslant 0\}$ be a stochastic process with values in B. Let P be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is P-almost surely continuous with respect to the norm and that for any $t\geqslant 0$, $h\geqslant 0$, $0<x^*\leqslant x$ there exist non-negative monotonically non-decreasing functions $\sigma_1(h)$ and $\sigma_2(h)$ such that*

$$P\{\|\Gamma(t+h)-\Gamma(t)\|\geqslant x\sigma_1(h)+\sigma_2(h)\}\leqslant K\exp(-\gamma x^\beta) \tag{3.3.51}$$

with some K, γ, $\beta>0$. Then for any given $2\leqslant\alpha<\mathrm{e}$, $0<\tau<1$,

$$\begin{aligned}P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t+s)-\Gamma(t)\|\geqslant x(\sigma_1(h+d(m,k)^{-1}h)+\sigma_1(h,m,k))\\ +\sigma_1^*(h,m,k)+\sigma_2(h+d(m,k)^{-1}h)+\sigma_2(h,m,k)\Big\}\end{aligned}$$

$$\leqslant 4K\left(\frac{T}{h}+1\right)d(m,k)^2\exp(-\gamma x^\beta) \tag{3.3.52}$$

for any $0\leqslant h\leqslant T$, $x\geqslant x^*$, integer $m\geqslant 3$ and $k=k(\tau)$ large enough, where

$$d(m,k)=\underbrace{\alpha^{\cdot^{\cdot^{\cdot^{\alpha^k}}}}}_{m\ times},$$

$$\sigma_1(h,m,k)=2^{2+\frac{1}{\beta}}d(m-3,k)^{-1}\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_1(h\alpha^{-\alpha^y})}{y}\mathrm{d}y,$$

$$\sigma_1^*(h,m,k)=2\left(\frac{2}{\gamma}\right)^{1/\beta}\left(1-\left(\frac{d(m-1,k+1-\tau)}{d(m-1,k+1)}\right)^{1/\beta}\right)^{-1}\times\int_{d(m-1,k+1-\tau)}^{\infty}\sigma_1(h\alpha^{-y^\beta})\mathrm{d}y,$$

$$\sigma_2(h,m,k)=4d(m-3,k)^{-1}\int_{d(m-2,k+1-\tau)}^{\infty}\frac{\sigma_2(h\alpha^{-\alpha^y})}{y}\mathrm{d}y.$$

Proof Fix a positive integer $m\geqslant 3$. Following the proof of

Lemma 3.1.3, for any positive real number t put

$$t_j=[td(m,j)/h]/(d(m,j)/h).$$

We have

$$\begin{aligned}\|\Gamma(t+s)-\Gamma(t)\|&\leqslant\|\Gamma((t+s)_k)-\Gamma(t_k)\|\\&+\sum_{j=0}^{\infty}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\|\\&+\sum_{j=0}^{\infty}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|.\end{aligned}$$

Since

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|(t+s)_k-t_k|\leqslant h+d(m,k)^{-1}h,$$

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}|(t+s)_{k+j+1}-(t+s)_{k+j}|\leqslant hd(m,k+j+1)^{-1},$$

we obtain from (3.3.51) for any $x\geqslant x^*$ and $x_j\geqslant x^*$

$$P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma((t+s)_k)-\Gamma(t_k)\|\geqslant x\sigma_1(h+d(m,k)^{-1}h)+\sigma_2(h+d(m,k)^{-1}h)\Big\}$$

$$\leqslant 2Kd(m,k)^2\left(\frac{T}{h}+1\right)\exp(-\gamma x^\beta),$$

$$P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma((t+s)_{k+j+1})-\Gamma((t+s)_{k+j})\|\geqslant x_j\sigma_1(hd(m,k+j+1)^{-1})+\sigma_2(hd(m,k+j+1)^{-1})\Big\}$$

$$\leqslant 2Kd(m,k+j+1)\left(\frac{T}{h}+1\right)\exp(-\gamma x_j^\beta)$$

as well as

$$P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|\Gamma(t_{k+j+1})-\Gamma(t_{k+j})\|\geqslant x_j\sigma_1(hd(m,k+j+1)^{-1})+\sigma_2(hd(m,k+j+1)^{-1})\Big\}$$

$$\leqslant 2Kd(m,k+j+1)\left(\frac{T}{h}+1\right)\exp(-\gamma x_j^\beta).$$

Now put $\gamma x_j^\beta=\gamma x^\beta+d(m-1,k+j+1)$. Then

$$\sum_{j=0}^{\infty}d(m,k+j+1)\exp(-\gamma x_j^\beta)$$

$$= \sum_{j=0}^{\infty} d(m,k+j+1)\exp(-d(m-1,k+j+1))\exp(-\gamma x^{\beta})$$
$$\leqslant \exp(-\gamma x^{\beta})$$
and
$$2\sum_{j=0}^{\infty} x_j \sigma_1(hd(m,k+j+1)^{-1})$$
$$\leqslant 2^{1+1/\beta} x \sum_{j=0}^{\infty} \sigma_1(hd(m,k+j+1)^{-1})$$
$$+ 2\left(\frac{2}{\gamma}\right)^{1/\beta} \sum_{j=0}^{\infty} d(m-1,k+j+1)^{1/\beta} \sigma_1(hd(m,k+j+1)^{-1})$$
$$\leqslant 2^{1+1/\beta} x \sum_{j=0}^{\infty} (d(m-3,k+j+1) - d(m-3,k+j+1-\tau))^{-1}$$
$$\times \int_{d(m-2,k+j+1-\tau)}^{d(m-2,k+j+1)} \frac{\sigma_1(h\alpha^{-\alpha^y})}{y} \mathrm{d}y / \log\alpha$$
$$+ 2\left(\frac{2}{\gamma}\right)^{1/\beta} \sum_{j=0}^{\infty} \left(1 - \left(\frac{d(m-1,k+j+1-\tau)}{d(m-1,k+j+1)}\right)^{1/\beta}\right)^{-1}$$
$$\times \int_{d(m-2,k+j+1-\tau)}^{d(m-2,k+j+1)} \sigma_1(h\alpha^{-\alpha^y}) \mathrm{d}\alpha^{\frac{1}{\beta}y}$$
$$\leqslant 2^{2+\frac{1}{\beta}} d(m-3,k)^{-1} x \int_{d(m-2,k+1-\tau)}^{\infty} \frac{\sigma_1(h\alpha^{-\alpha^y})}{y} \mathrm{d}y$$
$$+ 2\left(\frac{2}{\gamma}\right)^{\frac{1}{\beta}} \left(1 - \left(\frac{d(m-1,k+1-\tau)}{d(m-1,k+1)}\right)^{1/\beta}\right)^{-1}$$
$$\times \int_{d(m-1,k+1-\tau)}^{\infty} \sigma_1(h\alpha^{-y^{\beta}}) \mathrm{d}y$$
as well as
$$2\sum_{j=0}^{\infty} \sigma_2(hd(m,k+j+1)^{-1})$$
$$\leqslant 4d(m-3,k)^{-1} \int_{d(m-2,k+1-\tau)}^{\infty} \frac{\sigma_2(h\alpha^{-\alpha^y})}{y} \mathrm{d}y.$$
Combining all the above inequalities yields (3.3.52).

Remark 3.3.6 From the proof of Lemma 3.3.2, it is easy to

see that (3.3.51) implies
$$P\{\sup_{0\leqslant s\leqslant h} \| \Gamma(T+s) - \Gamma(T) \| \geqslant x(\sigma_1(h+d(m,k)^{-1}h) + \sigma_1(h,m,k))$$
$$+\sigma_1^*(h,m,k) + \sigma_2(h+d(m+k)^{-1}h) + \sigma_2(h,m,k)\}$$
$$\leqslant 4Kd(m,k)\exp(-\gamma x^{\beta})$$
for any $T\geqslant 0$, $h\geqslant 0$, $x\geqslant x^*$, $k>0$ and integer $m\geqslant 3$.

Using this lemma, we can show the following theorem, which is due to Lin (1997a). Put $L_m x = \underbrace{\log_a \cdots \log_a x}_{m \text{ times}}$.

Theorem 3.3.6 *Let $Y(\cdot)$ be defined as at the beginning of this section. Let a_T be a positive continuous quasi-increasing function of T with $a_T\to\infty$ as $T\to\infty$. Assume that*
$$\tilde{\sigma}(p,T)\to\tilde{\sigma}<\infty \text{ and } \sigma(p,T) = o\left(\left(\log\frac{T}{a_T}\right)^{1/2}\right) \text{ as } T\to\infty, \tag{3.3.53}$$
and
$$\int_1^{\infty} \tilde{\sigma}(p,\alpha^{-x^2})\mathrm{d}x < \infty,\ \int_1^{\infty} \sigma(p,\alpha^{-\alpha^x})\mathrm{d}x/x < \infty. \tag{3.3.54}$$
Assume that there are $0<\delta<1/\alpha$ and a positive integer $m\geqslant 1$ such that
$$a_T \leqslant Td(m,(L_m T)^{\delta+1/\alpha})^{-1} \tag{3.3.55}$$
and there is an $a_0>0$ such that for any $a\geqslant a_0$
$$\max_{k\geqslant 1} E(X_k(ia) - X_k((i-1)a))(X_k(ja) - X_k((j-1)a)) \leqslant 0 \tag{3.3.56}$$
for every $j>i\geqslant 1$. Then we have
$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \frac{\| Y(t+s) - Y(t) \|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.}, \tag{3.3.57}$$
$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T} \frac{\| Y(t+a_T) - Y(t) \|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.3.58}$$
and

$$\limsup_{T\to\infty}\frac{\| Y(T+a_T)-Y(T)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}=1\quad \text{a. s.} \tag{3. 3. 59}$$

If conditions (3. 3. 56) *and* (3. 3. 55) *are replaced by*

$$\limsup_{a\to\infty} m(a)\leqslant 0, \tag{3. 3. 56$'$}$$

where $m(a)=\max_{k\geqslant 1}\max_{j>i\geqslant 1}E(X_k(ia)-X_k((i-1)a))(X_k(ja)-X_k((j-1)a))$, *and*

$$a_T\leqslant T(T^{-\delta_T}\wedge d(m,(L_mT)^{\delta+1/\alpha})^{-1})\quad \textit{for some } 0<\delta<\frac{1}{\alpha}, \tag{3. 3. 55$'$}$$

where $\delta_T\to 0$ *as* $T\to\infty$ *with* $(0\vee m(T))/\delta_T\to 0$ *as* $T\to\infty$, *respectively, then* (3. 3. 57)—(3. 3. 59) *remain true.*

Remark 3. 3. 7 It is easy to see that

$$d(m,(L_mT)^{\delta+1/\alpha})=T^{(L_1T)^{-1+(L_2T)\cdots^{-1+(L_mT)^{-1+\delta+1/\alpha}}}}. \tag{3. 3. 60}$$

Put $d(m,(L_mT)^{\delta+1/\alpha})=T^{(L_1T)^{-1+o(T,m)}}$. Then $o(T,m)\to 0$ and $\frac{o(T,m+1)}{o(T,m)}\to 0$ as $T\to\infty$. Hence, as $T\to\infty$,

$$d(m,(L_mT)^{\delta+1/\alpha})=\alpha^{(L_1T)^{o(T,m)}}\leqslant\alpha^{(L_1T)^{\varepsilon}}\quad \text{for any } \varepsilon>0, \tag{3. 3. 61}$$

$$d(m+1,(L_{m+1}T)^{\delta+1/\alpha})/d(m,(L_mT)^{\delta+1/\alpha})\to 0.$$

Proof of Theorem 3. 3. 6 Note that condition (3. 3. 55) implies

$$\frac{\log(T/a_T)}{\log\log T}\to\infty\quad \text{as}\quad T\to\infty. \tag{3. 3. 62}$$

First we prove

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/a_T))^{1/2}}\leqslant 1\quad \text{a. s.} \tag{3. 3. 63}$$

Lemma 3. 1. 2 implies that (3. 3. 51) in Lemma 3. 3. 2 holds with

$K=2$, $\gamma=\frac{1}{2}$, $\beta=2$, $\sigma_1(h)=\tilde{\sigma}(p,h)$ and $\sigma_2(h)=\delta_p\sigma(p,h)$. For a given $\delta>0$ in (3. 3. 55), let $0<\tau<1$ in (3. 3. 52) and $1-1/\alpha-\delta<\delta_1<1-1/\alpha$ such that $\tau_1:=\tau-\log(1-\delta_1)<1$. Let $\varepsilon_T=d(m-1,(L_ma_T)^{1-\delta_1})/L_1a_T$. Take $k=L_{m+2}a_T^{\varepsilon_T}$ in (3. 3. 52) and $k_1=L_{m+2}a_T$. Then $d(m+2,k)=a_T^{\varepsilon_T}$ and

$$\begin{aligned} k-\tau &= L_m\left((L_2a_T)\left(1+\frac{L_1\varepsilon_T}{L_2a_T}\right)\right)-\tau \\ &= L_{m+2}a_T+L_1(1-\delta_1)-\tau \\ &= k_1-\tau_1. \end{aligned}$$

Hence using condition (3. 3. 54) we have

$$\begin{aligned} \sigma_1(a_T,m+2,k) &= 2^{5/2}d(m-1,k)^{-1}\int_{d(m,k+1-\tau)}^{\infty}\frac{\sigma_1(a_T\alpha^{-\alpha^x})}{x}\mathrm{d}x \\ &\leqslant 2^{5/2}d(m-1,k)^{-1}\int_{1}^{\infty}\frac{\sigma_1(\alpha^{\alpha^{d(m,k_1)}-\alpha^{xd(m,k_1+1-\tau_1)}})}{x}\mathrm{d}x \\ &\leqslant\frac{\varepsilon}{2}\tilde{\sigma}_p \end{aligned}$$

for any given $\varepsilon>0$ provided that T (equivalently, k_1) is large enough. Similarly, for large T

$$\begin{aligned} &\sigma_1^*(a_T,m+2,k) \\ &=4\left(1-\left(\frac{d(m+1,k+1-\tau)}{d(m+1,k+1)}\right)^{1/2}\right)^{-1}\int_{d(m+1,k+1-\tau)}^{\infty}\sigma_1(a_T\alpha^{-x^2})\mathrm{d}x \\ &\leqslant\varepsilon, \end{aligned}$$

$$\begin{aligned} \sigma_2(a_T,m+2,k) &= 4d(m-1,k)^{-1}\int_{d(m,k+1-\tau)}^{\infty}\frac{\sigma_2(a_T\alpha^{-\alpha^x})}{x}\mathrm{d}x \\ &\leqslant\varepsilon. \end{aligned}$$

Therefore, from (3. 3. 62), condition (3. 3. 53) and Lemma 3. 3. 2, it follows that for any given $c>0$ and any T large enough

$$P\left\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant ca_T}\frac{\| Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}_p(2\log(T/ca_T))^{1/2}}\geqslant 1+2\varepsilon\right\}$$

$$\leqslant P\Big\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant ca_T}\|Y(t+s)-Y(t)\|_{l^p}$$

$$\geqslant(1+\varepsilon)\Big(2\Big(\log\frac{T}{ca_T}+\log\log T\Big)\Big)^{1/2}$$

$$\times(\sigma_1(ca_T(1+d(m+2,k)^{-1}))+\sigma_1(ca_T,m+2,k))$$

$$+\sigma_1^*(ca_T,m+2,k)+\sigma_2(ca_T(1+d(m+2,k)^{-1}))$$

$$+\sigma_2(ca_T,m+2,k)\Big\}$$

$$\leqslant\frac{9T}{ca_T}d(m+2,k)^2\exp\Big\{-(1+\varepsilon)^2\Big(\log\frac{T}{ca_T}+\log\log T\Big)\Big\}$$

$$\leqslant 9c^{2\varepsilon}T^{-2\varepsilon}a_T^{2(\varepsilon+\varepsilon_T)}(\log T)^{-(1+2\varepsilon)}$$

$$\leqslant 9c^{2\varepsilon}(\log T)^{-(1+2\varepsilon)},$$

since $a_T^{\varepsilon+\varepsilon_T}\leqslant T^{\varepsilon+\varepsilon_T}d(m,(L_mT)^{\delta+1/\alpha})^{-\varepsilon}\leqslant T^\varepsilon$ for large T if (3.3.55) holds. Let $T_j=\theta^j$ for some $\theta>1$. Then by the Borel-Cantelli Lemma we have

$$\limsup_{j\to\infty}\sup_{0\leqslant t\leqslant T_j}\sup_{0\leqslant s\leqslant ca_{T_j}}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(T_j/ca_{T_j}))^{1/2}}\leqslant 1+2\varepsilon\quad\text{a.s.}\tag{3.3.64}$$

Noting that a_T is quasi-increasing we obtain (3.3.63) from (3.3.64). By (3.3.62), the inverse inequality holds from Proposition 3.3.2. So now (3.3.57) is proved.

In order to show (3.3.58), it suffices to prove

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}\geqslant 1\quad\text{a.s.}\tag{3.3.65}$$

Assume that conditions (3.3.55) and (3.3.56) are satisfied. Let $B_{nk}=\{T;kh\leqslant a_T<(k+1)h,n-1\leqslant T<n\}$ for some $h>0$, $a_n'=\inf\{a_T;n-1\leqslant T<n\}$, $a_n^*=\sup\{a_T;n-1\leqslant T<n\}$. Then

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}$$

$$\geqslant\liminf_{n\to\infty}\min_{a_n'/h-1\leqslant k\leqslant a_n^*/h}\inf_{T\in B_{nk}}\sup_{0\leqslant t\leqslant T}\frac{\|Y(t+a_T)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}$$

$$\geqslant\liminf_{n\to\infty}\min_{a_n'/h-1\leqslant k\leqslant a_n^*/h}\sup_{0\leqslant t\leqslant n-1}\frac{\|Y(t+kh)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(n/kh))^{1/2}}$$

$$-\limsup_{n\to\infty}\sup_{0\leqslant t\leqslant n}\sup_{0\leqslant s\leqslant h}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log((n-1)/a_n^*))^{1/2}}$$

$$=:L_1-L_2.$$

Note that $\tilde\sigma(p,h)/\tilde\sigma_p\to 0$ as $h\to 0$ and $a_n^*\leqslant ca_n$ since a_T is quasi-increasing. Then by imitating the proof of (3.3.63), we have

$$L_2\leqslant\varepsilon\quad\text{a.s.},\tag{3.3.66}$$

provided that h is small enough.

Consider L_1. Assume $1\leqslant p<2$. We have (cf. (3.1.17))

$$\|Y((j+1)kh)-Y(jkh)\|_{l^p}$$

$$\geqslant\frac{\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\left(\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{\frac{p-1}{p}}}.$$

Let

$$\xi(j,k)=\frac{\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\tilde\sigma(p,kh)\left(\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{\frac{p-1}{p}}}$$

$$=\frac{\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2(p-1)}{2-p}}(X_v((j+1)kh)-X_v(jkh))}{\left(\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2p}{2-p}}\right)^{1/2}}.$$

Then using condition (3.3.56), we have for $j>i\geqslant 1$

$$E\xi(i,k)\xi(j,k)$$

$$=\Big(\sum_{v=1}^\infty\sigma_v(kh)^{\frac{2p}{2-p}}\Big)^{-1}\Big(\sum_{v=1}^\infty\sigma_v(kh)^{\frac{4(p-1)}{2-p}}\Big)E(X_v((i+1)kh)$$

$$-X_v(ikh))(X_v((j+1)kh)-X_v(jkh))\leqslant 0,\tag{3.3.67}$$

provided that k is large enough. Therefore by Slepian's inequality, recalling the definition of B_{nk} and noting condition (3.3.55)

and that a_T is quasi-increasing, we obtain the result that there exists $C>0$ such that for large n

$$P\left\{\min_{a'_n/h-1\leqslant k\leqslant a_n^*/h}\max_{0\leqslant j\leqslant n/(2kh)}\xi(j,k)\leqslant(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}$$

$$\leqslant\sum_{k=[a'_n/h]-1}^{[a_n^*/h]}P\left\{\max_{0\leqslant j\leqslant n/(2kh)}\xi(j,k)\leqslant(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}$$

$$\leqslant\sum_{k=[a'_n/h]-1}^{[a_n^*/h]}\left(1-\exp\left\{-(1-\varepsilon)\log\frac{n}{kh}\right\}\right)^{n/(2kh)}$$

$$\leqslant Ca_n\exp\{-C(n/a_n)^{\varepsilon}\}$$

$$\leqslant Cn^{-2},\tag{3.3.68}$$

which implies

$$L_1\geqslant1-\varepsilon\quad\text{a.s.}\tag{3.3.69}$$

Assume $p\geqslant2$. Take N_k such that $\sigma_{N_k}(kh)=\sigma^*(kh)$. Clearly

$$\frac{\|Y((j+1)kh)-Y(jkh)\|_{l^p}}{\tilde\sigma_p}$$

$$\geqslant(1-\varepsilon)\frac{X_{N_k}((j+1)kh)-X_{N_k}(jkh)}{\sigma_{N_k}(kh)}$$

for large k. Following the way of the proof for the case of $1\leqslant p<2$, we have (3.3.69) as well. Now (3.3.65) is proved and hence we complete the proof of (3.3.57) and (3.3.58) under conditions (3.3.55) and (3.3.56).

Consider (3.3.59) now. It is enough to show

$$\limsup_{T\to\infty}\frac{\|Y(T+a_T)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}\geqslant1\quad\text{a.s.}\tag{3.3.70}$$

Put $a'_T=a_0[a_T/a_0]$, where a_0 is specified in condition (3.3.56). Then

$$\limsup_{T\to\infty}\frac{\|Y(T+a_T)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}$$

$$\geqslant\limsup_{T\to\infty}\frac{\|Y(T+a'_T)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a'_T))^{1/2}}$$

$$-\limsup_{T\to\infty}\sup_{0\leqslant s\leqslant a_0}\frac{\|Y(T+s)-Y(T)\|_{l^p}}{\tilde\sigma_p(2\log(T/a_T))^{1/2}}$$

$$=:I_1-I_2.\tag{3.3.71}$$

Noting Remark 3.3.5, and following the way of the proof of (3.3.63), we have

$$I_2=0\quad\text{a.s.}\tag{3.3.72}$$

Let $t_0=1$. Define t_k by $t_k=t_{k-1}+a'_{t_{k-1}}$, $k=1,2,\cdots$. Then

$$\frac{\|Y(t_k+a'_{t_k})-Y(t_k)\|_{l^p}}{\tilde\sigma(p,a'_{t_k})}$$

$$\geqslant\frac{\sum_{v=1}^{\infty}\sigma_v(a'_{t_k})^{\frac{2(p-1)}{2-p}}(X_v(t_k+a'_{t_k})-X_v(t_k))}{(\sum_{v=1}^{\infty}\sigma_v(a'_{t_k})^{\frac{2p}{2-p}})^{1/2}}=:\zeta_k.$$

Then using condition (3.3.56) again, we obtain for $j>i$

$$E\zeta_i\zeta_j\leqslant0$$

provided that i is large enough. Put $D_n=\{k;\frac{1}{2}n\leqslant t_k\leqslant n-1\}$. Obviously, by condition (3.3.55), for $k\in D_n$, $a_{t_k}=o(n)$ as $n\to\infty$. Hence

$$\sum_{k\in D_n}a_{t_k}\geqslant\sum_{k\in D_n}(t_k-t_{k-1})-\max_{k\in D_n}a_{t_k}$$

$$\geqslant\frac{1}{3}n$$

for large n. Moreover, condition (3.3.55) implies

$$a_T\leqslant T(\log T)^{-A}\quad\text{for any }A>0\text{ and all large }T.\tag{3.3.73}$$

By Slepian's inequality, for large n, $n-1<T\leqslant n$,

$$P\left\{\sup_{T/2\leqslant t\leqslant T}\frac{\|Y(t+a'_t)-Y(t)\|_{l^p}}{\tilde\sigma_p(2\log(t/a_t))^{1/2}}\leqslant1-\varepsilon\right\}$$

$$\leqslant P\left\{\max_{k\in D_n}\zeta_k/(2\log(t_k/a_{t_k}))^{1/2}\leqslant1-\frac{\varepsilon}{2}\right\}$$

$$\leqslant \prod_{k\in D_n} P\left\{\zeta_k \leqslant (1-\frac{\varepsilon}{2})(2\log(t_k/a_{t_k}))^{1/2}\right\}$$
$$\leqslant \prod_{k\in D_n}\left\{1-\exp\left(-(1-\frac{\varepsilon}{2})\log(t_k/a_{t_k})\right)\right\}$$
$$\leqslant \exp\{-\sum_{k\in D_n}(a_{t_k}/t_k)^{1-\varepsilon/2}\}$$
$$\leqslant \exp\{-\frac{1}{3}\log n\}\to 0$$
as $n\to\infty$, which implies
$$I_1 \geqslant 1-\varepsilon \quad \text{a.s.} \tag{3.3.74}$$
Substituting (3.3.72) and (3.3.74) into (3.3.71) yields (3.3.70), and hence (3.3.59) is proved.

When conditions (3.3.55) and (3.3.56) are replaced by (3.3.55)′ and (3.3.56)′, respectively, the proof of (3.3.69) is similar. We consider only the case of $1\leqslant p<2$. Noting the fact we have shown, without loss of generality we assume that $m(a)\geqslant 0$ for all large a. Put
$$m_1(a) = m(a)\Big(\sum_{v=1}^{\infty}\sigma_v(a)^{\frac{2p}{2-p}}\Big)^{-1}\sum_{v=1}^{\infty}\sigma_v(a)^{\frac{4(p-1)}{2-p}}.$$
Let $\eta_j=\eta_j^{(k)}$, $j=0,1,\cdots,[n/2kh]$, and $\tau=\tau^{(k)}$ be independent normal random variables with means being zero and $E\eta_j^2=1-m_1(kh)$ and $E\tau^2=m_1(kh)$. Define $\xi_i=\eta_i+\tau$. Then $E\xi_i^2=1$ and
$$E\xi(i,k)\xi(j,k)=E\xi_i\xi_j=m_1(kh),\quad j-i\geqslant 1.$$
Therefore we have for large n
$$P\left\{\min_{a_n'/h-1\leqslant k\leqslant a_n^*/h}\ \max_{0\leqslant j\leqslant n/(2kh)}\xi(j,k)\leqslant(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}$$
$$\leqslant \sum_{k=[a_n'/h]-1}^{[a_n^*/h]} P\left\{\max_{0\leqslant j\leqslant n/(2kh)}\xi_j\leqslant(1-\varepsilon)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}$$

$$\leqslant \sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\left(P\left\{\max_{0\leqslant j\leqslant n/(2kh)}\eta_j\leqslant\left(1-\frac{\varepsilon}{2}\right)\left(2\log\frac{n}{kh}\right)^{1/2}\right\}\right.$$
$$\left.+P\left\{\tau\geqslant\frac{\varepsilon}{2}\left(2\log\frac{n}{kh}\right)^{1/2}\right\}\right)$$
$$\leqslant \sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\left(\left(1-\exp\left\{-\left(1-\frac{\varepsilon}{2}\right)\log\frac{n}{kh}\right\}\right)^{\frac{n}{2kh}}\right.$$
$$\left.+\exp\left\{-\frac{\varepsilon^2}{4m_1(kh)}\log\frac{n}{kh}\right\}\right)$$
$$\leqslant \sum_{k=[a_n'/h]-1}^{[a_n^*/h]}\left(\exp\left\{-\left(\frac{n}{2kh}\right)^{\varepsilon/2}\right\}+\left(\frac{n}{kh}\right)^{-\varepsilon^2/4m_1(kh)}\right)$$
$$\leqslant h^{-1}n\exp(-(\log n)^2)+\sum_{k=[a_n'/h]-1}^{[a_n^*/h]} n^{-\varepsilon^2\delta_{kh}/4m_1(kh)},$$
which implies (3.3.69) by noting $m(T)/\delta_T\to 0$ as $T\to\infty$. The proof of (3.3.58) is completed. Similarly, we can prove (3.3.59) under conditions (3.3.55)′ and (3.3.56)′. The details are omitted.

Using the conclusion of Theorem 3.3.6 we can give a consequence.

Corollary 3.3.5 *Let $\{Y(t);t\geqslant 0\}=\{X_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent fractional O-U processes of order γ with coefficients γ_k and λ_k, where $0<\gamma<1$, $\gamma_k\geqslant 0$, $\lambda_k>0$. Assume that conditions (3.3.53) and (3.3.54) are satisfied. Let a_T be defined as in Theorem 3.3.5, and satisfy condition (3.3.55). Then (3.3.57)—(3.3.59) hold true.*

3.4 The Increments for l^∞-valued Gaussian Processes

Let $\{Y(t);t\geqslant 0\}=\{X_k(t);t\geqslant 0\}_{k=1}^\infty$ be a sequence of Gaussian processes with $EX_k(t)=0$ and $\sigma_k^2(h)=E(X_k(t+h)-X_k(t))^2$, which is non-decreasing and continuous. We will keep the notations introduced above.

3.4.1 Moduli of continuity

We assume, without loss of generality, that for every $k\geqslant 1$, $\sigma_k(h)>0$ for $h>0$. Let y_h be the solution of the equation

$$\sum_{k=1}^\infty (hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)}=h. \tag{3.4.1}$$

The following result is due to Csörgő, Lin and Shao (1994a).

Theorem 3.4.1 *Suppose that $\sigma^{*2}(h)/h^\alpha$ is quasi-increasing for some $\alpha>0$ and that there exist postitive numbers A and h_0 such that*

$$\sum_{k=1}^\infty \sigma_k^A(h_0)<\infty. \tag{3.4.2}$$

Then

$$\lim_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\leqslant 1\quad \text{a.s.} \tag{3.4.3}$$

If condition (3.4.2) is replaced by conditions for $0<h\leqslant h_0$ so that

$$\inf_{0<s\leqslant h}\frac{\sigma^*(s)}{\sigma_k(s)}\geqslant c_1\frac{\sigma^*(h)}{\sigma_k(h)}\quad \textit{for some } c_1>0 \textit{ and every } k\geqslant 1 \tag{3.4.4}$$

and

$$\sum_{k=1}^\infty \exp\left\{-\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\log\frac{1}{h}\right\}<\infty, \tag{3.4.5}$$

then (3.4.3) remains true with $y_h=1$. If, in addtion, $X_k(\cdot)$, $k=1,2,\cdots$ are independent and for $0\leqslant t_1<t_2\leqslant t_3<t_4$,

$$E(X_k(t_2)-X_k(t_1))(X_k(t_4)-X_k(t_3))\leqslant 0, \tag{3.4.6}$$

then

$$\lim_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}=1\quad \text{a.s.} \tag{3.4.7}$$

and

$$\lim_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\max_{k\geqslant 1}\frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}=1\quad \text{a.s.} \tag{3.4.8}$$

Proof At first, we list the following facts. We have $0<y_h\leqslant 1$, since

$$h=\sum_{k=1}^\infty (hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)}\geqslant hy_h.$$

Moreover, by elementary calculations, it is easy to see that condition (3.4.2) implies condition (3.4.5), and the latter condition guarantees that the solution of equation (3.4.1) exists and is unique. We have also the following property: There exists a constant $d>0$, such that

$$\sigma^{*2}(h)\geqslant dh^2. \tag{3.4.9}$$

In fact, noting the definition of $\sigma_k^2(h)$, we have $\sigma_k^2(2h)\leqslant 4\sigma_k^2(h)$. So, inductively,

$$\sigma_k^2(h)\geqslant\frac{1}{4}\sigma_k^2(2h)\geqslant\cdots\geqslant\frac{1}{4^l}\sigma_k^2(2^lh)\geqslant h^2\sigma_k^2\left(\frac{1}{2}\right)\quad\text{for }\frac{1}{2}\leqslant 2^lh\leqslant 1,\tag{3.4.10}$$

which implies (3.4.9). Furthermore, combining this with condition (3.4.2), we arrive at

$$\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A\leqslant\frac{c_2}{h^A}\quad\text{for }0<h\leqslant h_0,\tag{3.4.11}$$

where $c_2=d^{-A/2}\sum_{k=1}^{\infty}\sigma_k^A(h_0)$.

As the first step, we prove that for a given $\varepsilon>0$ there exists a constant $C=C(\varepsilon)>0$ such that

$$P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1+\varepsilon\right\}\leqslant Ch^{\varepsilon}y_h^{\varepsilon}.\tag{3.4.12}$$

It is easy to see that for $0<s\leqslant h$

$$\begin{aligned}&P\left\{\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1+\varepsilon\right\}\\&\leqslant\sum_{k=1}^{\infty}\exp\left\{-(1+\varepsilon)^2\left(\log\frac{1}{hy_h}\right)\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\right\}\\&=\sum_{k=1}^{\infty}(hy_h)^{(1+\varepsilon)^2\sigma^{*2}(h)/\sigma_k^2(h)}\leqslant h^{1+2\varepsilon}y_h^{2\varepsilon}\end{aligned}\tag{3.4.13}$$

by (3.4.1). For any positive number t and positive integer $r=r(\varepsilon)$, put $r_1=h/2^r$ and $t_r=[t/r_1]r_1$. We have

$$\begin{aligned}&|X_k(t+s)-X_k(t)|\\&\leqslant|X_k((t+s)_r)-X_k(t_r)|+\sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1})\\&\quad-X_k((t+s)_{r+j})|+\sum_{j=0}^{\infty}|X_k(t_{r+j+1})-X_k(t_{r+j})|.\end{aligned}$$

Then, from (3.4.13)

$$\begin{aligned}&P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h-r_1}\max_{k\geqslant 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1+\frac{\varepsilon}{2}\right\}\\&\leqslant\frac{4}{h}2^{2r}h^{1+\varepsilon}y_h^{\varepsilon}\leqslant Ch^{\varepsilon}y_h^{\varepsilon}.\end{aligned}\tag{3.4.14}$$

Since $\sigma^{*2}(h)/h^{\alpha}$ is quasi-increasing, there exists a $c_0>0$ such that

$$\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}\geqslant c_0 2^{\alpha(r-1)}\frac{\sigma^{*2}(2h/2^r)}{\sigma_k^2(2h/2^r)}\geqslant c_0 2^{\alpha(r-1)}.$$

If condition (3.4.2) is satisfied, then from (3.4.11), for r large enough and h small enough, similar to (3.4.13), we get

$$\begin{aligned}p_1:&=P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{h-r_1\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k((t+s)_r)-X_k((t+h-r_1)_r)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant\frac{\varepsilon}{4}\right\}\\&\leqslant\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{\varepsilon^2\sigma^{*2}(h)/(16\sigma_k^2(2h/2^r))}\\&\leqslant 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\exp\left\{-\left(\frac{\varepsilon^2}{16}\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}-2-A\right)\log\frac{1}{hy_h}\right\}\\&\leqslant 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\exp\left\{-\left(\frac{\sigma^{*2}(h)}{\sigma_k^2(2h/2^r)}-1\right)\frac{A}{2}\right\}\\&\leqslant 2^{r+1}h^{1+A}y_h\sum_{k=1}^{\infty}\left(\frac{\sigma_k(2h/2^r)}{\sigma^*(h)}\right)^A\\&\leqslant c_2 2^{r+1}h^{1+A}y_h\left(\frac{\sigma^*(2h/2^r)}{\sigma^*(h)}\right)^A\left(\frac{2h}{2^r}\right)^{-A}\\&\leqslant Chy_h.\end{aligned}\tag{3.4.15}$$

If conditions (3.4.4) and (3.4.5) are satisfied, then noting that $\sigma^{*2}(h)/h^{\alpha}$ is quasi-increasing and taking r to be large enough and h to be small enough, we obtain

$$\begin{aligned}p_1&\leqslant\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{(\sigma^{*2}(h)/\sigma_k^2(h))(\varepsilon^2/16)\sigma_k^2(h)/\sigma_k^2(2h/2^r)}\\&\leqslant\frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{(\sigma^{*2}(h)/\sigma_k^2(h))(\varepsilon^2/16)c_1\sigma^{*2}(h)/\sigma^{*2}(2h/2^r)}\end{aligned}$$

$$\leqslant \frac{2}{h}2^r\sum_{k=1}^{\infty}(hy_h)^{2\sigma^{*2}(h)/\sigma_k^2(h)}$$
$$\leqslant Chy_h. \qquad (3.4.16)$$

Furthermore, let $x_j^2=2B\log\frac{1}{hy_h}+2(1+A)j$, where $B=2/c_1$. If condition (3.4.2) is satisfied, then similar to (3.4.15),

$$p_2:=P\Big\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1})$$
$$-X_k((t+s)_{r+j})|\geqslant\sum_{j=0}^{\infty}x_j\sigma^*(h/2^{r+j+1})\Big\}$$
$$\leqslant\frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{r+j+1}\exp\Big\{-\frac{x_j^2}{2}\frac{\sigma^{*2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})}\Big\}$$
$$\leqslant 2\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{r+j+1}e^{-(1+A)j}h^{1+A}y_h$$
$$\times\exp\Big\{-\Big(\frac{B\sigma^{*2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})}-2-A\Big)\log\frac{1}{hy_h}\Big\}$$
$$\leqslant 4\times 2^rh^{1+A}y_h\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^je^{-(1+A)j}\Big(\frac{\sigma_k(h/2^{r+j+1})}{\sigma^*(h/2^{r+j+1})}\Big)^A$$
$$\leqslant 4c_22^{r(1+A)+A}hy_h\sum_{j=0}^{\infty}2^{(1+A)j}e^{-(1+A)j}$$
$$\leqslant Chy_h. \qquad (3.4.17)$$

Also, if conditions (3.4.4) and (3.4.5) are satisfied, then, similar to (3.4.16) and (3.4.17),

$$p_2\leqslant\frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}2^{2(r+j+1)}e^{-(1+A)j}\exp\Big\{-B\Big(\log\frac{1}{hy_h}\Big)c_1\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}\Big\}$$
$$\leqslant Chy_h. \qquad (3.4.18)$$

Similarly, in both cases, we have

$$P\Big\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\sum_{k=1}^{\infty}|X_k(t_{r+j+1})-X_k(t_{r+j})|$$

$$\geqslant\sum_{j=0}^{\infty}x_j\sigma^*(h/2^{r+j+1})\Big\}$$
$$\leqslant Chy_h. \qquad (3.4.19)$$

Moreover, since $\sigma^{*2}(h/2^{r+j+1})/\sigma^{*2}(h)\leqslant c_0^{-1}2^{-\alpha(r+j+1)}$, we have

$$\sum_{j=0}^{\infty}x_j\sigma^*\Big(\frac{h}{2^{r+j+1}}\Big)$$
$$=\sigma^*(h)\Big\{(2Bc_0^{-1})^{1/2}\Big(\log\frac{1}{hy_h}\Big)^{1/2}\sum_{j=0}^{\infty}2^{-\alpha(r+j+1)/2}$$
$$+c_0^{-1/2}\sum_{j=0}^{\infty}\frac{(2(1+A)j)^{1/2}}{2^{\alpha(r+j+1)/2}}\Big\}$$
$$\leqslant\frac{\varepsilon}{8}\sigma^*(h)\Big(\log\frac{1}{hy_h}\Big)^{1/2}, \qquad (3.4.20)$$

provided r is large enough. Combining these estimations, we get (3.4.12).

In order to prove (3.4.3), we use (3.4.9) again and obtain from (3.4.12)

$$P\Big\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1+\varepsilon\Big\}$$
$$\leqslant C(hy_h)^{\varepsilon/2}(\log\sigma^{*-1}(h))^{-2}, \qquad (3.4.21)$$

provided h is small enough.

Let $\theta>1$. Define $A_i=\{h;\theta^{-i-1}\leqslant\sigma^*(h)<\theta^{-i}\}$, $A_{ij}=\{h;\theta^{-j-1}\leqslant hy_h<\theta^{-j},h\in A_i\}$, and $h_{ij}=\sup\{h;h\in A_{ij}\}$. Then

$$\limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}$$
$$\leqslant\limsup_{i\to\infty}\sup_{j\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h_{ij}}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{-i-1}(2\log\theta^j)^{1/2}}$$
$$\leqslant\limsup_{i\to\infty}\sup_{j\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h_{ij}}\max_{k\geqslant 1}$$
$$\frac{\theta^2|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}}.$$

Using (3.4.21), we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h_{ij}}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}}\geqslant 1+\varepsilon\right\}$$

$$\leqslant C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(h_{ij}y_{h_{ij}})^{\varepsilon/2}(\log\sigma^{*-1}(h_{ij}))^{-2}$$

$$\leqslant C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{-j\varepsilon/2}(i\log\theta)^{-2}<\infty.$$

Hence (3.4.3) is proved by the Borel-Cantelli lemma if only we can show that $y_h=1$ under conditions (3.4.4) and (3.4.5). It suffices to prove that $\log\dfrac{1}{y_h}=o(\log\dfrac{1}{h})$ in this case. Consider the equation

$$\sum_{k=1}^{\infty}x^{c_1(\log 2)\sigma^{*2}\left(\frac{1}{2}\right)/\sigma_k^2\left(\frac{1}{2}\right)}=1.$$

From condition (3.4.5), its solution $x=x_0>0$ exists. Then for any $0<h\leqslant 1/2$,

$$1=\frac{1}{h}\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)}\leqslant\sum_{k=1}^{\infty}y^{c_1\sigma^{*2}\left(\frac{1}{2}\right)/\sigma_k^2\left(\frac{1}{2}\right)}.$$

Consequently, $y_h\geqslant x_0^{\log 2}$ for $0<h\leqslant 1/2$, as required.

Next, we prove (3.4.8) under the assumption of independence and condition (3.4.6). Having (3.4.3), it is enough to show

$$\limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\max_{k\geqslant 1}\frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1\quad\text{a.s.}\tag{3.4.22}$$

To this end, it suffices to prove that for any $h_n\downarrow 0$, $0<\varepsilon<1$,

$$\lim_{n\to\infty}P\left\{\sup_{0\leqslant t\leqslant 1}\max_{k\geqslant 1}\frac{|X_k(t+h_n)-X_k(t)|}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}\geqslant 1-\varepsilon\right\}=1.\tag{3.4.23}$$

In fact, for n large enough, i.e. h_n small enough, we have

$$P\left\{\sup_{0\leqslant t\leqslant 1}\max_{k\geqslant 1}\frac{X_k(t+h_n)-X_k(t)}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}<1-\varepsilon\right\}$$

$$\leqslant P\left\{\max_{0\leqslant j\leqslant 1/h_n}\max_{k\geqslant 1}\frac{X_k((j+1)h_n)-X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}<1-\varepsilon\right\}$$

$$\leqslant\prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}P\left\{\frac{X_k((j+1)h_n)-X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}<1-\varepsilon\right\}$$

$$\leqslant\prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}\left\{1-\exp\left(-\frac{(1-\varepsilon)\sigma^{*2}(h_n)}{\sigma_k^2(h_n)}\log\frac{1}{h_ny_{h_n}}\right)\right\}$$

$$=\prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}\left(1-(h_ny_{h_n})^{(1-\varepsilon)\sigma^{*2}(h_n)/\sigma_k^2(h_n)}\right)$$

$$\leqslant\prod_{j=0}^{[1/h_n]}\exp\left\{-\sum_{k=1}^{\infty}(h_ny_{h_n})^{(1-\varepsilon)\sigma^{*2}(h_n)/\sigma_k^2(h_n)}\right\}$$

$$\leqslant\exp(-h_n^{-\varepsilon})\to 0$$

as $n\to\infty$, where in the second inequality we used independence and Slepian's inequality. Hence (3.4.8) is proved.

Finally we prove (3.4.7). With the help of (3.4.3), it suffices to show

$$\liminf_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\geqslant 1\quad\text{a.s.}\tag{3.4.24}$$

Define A_{ij} as above, and put $h'_{ij}=\inf\{h;h\in A_{ij}\}$. Then

$$\liminf_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}$$

$$\geqslant\liminf_{i\to\infty}\inf_{j\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h'_{ij}}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{-i}(2\log\theta^{j+1})^{1/2}}$$

$$\geqslant\liminf_{i\to\infty}\inf_{j\geqslant 0}\max_{0\leqslant l\leqslant 1/h'_{ij}}\max_{k\geqslant 1}\frac{|X_k((l+1)h'_{ij})-X_k(lh'_{ij})|}{\theta^2\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}.\tag{3.4.25}$$

Using Slepian's inequality again, we obtain

$$P\left\{\max_{0\leqslant l\leqslant 1/h'_{ij}}\max_{k\geqslant 1}\frac{|X_k((l+1)h'_{ij})-X_k(lh'_{ij})|}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}\leqslant 1-\varepsilon\right\}$$
$$\leqslant\prod_{l=0}^{[1/h'_{ij}]}\prod_{k=1}^{\infty}\left\{1-\exp\left(-\frac{(1-\varepsilon)\sigma^{*2}(h'_{ij})}{\sigma_k^2(h'_{ij})}\log\frac{1}{h'_{ij}y_{h'_{ij}}}\right)\right\}$$
$$\leqslant\exp\{-(h'_{ij}y_{h'_{ij}})^{-\varepsilon}\}$$
$$\leqslant\exp\{-(h'_{ij}y_{h'_{ij}})^{-\varepsilon/2}\log\sigma^{*-1}(h'_{ij})\}$$
$$\leqslant\exp\{-\theta^{i\varepsilon/2}(i\log\theta)\}.$$

The last but one inequality is due to (3.4.9). So we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P\left\{\max_{0\leqslant l\leqslant 1/h'_{ij}}\max_{k\geqslant 1}\frac{|X_k((l+1)h'_{ij})-X_k(lh'_{ij})|}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}\leqslant 1-\varepsilon\right\}$$
$$<\infty. \tag{3.4.26}$$

Now (3.4.25) and (3.4.26) imply (3.4.24). This completes the proof of Theorem 3.4.1.

Corollary 3.4.1 *Suppose that there exist constants $0<c_1\leqslant c_2<\infty$, a sequence of positive numbers $\{a_k;k\geqslant 1\}$, and a non-decreasing function $\sigma(h)$ such that*

$$c_1a_k\sigma(h)\leqslant\sigma_k(h)\leqslant c_2a_k\sigma(h) \tag{3.4.27}$$

for any $h>0$ and every $k\geqslant 1$. Moreover, suppose that $\sigma^2(h)/h^\alpha$ is quasi-increasing for some $\alpha>0$ and

$$\sum_{k=1}^{\infty}\exp\{-Aa^{*2}/a_k^2\}<\infty \tag{3.4.28}$$

for some $A>0$, where $a^*=\max_{k\geqslant 1}a_k$. Then (3.4.3) holds with $y_h=1$. If, in addition, $\{X_k(\cdot)\}_{k=1}^{\infty}$ are independent and (3.4.6) is satisfied, then (3.4.7) and (3.4.8) hold with $y_h=1$.

Proof It is clear that (3.4.27) implies (3.4.4). Now the conclusion follws from Theorem 3.4.1 immediately.

Remark 3.4.1 We give upper and lower estimates for y_h when

(3.4.2) is satisfied.

Note that for any $0<h\leqslant e^{-A/2}$

$$1\leqslant y_h\sum_{k=1}^{\infty}h^{\sigma^{*2}(h)/\sigma_k^2(h)-1}$$
$$=y_h\sum_{k=1}^{\infty}\exp\left\{-\left(\log\frac{1}{h}\right)\left(\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}-1\right)\right\}$$
$$\leqslant y_h\sum_{k=1}^{\infty}\exp\left\{-\frac{A}{2}\left(\frac{\sigma^{*2}(h)}{\sigma_k^2(h)}-1\right)\right\}\leqslant y_h\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A,$$

that is, we have

$$y_h\geqslant\left(\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A\right)^{-1}. \tag{3.4.29}$$

Moreover, note that for any $h>0$ and $\theta>1$

$$h=\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*2}(h)/\sigma_k^2(h)}\geqslant\sum_{k\in A_\theta(h)}(hy_n)^\theta=\mathrm{Card}\{A_\theta(h)\}(hy_h)^\theta,$$

where $A_\theta(h)=\{k;\sigma_k^2(h)/\sigma^{*2}(h)\geqslant 1/\theta\}$. Hence we have

$$y_h\leqslant(\mathrm{Card}\{A_\theta(h)\}/h)^{1/\theta}/h. \tag{3.4.30}$$

Theorem 3.4.1 in combination with (3.4.29) yields the next result.

Corollary 3.4.2 *Suppose that $\sigma^{*2}(h)/h^\alpha$ is quasi-increasing for some $\alpha>0$ and (3.4.2) is satisfied. And suppose that*

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A=o\left(\frac{1}{h}\right)\quad as\ h\to 0. \tag{3.4.31}$$

Then (3.4.3) holds with $y_h=1$. If, in addition, $X_k(\cdot)$, $k=1,2,\cdots$ are independent and (3.4.6) is satisfied, then (3.4.7) and (3.4.8) hold with $y_h=1$.

As an application of this corollary, we have

Corollary 3.4.3 *Let $\{X_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, $k=$*

$1,2,\cdots$. *Suppose that* $\sigma^{*2}(h)/h^{\alpha}$ *is quasi-increasing for some* $\alpha>0$ *and that*

$$\sum_{k=1}^{\infty}\gamma_k^A<\infty \tag{3.4.32}$$

for some $A\geq 2$. *Then* (3.4.7) *and* (3.4.8) *hold with* $y_h=1$.

Proof (3.4.6) is satisfied for $\{X_k(\cdot)\}$. It follows from (3.4.32) that

$$\sum_{k=1}^{\infty}\sigma_k^{2A}(h)\leq(2h)^A\sum_{k=1}^{\infty}\gamma_k^A.$$

On the other hand, it is easy to see that

$$\liminf_{h\downarrow 0}\sigma^{*2}(h)/h>0,$$

by recalling the proof of (3.4.9) and noting that $E(X_k(t+2h)-X_k(t+h))(X_k(t+h)-X_k(t))\leq 0$. So we have

$$\limsup_{h\downarrow 0}\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^{2A}<\infty,$$

that is, (3.4.31) is satisfied. We obtain the conclusion of Corollary 3.4.3 from Corollary 3.4.2.

3.4.2 The large increments

At first, we still consider the case that $\sigma^{*2}(h)/h^{\alpha}$ is quasi-increasing for some $\alpha>0$.

Let a_T, $0<a_T\leq T$, be a continuous function satisfying $a_T\to\infty$ as $T\to\infty$. Let y_T be the solution of the equation

$$\sum_{k=1}^{\infty}\left(\frac{a_Ty_T}{T\log\sigma^*(a_T)}\right)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)}=\frac{a_T}{T\log\sigma^*(a_T)}. \tag{3.4.33}$$

The following theorem due to Lin (1998) is an analogue of Theorem 3.4.1 in the case of large increments.

Theorem 3.4.2 *Suppose that* $\sigma^{*2}(h)/h^{\alpha}$ *is quasi-increasing for some* $\alpha>0$ *and that there exist positive numbers* h_0, A *and* B *such that for any* $h\geq h_0$

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A<B. \tag{3.4.34}$$

Then

$$\limsup_{T\to\infty}\sup_{0\leq t\leq T-a_T}\sup_{0\leq s\leq a_T}\max_{k\geq 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}}\leq 1\quad\text{a.s} \tag{3.4.35}$$

If condition (3.4.34) *is replaced by conditions that there exist positive numbers* h_1, c_1, T_0 *and* C *so that*

$$\inf_{0\leq s\leq h}\frac{\sigma^*(s)}{\sigma_k(s)}\geq c_1\frac{\sigma^*(h)}{\sigma_k(h)} \tag{3.4.36}$$

for any $h\geq h_1$ *and every* $k\geq 1$ *and*

$$\sum_{k=1}^{\infty}\left(\frac{a_T}{T\log\sigma^*(a_T)}\right)^{\sigma^{*2}(a_T)/\sigma_k^2(a_T)}<C \tag{3.4.37}$$

for any $T\geq T_0$, *then* (3.4.35) *remains true. If, in addition,* $X_k(\cdot)$, $k=1,2,\cdots$ *are independent and for* $0\leq t_1<t_2\leq t_3<t_4$,

$$E(X_k(t_2)-X_k(t_1))(X_k(t_4)-X_k(t_3))\leq 0 \tag{3.4.38}$$

and

$$\lim_{T\to\infty}\frac{\log(T/a_T)}{\log\log\sigma^*(a_T)}=\infty, \tag{3.4.39}$$

then

$$\lim_{T\to\infty}\sup_{0\leq t\leq T-a_T}\sup_{0\leq s\leq a_T}\max_{k\geq 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}=1\quad\text{a.s.} \tag{3.4.40}$$

and

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\max_{k\geqslant 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}=1\quad \text{a. s.}$$

(3. 4. 41)

Remark 3. 4. 2 As an analogue of Remark 3. 4. 1, we have

$$y_T\geqslant\left(\sum_{k=1}^{\infty}\left(\frac{\sigma_k(a_T)}{\sigma^*(a_T)}\right)^A\right)^{-1}$$

for T satisfying $\sigma^*(a_T)\geqslant e^{A/2}$. Hence

$$y_T\geqslant B^{-1}>0 \tag{3. 4. 42}$$

by (3. 4. 34) for large T. Moreover, we can also show the following facts: $y_T\leqslant 1$ and the solution of equation (3. 4. 33) exists and is unique under condition (3. 4. 37), which is implied by (3. 4. 34).

Proof of Theorem 3. 4. 2 There is $d>0$ such that $\sigma^{*^2}(h)\geqslant dh^2$ for any $0<h\leqslant 1$ (cf. (3. 4. 9)). If $h_0>1$, then for any $1<h\leqslant h_0$, $\sigma^{*^2}(h)\geqslant\sigma^{*^2}(1)\geqslant h^2\sigma^{*^2}(1)/h_0^2$. Hence, putting $d'=d\wedge(\sigma^{*^2}(1)/h_0^2)$, we have $\sigma^{*^2}(h)\geqslant d'h^2$ for any $h\leqslant h_0$, which implies that for $h\leqslant h_0$

$$\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A\leqslant d'^{-A/2}h^{-A}\sum_{k=1}^{\infty}\sigma_k^A(h_0)=:d_1h^{-A},$$

(3. 4. 43)

where $d_1=d'^{-A/2}\sum_{k=1}^{\infty}\sigma_k^A(h_0)$.

Let $\theta>1$, define $A_i=\{T;\theta^{i-1}\leqslant\sigma^*(a_T)<\theta^i\}$, $A_{ij}=\{T;\theta^{j-1}\leqslant T/a_T<\theta^j, T\in A_i\}$, $a_{ij}=\sup\{a_T;T\in A_{ij}\}$, $T_{ij}=\sup\{T;a_T=a_{ij},T\in A_{ij}\}$, $T'_{ij}=\sup\{T;T-a_T=\sup_{T\in A_{ij}}(T-a_T),T\in A_{ij}\}$ and $J=\max\{j;\theta^j\leqslant\max_{T>0}T/a_T\}$. Then $J\leqslant\infty$ and

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log((T\log\sigma^*(a_T))/a_T))^{1/2}}$$

$$\leqslant\limsup_{i\to\infty}\max_{1\leqslant j\leqslant J}\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\sup_{0\leqslant s\leqslant a_{ij}}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{i-1}(2\log(\theta^{j-1}\log\theta^{i-1}))^{1/2}}$$

$$\leqslant\limsup_{i\to\infty}\max_{1\leqslant j\leqslant J}\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\sup_{0\leqslant s\leqslant a_{ij}}\max_{k\geqslant 1}$$

$$\frac{\theta^2|X_k(t+s)-X_k(t)|}{\sigma^*(a_{ij})(2\log((T_{ij}\log\sigma^*(a_{ij}))/a_{ij}))^{\frac{1}{2}}}. \tag{3. 4. 44}$$

For any $\varepsilon>0$, $r=r(\varepsilon)>0$ will be specified later on. Put $r_{ij}=a_{ij}/2^r$. For any $t>0$, put $t_r:=t_{r_{ij}}=[t/r_{ij}]r_{ij}$. Write

$$|X_k(t+s)-X_k(t)|\leqslant|X_k((t+s)_r)-X_k(t_r)|$$

$$+\sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1})-X_k((t+s)_{r+l})|$$

$$+\sum_{l=0}^{\infty}|X_k(t_{r+l+1})-X_k(t_{r+l})|. \tag{3. 4. 45}$$

Similar to (3. 4. 14) and noting (3. 4. 42), for large T, we have

$$p_0:=P\left\{\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\sup_{0\leqslant s\leqslant a_{ij}-r_{ij}}\max_{k\geqslant 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(a_{ij})\left(2\log\left(\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\right)\right)^{1/2}}\right.$$

$$\left.\geqslant 1+\varepsilon\right\}$$

$$\leqslant P\left\{\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\sup_{0\leqslant s\leqslant a_{ij}-r_{ij}}\max_{k\geqslant 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(a_{ij})\left(2\log\left(\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\right)\right)^{1/2}}\right.$$

$$\left.\geqslant 1+\varepsilon/2\right\}$$

$$\leqslant\frac{4T'_{ij}}{a_{ij}}2^{2r}\sum_{k=1}^{\infty}\exp\left\{-\left(1+\frac{\varepsilon}{2}\right)^2\left(\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\right)\frac{\sigma^{*^2}(a_{ij})}{\sigma_k^2(a_{ij})}\right\}$$

$$\leqslant\frac{4T'_{ij}}{a_{ij}}2^{2r}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^{1+\varepsilon}$$

$$\leqslant c\left(\frac{T_{ij}}{a_{ij}}\right)^{-\varepsilon}(\log\sigma^*(a_{ij}))^{-1-\varepsilon}$$

$$\leqslant c\theta^{-\varepsilon j}(i\log\theta)^{-1-\varepsilon}, \tag{3.4.46}$$

if $T'_{ij}\leqslant T_{ij}$. In the contrary case we have

$$T_{ij}\leqslant T'_{ij}\leqslant \theta^j a_{T'_{ij}}\leqslant \theta T_{ij},$$

and hence (3.4.46) is also true in either case. Combining the lines of proofs of (3.4.15) and (3.4.46) and using condition (3.4.34) instead of (3.4.11) we have

$$p_1:=$$

$$P\left\{\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\ \sup_{a_{ij}-r_{ij}\leqslant s\leqslant a_{ij}}\ \max_{k\geqslant 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(a_{ij})\left(2\log\left(\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\right)\right)^{1/2}}\right.$$

$$\left.\geqslant\varepsilon/2\right\}$$

$$\leqslant c\frac{T'_{ij}}{a_{ij}}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^{2+A}\left(\frac{\sigma^*(2a_{ij}/2^r)}{\sigma^*(a_{ij})}\right)^{A}\left(\frac{2a_{ij}}{2^r}\right)^{-A}$$

$$\leqslant\frac{cT'_{ij}a_{ij}}{(T_{ij}\log\sigma^*(a_{ij}))^{2+A}}\leqslant c\theta^{-j}i^{-2-A}, \tag{3.4.47}$$

provided r and T are large enough. Under conditions (3.4.36) and (3.4.37) we also have the same bound.

Consider the first series in the right hand side of (3.4.45). Let $b_{r_{ij}}=\sup\{b;a_{ij}/2^{r+[\log_2 ba_{ij}]+1}\geqslant h_0\}$ and $l_0=[\log_2(b_{r_{ij}}a_{ij})]$. Let $D=3/c_1$, $x_l^2:=x_{l_{ij}}^2=2D\log(T_{ij}\log\sigma^*(a_{ij})/a_{ij})+2(1+A)l$. If condition (3.4.34) is satisfied, then by (3.4.34) and (3.4.43)

$$p_2:=$$

$$P\left\{\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\ \sup_{0\leqslant s\leqslant a_{ij}}\ \max_{k\geqslant 1}\sum_{l=0}^{\infty}|X_k((t+s)_{r+l+1})-X_k((t+s)_{r+l})|\right.$$

$$\left.\geqslant\sum_{l=0}^{\infty}x_l\sigma^*(a_{ij}/2^{r+l+1})\right\}$$

$$\leqslant\frac{2T'_{ij}}{a_{ij}}\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}2^{r+l+1}e^{-(1+A)l}\left(\frac{a_{ij}y_{T_{ij}}}{T_{ij}\log\sigma^*(a_{ij})}\right)^2$$

$$\times\exp\left\{-\left(\frac{D\sigma^{*2}(a_{ij}/2^{r+l+1})}{\sigma_k^2(a_{ij}/2^{r+l+1})}-2\right)\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}y_{T_{ij}}}\right\}$$

$$\leqslant c\frac{T'_{ij}}{a_{ij}}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^2$$

$$\times\left(\sum_{l=0}^{l_0}+\sum_{l=l_0+1}^{\infty}\right)\sum_{k=1}^{\infty}2^l e^{-(1+A)l}\left(\frac{\sigma_k(a_{ij}/2^{r+l+1})}{\sigma^*(a_{ij}/2^{r+l+1})}\right)^A$$

$$\leqslant c\frac{T'_{ij}}{a_{ij}}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^2\left(\sum_{l=0}^{l_0}B2^l e^{-(1+A)l}\right.$$

$$\left.+\sum_{l=l_0+1}^{\infty}d_1 2^l e^{-(1+A)l}(a_{ij}/2^{r+l+1})^{-A}\right)$$

$$\leqslant c\frac{T'_{ij}}{a_{ij}}\left(\frac{a_{ij}}{T_{ij}\log\sigma^*(a_{ij})}\right)^2\leqslant c\theta^{-j}(i\log\theta)^{-2} \tag{3.4.48}$$

for large r and T. Also, if conditions (3.4.36) and (3.4.37) are satisfied, we have the same bound.

For the second series in the right hand side of (3.4.45), we also have the same conclusion.

Moreover, similar to (3.4.20), we have

$$\sum_{l=0}^{\infty}x_l\sigma^*(a_{ij}/2^{r+l+1})\leqslant\frac{\varepsilon}{4}\sigma^*(a_{ij})\left(\log\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\right)^{1/2} \tag{3.4.49}$$

provided r is large enough. Combining these estimates, we obtain

$$\sum_{i=1}^{\infty}\sum_{j=1}^{J}P\left\{\sup_{0\leqslant t\leqslant T'_{ij}-a_{T'_{ij}}}\ \sup_{0\leqslant s\leqslant a_{ij}}\ \max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_{ij})\left(2\log\left(\frac{T_{ij}\log\sigma^*(a_{ij})}{a_{ij}}\right)\right)^{1/2}}\right.$$

$$\left.\geqslant 1+2\varepsilon\right\}<\infty.$$

Therefore by the Borel-Cantelli lemma, the right hand side of (3.4.44) is bounded by one almost surely. (3.4.35) is proved.

Now we prove (3.4.41). Noting (3.4.35), it is enough to show

$$\limsup_{T\to\infty}\sup_{0\leq t\leq T-a_T}\max_{k\geq 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}\geq 1\quad \text{a.s.}\tag{3.4.50}$$

Using (3.4.39), Slepian's inequality and independence of $\{X_k(\cdot)\}_{k=1}^{\infty}$, we have for $T_n(\uparrow\infty)$ large enough

$$P\left\{\sup_{0\leq t\leq T_n-a_{T_n}}\max_{k\geq 1}\frac{|X_k(t+a_{T_n})-X_k(t)|}{\sigma^*(a_{T_n})(2\log(T_n/a_{T_n}))^{1/2}}<1-\varepsilon\right\}$$

$$\leq P\left\{\max_{0\leq j\leq\frac{T_n}{a_{T_n}}}\max_{k\geq 1}\frac{X_k((j+1)a_{T_n})-X_k(ja_{T_n})}{\sigma^*(a_{T_n})\left(2\log\left(\frac{T_n\log\sigma^*(a_{T_n})}{a_{T_n}y_{T_n}}\right)\right)^{1/2}}<1-\varepsilon\right\}$$

$$\leq\prod_{j=0}^{[T_n/a_{T_n}]}\prod_{k=1}^{\infty}\left\{1-\exp\left\{-\frac{(1-\varepsilon)\sigma^{*2}(a_{T_n})}{\sigma_k^2(a_{T_n})}\log\frac{T_n\log\sigma^*(a_{T_n})}{a_{T_n}y_{T_n}}\right\}\right\}$$

$$\leq\prod_{j=0}^{[T_n/a_{T_n}]}\exp\left\{-\sum_{k=1}^{\infty}\left(\frac{a_{T_n}y_{T_n}}{T_n\log\sigma^*(a_{T_n})}\right)^{(1-\varepsilon)\sigma^{*2}(a_{T_n})/\sigma_k^2(a_{T_n})}\right\}$$

$$\leq\exp\left\{-\frac{T_n}{a_{T_n}}\left(\frac{a_{T_n}}{T_n\log\sigma^*(a_{T_n})}\right)^{1-\varepsilon}\right\}$$

$$\leq\exp\left\{-\left(\frac{T_n}{a_{T_n}}\right)^{\varepsilon/2}\right\}\to 0\tag{3.4.51}$$

as $n\to\infty$. Hence (3.4.41) is proved.

Finally we prove (3.4.40). By (3.4.35), it suffices to show that the "lim inf" is not less than one almost surely. Let $a'_{ij}=\inf\{a_T;T\in A_{ij}\}$, $t'_{ij}=\inf\{T;a_T=a'_{ij},T\in A_{ij}\}$, $t_{ij}=\inf\{T;T-a_T=\inf_{T\in A_{ij}}(T-a_T),T\in A_{ij}\}$. We have

$$\liminf_{T\to\infty}\sup_{0\leq t\leq T-a_T}\sup_{0\leq s\leq a_T}\max_{k\geq 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(a_T)(2\log(T/a_T))^{1/2}}$$

$$\geq\liminf_{i\to\infty}\min_{1\leq j\leq J}\sup_{0\leq t\leq t_{ij}-a_{t_{ij}}}\sup_{0\leq s\leq a'_{ij}}\max_{k\geq 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{i}(2\log\theta^{i})^{1/2}}$$

$$\geq\liminf_{i\to\infty}\min_{1\leq j\leq J}\max_{1\leq l\leq t_{ij}/a'_{ij}}\max_{k\geq 1}\frac{|X_k((l+1)a'_{ij})-X_k(la'_{ij})|}{\theta^2\sigma^*(a'_{ij})\left(2\log\left(\frac{t'_{ij}\log\sigma^*(a'_{ij})}{a'_{ij}y_{t'_{ij}}}\right)\right)^{1/2}}.$$

Then, similar to (3.4.51) we have

$$P\left\{\max_{0\leq l\leq t_{ij}/a'_{ij}}\max_{k\geq 1}\frac{|X_k((l+1)a'_{ij})-X_k(la'_{ij})|}{\sigma^*(a'_{ij})\left(2\log\left(\frac{t'_{ij}\log\sigma^*(a'_{ij})}{a'_{ij}y_{t'_{ij}}}\right)\right)^{1/2}}<1-\varepsilon\right\}$$

$$\leq\exp\left\{-\frac{t_{ij}}{a'_{ij}}\left(\frac{a'_{ij}}{t'_{ij}\log\sigma^*(a'_{ij})}\right)^{1-\varepsilon}\right\}$$

$$\leq\exp\{-(t'_{ij}/a'_{ij})^{\varepsilon/2}\log\sigma^*(a'_{ij})\}$$

$$\leq\exp\{-\theta^{(j-1)\varepsilon/2}(i\log\theta)\}.$$

The second inequality is due to (3.4.39) again. Hence by the Borel-Cantelli lemma, (3.4.40) is proved. This completes the proof of Theorem 3.4.2.

Now we consider the case that $\sigma^{*2}(h)\to\sigma^{*2}<\infty$ as $h\to\infty$. Let $\sigma_k^2=\lim_{h\to\infty}\sigma_k^2(h)$ and z_T be the solution of the equation

$$\sum_{k=1}^{\infty}\left(\frac{a_Tz_T}{T}\right)^{\sigma^{*2}/\sigma_k^2}=\frac{a_T}{T}.\tag{3.4.52}$$

Continue employing the notations $d(m,k)$, L_mx etc. introduced in the previous section.

Theorem 3.4.3 *Suppose that $\sigma^*(h)\to\sigma^*$ as $h\to\infty$ and that there exists $A>0$ such that*

$$\sum_{k=1}^{\infty}\sigma_k^A<\infty.\tag{3.4.53}$$

And suppose that there exists $2\leq\alpha<e$, $0<\delta<1-1/\alpha$ and integer $m\geq 1$ such that

$$\int_1^{\infty}\sigma^*(\alpha^{-x^2})dx<\infty,\tag{3.4.54}$$

$$a_T \leqslant Td(m,(L_mT)^{\delta+1/\alpha})^{-1}. \tag{3.4.55}$$

Then

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \max_{k\geqslant 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} \leqslant 1 \quad \text{a.s.} \tag{3.4.56}$$

If, in addition, $X_k(\cdot)$, $k=1,2,\cdots$ *are independent and* (3.4.38) *is satisfied, then*

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \max_{k\geqslant 1} \frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.4.57}$$

and

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \max_{k\geqslant 1} \frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.4.58}$$

Proof Similar to Theorem 3.4.2, condition (3.4.53) guarantees that the solution of equation (3.4.52) exists and is unique. Moreover, there exists $0<b<1$ such that $z_T \geqslant b$ and (3.3.62) remains true from condition (3.4.55).

For $\theta>1$, define $A_j=\{T;\theta^{j-1}\leqslant T/a_T<\theta^j\}$, $a_j=\sup\{a_T;T\in A_j\}$, $T_j=\sup\{T;a_T=a_j,T\in A_j\}$ and $T'_j=\sup\{T;T-a_T=\sup_{T\in A_j}(T-a_T),T\in A_j\}$. Given $r>0$ for any $t>0$ put $t_r^j := t_r(a_j)=[td(m+2,r)/a_j](a_j/d(m+2,r))$. Write

$$\begin{aligned}|X_k(t+s)-X_k(t)| &\leqslant |X_k((t+s)_r^j)-X_k(t_r^j)| \\ &+ \sum_{l=0}^{\infty} |X_k((t+s)_{r+l+1}^j)-X_k((t+s)_{r+l}^j)| \\ &+ \sum_{l=0}^{\infty} |X_k(t_{r+l+1}^j)-X_k(t_{r+l}^j)|.\end{aligned} \tag{3.4.59}$$

For $\delta>0$ in (3.4.55), let δ_1 satisfy $1-1/\alpha-\delta<\delta_1<1-1/\alpha$.

And let $\varepsilon(a_T)=d(m-1,(L_ma_T)^{1-\delta_1})/L_1a_T$, $r:=r(a_j)=L_{m+2}a_j^{\varepsilon(a_j)}$, $r':=r'(a_j)=L_{m+2}a_j$, Then $d(m+2,r)=a_j^{\varepsilon(a_j)}$ and $d(m+2,r')=a_j$, moreover,

$$\begin{aligned}0<r'-r &= L_{m+2}a_j-L_m\left[(L_2a_j)\left(1+\frac{L_1\varepsilon(a_j)}{L_2a_j}\right)\right] \\ &= -L_1(1-\delta_1)<1.\end{aligned} \tag{3.4.60}$$

Similar to (3.4.46), we have

$$\begin{aligned}p'_0 &:= P\left\{\sup_{0\leqslant t\leqslant T'_j-a_{T'_j}} \sup_{0\leqslant s\leqslant a_j} \max_{k\geqslant 1} \frac{|X_k((t+s)_r^j)-X_k(t_r^j)|}{\sigma^*(2\log(T'_j/a_j))^{1/2}} \geqslant 1+\varepsilon\right\} \\ &\leqslant \frac{4T'_j}{a_j} d(m+2,r)^2\left(\frac{a_j}{T_j}\right)^{1+\varepsilon} \\ &\leqslant 4a_j^{2\varepsilon(a_j)+\varepsilon}\left(\frac{T'_j}{T_j}\right)T_j^{-\varepsilon} \\ &\leqslant 5a_j^{2\varepsilon(a_j)+\varepsilon}T_j^{-\varepsilon}\end{aligned} \tag{3.4.61}$$

if $T'_j/T_j\leqslant 1$. In the case of $T'_j/T_j>1$, we have

$$T_j<T'_j\leqslant \theta^j a_{T'_j}\leqslant \theta^j a_j\leqslant \theta T_j.$$

Hence (3.4.61) holds true in either case provided $\theta<5/4$. By condition (3.4.55) we obtain

$$a_j^{2\varepsilon(a_j)+\varepsilon/2}\leqslant T_j^{2\varepsilon(a_j)+\varepsilon/2}d(m,(L_mT_j)^{\delta+1/\alpha})^{-\varepsilon/2}\leqslant T_j^{\varepsilon/2}.$$

Inserting it into (3.4.61) yields

$$p'_0\leqslant 5(T_j/a_j)^{-\varepsilon/2}\leqslant 5\theta^{-(j-1)\varepsilon/2}.$$

Let $x'^2_l := x'^2_{lj} = 4\log(T_j/a_j)+2(1+A)d(m+1,r+l+1)$. Then we have

$$\begin{aligned}p'_2 &:= P\left\{\sup_{0\leqslant t\leqslant T'_j-a_{T'_j}} \sup_{0\leqslant s\leqslant a_j} \max_{k\geqslant 1}\left|\sum_{l=0}^{\infty}(X_k((t+s)_{r+l+1}^j)\right.\right. \\ &\quad \left.\left. -X_k((t+s)_{r+l}^j)\right| \geqslant \sum_{l=0}^{\infty} x'_l\sigma^*(a_j/d(m+2,r+l+1))\right\} \\ &\leqslant \frac{4T'_j}{a_j}\left(\frac{a_j}{T_j}\right)^2 \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} d(m+2,r+l+1)e^{-(1+A)d(m+1,r+l+1)}\end{aligned}$$

$$\times \exp\left\{-\left(\frac{2\sigma^{*2}(a_j/d(m+2,r+l+1))}{\sigma_k^2(a_j/d(m+2,r+l+1))}-2\right)\log\frac{T_j}{a_j}\right\}$$
$$\leqslant \frac{4a_j}{T_j}\frac{T_j'}{T_j}\sum_{l=0}^{\infty} d(m+2,r+l+1)\mathrm{e}^{-(1+A)d(m+1,r+l+1)}$$
$$\times \sum_{k=1}^{\infty}\left(\frac{\sigma_k(a_j/d(m+2,r+l+1))}{\sigma^*(a_j/d(m+2,r+l+1))}\right)^A. \qquad (3.4.62)$$

Noting $0<r'-r<1$ (see (3.4.60)), we have

$$a_j/d(m+2,r+l+1)\leqslant d(m+2,r')/d(m+2,r+1)\leqslant 1$$

for $l\geqslant 0$. Hence, using the fact that $\sigma^{*2}(h)\geqslant dh^2$ for $0<h\leqslant 1$, we obtain

$$\sigma^*(a_j/d(m+2,r+l+1))^{-A}$$
$$\leqslant d^{-A/2}(a_j/d(m+2,r+l+1))^{-A}$$

and further

$$p_2'\leqslant \frac{5d^{-A/2}a_j^{1-A}}{T_j}\sum_{l=0}^{\infty}(\alpha/\mathrm{e})^{(1+A)d(m+1,r+l+1)}\leqslant c\theta^{-j}.$$

For the second series in the right hand side of (3.4.59), we have the same estimate.

Let $\beta>0$ satisfy $r'-r+\beta<1$. Then

$$\sum_{l=0}^{\infty} d(m+1,r+l+1)^{1/2}\sigma^*(a_j/d(m+2,r+l+1))$$
$$\leqslant \sum_{l=0}^{\infty}\left(1-\left(\frac{d(m+1,r+l+1-\beta)}{d(m+1,r+l+1)}\right)^{1/2}\right)^{-1}$$
$$\times\int_{d(m+1,r+l+1-\beta)^{1/2}}^{d(m+1,r+l+1)^{1/2}}\sigma^*(a_j\alpha^{-y^2})\mathrm{d}y$$
$$\leqslant\left(1-\left(\frac{d(m+1,r+1-\beta)}{d(m+1,r+1)}\right)^{1/2}\right)^{-1}$$
$$\times\int_{d(m+1,r+1-\beta)^{1/2}}^{\infty}\sigma^*(\alpha^{d(m+1,r')-y^2})\mathrm{d}y$$
$$\leqslant\left(1-\left(\frac{d(m+1,r+1-\beta)}{d(m+1,r+1)}\right)^{1/2}\right)^{-1}$$

$$\times\int_{d(m+1,r+1-\beta)^{1/2}-d(m+1,r')^{1/2}}^{\infty}\sigma^*(\alpha^{-y^2})\mathrm{d}y\leqslant\varepsilon \qquad (3.4.63)$$

provided T is large enough by condition (3.4.54). Furthermore, obviously, (3.4.63) implies

$$\sum_{l=0}^{\infty}\sigma^*(a_j/d(m+2,r+l+1))\leqslant\frac{\varepsilon}{8}\sigma^*$$

for large T. Then

$$\sum_{l=0}^{\infty}x_l'\sigma^*(a_j/d(m+2,r+l+1))\leqslant\frac{\varepsilon}{2}\sigma^*\left(2\log\frac{T_j}{a_j}\right)^{1/2}.$$

Combining these results, we obtain

$$\sum_{j=1}^{\infty}P\left\{\sup_{j\geqslant 1}\sup_{0\leqslant t\leqslant T_j'-a_{T_j'}}\sup_{0\leqslant s\leqslant a_j}\max_{k\geqslant 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T_j/a_j))^{1/2}}\leqslant 1+2\varepsilon\right\}$$
$$<\infty,$$

which implies (3.4.56).

The proofs of (3.4.57) and (3.4.58) are similar to that of (3.4.40) and (3.4.41) respectively when we use $(1+\varepsilon)\sigma^*(a_T)$ instead of σ^* (noting $\sigma^*\leqslant(1+\varepsilon)\sigma^*(a_T)$ for large T), so they are omitted. Theorem 3.4.3 is proved.

As an application of Theorem 3.4.3, we establish the large increment result for l^∞-valued Ornstein-Uhlenbeck process. Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients $\gamma_k\geqslant 0$ and $\lambda_k>0$.

Corollary 3.4.4 *Suppose that there exists $A>0$ such that*

$$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^A<\infty$$

and that there exist $2\leqslant\alpha<\mathrm{e}$, $0<\delta<1-1/\alpha$ and integer $m\geqslant 1$ such that (3.4.54) and (3.4.55) are satisfied. Then

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1\quad \text{a.s.},$$

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\max_{k\ge 1}\frac{|X_k(t+a_T)-X_k(t)|}{\sigma^*(2\log(T/a_T))^{1/2}}=1\quad \text{a.s.}$$

Chapter 4

The Law of the Iterated Logarithm and Almost Sure Limit Inferior of Increments for Gaussian Processes

Since Chung's law of the iterated logarithm (1948) and Strassen's functional law of the iterated logarithm (1964) appeared, the functional law of the iterated logarithm (LIL) and its rates for some classes of Gaussian processes have been discussed by Oodaira (1972), Bolthausen (1978), Grill (1987), Goodman and Kuelbs (1988, 1991a, b) etc. Let $\eta(t)=\sup\{s;0\le s\le t, W(s)\ge(2s\log\log s)^{1/2}\}$, where $\{W(t);t\ge 0\}$ is a standard Wiener process. Erdős-Révész gave a new law of the iterated logarithm of $\{\eta(t)\}$. Shao (1992) extended this type LIL to a kind of stationary Gaussian processes which include the infinite series of independent Ornstein-Uhlenbeck processes and the fractional Wiener process of order α. In terms of the small ball probability estimates, Chung's law of the iterated logarithm for various Gaussian processes and Gaussian fields have been studied by Shao (1993), Kuelbs, Li and Talagrand (1994), Kuelbs, Li and

Shao (1995), Monrad and Rootzen (1995), Shao and Wang (1995a) etc. The other laws of the iterated logarithm, small increments and moduli of non-differentiability for Gaussian processes are also discussed by some authors. In recent ten years, many authors are paying a good deal of attention to all these problems, especially they pay more attention to the "liminf" behavior. In this chapter, we will intend to summarize and elaborate on a number of recent laws of the iterated logarithm and almost sure limit inferiors of increments for various Gaussian processes.

The first two sections deal with the law of the iterated logarithm of Gaussian processes. In the first section, we investigate a Strassen's functional law of the iterated logarithm and its exact convergence rate for a class of Gaussian processes, especially for the fractional Wiener process. An Erdös and Révész's law of the iterated logarithm for Gaussian process will be discussed in Section 2.2. The second two sections deal with Chung's law of iterated logarithm. The small ball probabilities of Gaussian processes are established, and from which Chung's law of the iterated logarithm of Gaussian processes is proved in Section 4.3. Similar results for Gaussian fields are introduced in Section 4.4. In the last two sections, we study some limit inferiors of increments for Gaussian processes. An almost sure limit inferior for increments of one-parameter Gaussian processes with stationary increments is given in Section 4.5 and the liminfs for two-parameter Gaussian processes are investigated in Section 4.6.

4.1 The Strassen Laws of the Iterated Logarithm and Its Rates for Gaussian Processes

4.1.1 Strassen's law of the iterated logarithm

Let $\{W(t);t\geqslant 0\}$ be a Wiener process in $\mathbf{R}^1$, let $C[0,1]$ be the space of continuous functions on $[0,1]$, and

$$\boldsymbol{K} = \left\{f(t) = \int_0^t g(s)\mathrm{d}s; 0\leqslant t\leqslant 1, \int_0^1 g^2(s)\mathrm{d}s\leqslant 1\right\}.$$

Then $\boldsymbol{K}$ is a compact convex symmetric subset of $C[0,1]$ and $\boldsymbol{K}$ is also the unit ball of reproducing kernel (r. k.) Hilbert space with r. k. function $R(s,t)=s\wedge t$. Define

$$f_n(t) = W(nt)/\sqrt{2n\,\mathrm{loglog}\,n},\quad 0\leqslant t\leqslant 1.$$

Then $\{f_n(t); n\geqslant 3\}$ is a sequence of stochastic processes with sample paths almost surely in $C[0,1]$ and with the usual sup norm $\|\cdot\|$. Strassen (1964) showed the following theorem.

Theorem S *The sequence* $\{f_n(t)\}$ *is relatively compact in* $C[0,1]$ *with probability one, and the set of its limit points coincides with set* $\boldsymbol{K}$.

Oodaira (1972) generalized the above theorem to a certain class of Gaussian processes including Wiener process. Let $\{X(t); t\geqslant 0\}$ be a separable, measurable, real valued Gaussian process with $X(0)=0$, $EX(t)=0$ and the covariance

$$R(s,t) = EX(s)X(t).$$

Denote $\sigma^2(t)=R(t,t)$. Suppose that the following conditions are satisfied

(I) For any $T>0$, there exists a positive, non-decreasing function $g(h,T)$, $h>0$, such that

$$|R(t+h,t+h)-2R(t+h,t)+R(t,t)|\leqslant g(h,T)\to 0$$
$$\text{as } h\to 0,\ \text{for all } t,t+h\in[0,T], \tag{4.1.1}$$

$$\{g(1,T)\}^{-1/2}\int_1^\infty g^{1/2}(e^{-u^2},T)du \leqslant C<\infty, \tag{4.1.2}$$

and

$$\sigma^2(T)/g(1,T)\uparrow\infty \quad \text{as} \quad T\to\infty; \tag{4.1.3}$$

(II) There is a positive function $v(r)$, $r>0$, such that $v(r)\uparrow$ and

$$R(rs,rt)=v(r)R(s,t) \quad \text{for all} \quad r>0,s,t\geqslant 0. \tag{4.1.4}$$

Under condition (I), the process $\{X(t);0\leqslant t\leqslant T\}$ has continuous sample paths a.s. for any $T>0$ by Theorem 2.1.3.

Examples Gaussian processes having covariance kernels

$$R(s,t)=\int_0^{s\wedge t}(s-\lambda)^\beta(t-\lambda)^\beta d\lambda, \quad -1/2<\beta<\infty$$

satisfy conditions (I) and (II). This class includes the Wiener process $\{W(t)\}$ (with $\beta=0$) and the process $\left\{\int_0^t W(u)du\right\}$ (with $\beta=1$). Fractional Wiener processes are Gaussian processes with stationary increments having covariance kernels

$$R(s,t)=(s^{2\alpha}+t^{2\alpha}-|s-t|^{2\alpha})/2, \quad 0<\alpha\leqslant 1 \tag{4.1.5}$$

satisfy conditions (I) and (II).

Define

$$\eta_n(t,\omega)=X(nt,\omega)/(2\sigma^2(n)\log\log n)^{1/2}, \quad 0\leqslant t\leqslant 1,\ n\geqslant 3. \tag{4.1.6}$$

Let $H(R)$ be the reproducing kernel (r.k.) Hilbert space with r.k. function $R(s,t)$, $0\leqslant s,t\leqslant 1$. Denote

$$\boldsymbol{K}=\{h\in H(R);\ \|h\|_H\leqslant 1/\sigma(1)\},$$

where $\|\cdot\|_H$ denotes the norm of $H(R)$. Oodaira (1972) proved the following results.

Theroem 4.1.1 *If the conditions (I) and (II) are fulfilled, then the set of limit points of sequence $\{\eta_n(t)\}$ is contained in set $\boldsymbol{K}$ with probability one.*

Theorem 4.1.2 *Let $\{Z(t);t\geqslant 0\}$ be a fractional Wiener process, then the set of limit points of $\{\eta_n(t,\omega)\}$ coincides with the set $\boldsymbol{K}$ with probability one.*

The proofs of theorems need the following lemmas.

Lemma 4.1.1 *Let $\{X(t);t\geqslant 0\}$ be a Gaussian process as above which satisfies conditions (I) and (II), then the sequence of functions $\{\eta_n(t,\omega);n\geqslant 3\}$ is equicontinuous with probability one.*

Proof We need only to show that for any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that for almost sure ω and for some integer $N=N(\omega)\geqslant 3$, we have

$$|\eta_n(t,\omega)-\eta_n(s,\omega)|<\varepsilon \tag{4.1.7}$$

if $|t-s|<\delta$ and $n\geqslant N$. Let $q=q(\varepsilon)$ be an integer, which will be specified later on, and put $\delta(\varepsilon)=2^{-q}$. It is clear that (4.1.7) is equivalent to

$$|X(nt)-X(ns)|<\varepsilon(2\sigma^2(n)\log\log n)^{1/2},$$

where $|t-s|<\delta=2^{-q}$, $0\leqslant t,s\leqslant 1$. Let

$$A_n=\{\sup_{\substack{|t-s|<2^{-q}\\ 0\leqslant s,t\leqslant 1}}|X(nt)-X(ns)|\geqslant\varepsilon(2\sigma^2(n)\log\log n)^{1/2}\}.$$

Consider the subsequence $\{n_r=2^r;r\geqslant\max(q,3)\}$ and let

$$B_r=\{\max_{2^r\leqslant n<2^{r+1}}\sup_{\substack{|t-s|<2^{-q}\\0\leqslant s,t\leqslant 1}}|X(nt)-X(ns)|$$
$$\geqslant\varepsilon(2\sigma^2(2^r)\log\log 2^r)^{1/2}\}.$$

Then it is enough to prove that $P\{\limsup_r B_r\}=0$. Let

$$C_r=\{\sup_{\substack{0\leqslant h\leqslant 2^{r+1-q}\\0\leqslant t<t+h\leqslant 2^{r+1}}}|X(t+h)-X(t)|\geqslant\varepsilon(2\sigma^2(2^r)\log\log 2^r)^{1/2}\},$$

$$C_r^{(v)}=\{\sup_{t,t+h\in I(r,v)}|X(t+h)-X(t)|\geqslant\varepsilon(2\sigma^2(2^r)\log\log 2^r)^{1/2}\},$$

where

$I(r,v)=[(v-1)2^{r-q+1},(v+1)2^{r-q+1}],\quad v=1,2,\cdots,2^q-1.$

Since $B_r\subset C_r\subset\bigcup_{v=1}^{2^q-1}C_r^{(v)}$, it suffices to show that for each fixed v, $P\{\limsup_r C_r^{(v)}\}=0$.

Denote $t_v=(v-1)2^{r-q+1}$, put

$$D_r^{(v)}=\left\{\sup_{t\in I(r,v)}|X(t)-X(t_v)|\geqslant\frac{\varepsilon}{2}(2\sigma^2(2^r)\log\log 2^r)^{1/2}\right\}.$$

Then we have $P\{C_r^{(v)}\}\leqslant 2P\{D_r^{(v)}\}$. In order to evaluate $P\{D_r^{(v)}\}$, let

$$Y(s)=X(s2^{r-q+2}+t_v),\quad 0\leqslant s\leqslant 1.$$

Then

$$E(Y(t)-Y(s))^2=E\{X(t2^{r-q+2}+t_v)-X(s\cdot 2^{r-q+2}+t_v)\}^2$$
$$=v(2^{r-q+2})\left\{R\left(t+\frac{v-1}{2},t+\frac{v-1}{2}\right)\right.$$
$$\left.-2R\left(t+\frac{v-1}{2},s+\frac{v-1}{2}\right)+R\left(s+\frac{v-1}{2},s+\frac{v-1}{2}\right)\right\}$$
$$\leqslant v(2^{r-q+2})g(|t-s|,2^{q-1})$$

and the covariance $\Gamma(t,s)$ of $Y(\cdot)$ satisfies

$$|\Gamma(t,s)|\leqslant\{E(X(t2^{r-q+2}+t_v)-X(t_v))^2\}^{1/2}$$
$$\times\{E(X(s2^{r-q+2}+t_v)-X(t_v))^2\}^{1/2}$$
$$\leqslant v(2^{r-q+2})g^{1/2}\left(t,\frac{v+1}{2}\right)g^{1/2}\left(s,\frac{v+1}{2}\right)$$

$$\leqslant v(2^{r-q+2})g(1,2^{q-1}).$$

Hence, by Fernique's inequality (cf. Theorem 1.1.3), we have

$$P\{D_r^{(v)}\}=P\left\{\|Y(t)\|\geqslant\frac{\varepsilon}{2}(2\sigma^2(2^r)\log\log 2^r)^{1/2}\right\}$$
$$\leqslant 4p^2\int_{y_r}^{\infty}e^{-u^2/2}du,$$

where $\|Y(t)\|=\sup_t|Y(t)|$,

$$y_r=(\varepsilon/2)(2\log\log 2^r)^{1/2}\{\sigma(2^r)v^{-1/2}(2^{r-q+2})g^{-1/2}(1,2^{q-1})\}$$
$$\times\left\{1+4v^{-\frac12}(2^{r-q+2})g^{-\frac12}(1,2^{q-1})\right.$$
$$\left.\times\int_1^{\infty}v^{\frac12}(2^{r-q+2})g^{\frac12}(p^{-u^2},2^{q-1})du\right\}^{-1}.$$

By assumptions (4.1.2) and (4.1.4), we obtain

$$y_r=(\varepsilon/2C)\{\sigma(2^{q-2})g^{-1/2}(1,2^{q-1})\}(2\log\log 2^r)^{1/2}\to\infty$$

as $r\to\infty$. Choose q sufficiently large such that

$$(\varepsilon/2C)^2\sigma^2(2^{q-2})g^{-1}(1,2^{q-1})=\theta>1,$$

which is possible because of the assumption (4.1.4). Thus we have

$$P\{C_r^{(v)}\}\leqslant 8p^2\int_{y_r}^{\infty}e^{-u^2/2}du\leqslant c(\log 2^r)^{-\theta}$$

and $\sum_r P\{C_r^{(v)}\}<\infty$. Hence, by the Borel-Cantelli lemma, $P\{\limsup_r C_r^{(v)}\}=0$. This completes the proof of Lemma 4.1.1.

Corollary 4.1.1 *For any* $\varepsilon>0$ *there is a* $\delta=\delta(\varepsilon)$ *such that for a.s.* ω *and for some integer* $N=N(\varepsilon)$ *we have* $\sup_{0\leqslant t\leqslant 1}|\eta_m(t)-\eta_n(t)|<\varepsilon$ *for all* $m,n\geqslant N$ *with* $|1-(m/n)|<\delta$.

Proof From the definition of $\eta_n(t)$, we see that

$$\eta_m\left(\frac{n}{m}t,\omega\right)=\sqrt{\frac{\sigma(n)\log\log n}{\sigma(m)\log\log m}}\eta_n(t,\omega)$$

for $3\leqslant n\leqslant m$, at the points of the form $t=k/n$, $k=0,1,2,\cdots,n$.

By Lemma 4.1.1 for a.s. ω and for $\varepsilon>0$ there is an integer n_0 such that for $m\geqslant n\geqslant n_0(\omega,\varepsilon)$ we have

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant 1}|\eta_n(t)-\eta_m(t)|\leqslant\max_{1\leqslant k\leqslant n}\left|\eta_n\left(\frac{k}{n}\right)-\eta_m\left(\frac{k}{n}\right)\right|+\frac{1}{3}\varepsilon\\
&=\max_{1\leqslant k\leqslant n}\left|\sqrt{\frac{\sigma(m)\mathrm{loglog}m}{\sigma(n)\mathrm{loglog}n}}\eta_m\left(\frac{n}{m}\frac{k}{n}\right)-\eta_m\left(\frac{k}{n}\right)\right|+\frac{1}{3}\varepsilon\\
&\leqslant\left|\sqrt{\frac{\sigma(m)\mathrm{loglog}m}{\sigma(n)\mathrm{loglog}n}}-1\right|\ \|\eta_m(t)\|\\
&\quad+\max_{1\leqslant k\leqslant n}\left|\eta_m\left(\frac{n}{m}\frac{k}{n}\right)-\eta_m\left(\frac{k}{n}\right)\right|+\frac{1}{3}\varepsilon.
\end{aligned}$$

By the equicontinuity asserted in Lemma 4.1.1, we conclude that

$$\max_{1\leqslant k\leqslant n}\left|\eta_m\left(\frac{n}{m}\frac{k}{n}\right)-\eta_m\left(\frac{k}{n}\right)\right|<\frac{1}{3}\varepsilon$$

by choosing m/n sufficiently close to 1, and that the $\|\eta_n(t)\|$ are bounded, hence $\left|\sqrt{\sigma(m)\mathrm{loglog}m/(\sigma(n)\mathrm{loglog}n)}-1\right|\times\|\eta_m(t)\|$ can be made less than $\varepsilon/3$ as well.

Consider the r.k. Hilbert space $H(R)$ and $H(R_n)$ with r.k. $R_n=R(s,t)$, $0\leqslant s,t\leqslant n$. From assumption (4.1.5) it follows that

$$\begin{aligned}
&\langle R(*,nt),R(*,ns)\rangle_n=R(nt,ns)=v(n)R(s,t)\\
&=v(n)\langle R(\cdot,t),R(\cdot,s)\rangle_1\\
&=\langle v^{1/2}(n)R(\cdot,t),v^{1/2}(n)R(\cdot,s)\rangle_1\quad\text{for }0\leqslant s,t\leqslant 1,
\end{aligned}\tag{4.1.8}$$

where $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_n$ denote respectively the inner products of $H(R)$ and $H(R_n)$. (4.1.8) implies that there is an isometric isomorphism θ_n from $H(R)$ to $H(R_n)$ such that

$$\theta_n(v^{1/2}(n)R(\cdot,t))=R(*,nt),\quad 0\leqslant t\leqslant 1.$$

Note that for any $h\in H(R)$

$$\begin{aligned}
\theta_n h(nt)&=\langle\theta_n h(*),R(*,nt)\rangle_n\\
&=v^{1/2}(n)\langle h(\cdot),R(\cdot,t)\rangle_1\\
&=v^{1/2}(n)h(t),\quad 0\leqslant t\leqslant 1.
\end{aligned}$$

And if $\{e_j(\cdot);j=1,2,\cdots,J\}$ is a system of orthonormal functions in $H(R)$, so is $\{e_{nj}(*)=\theta_n e_j(*);j=1,2,\cdots,J\}$ in $H(R_n)$.

It is well known that there is an isomorphism between $H(R_n)$ and the closed linear manifold $L_n^2(X)$ spanned by $\{X(t);0\leqslant t\leqslant n\}$, and if $\xi_{nj};j=1,2,\cdots,J$ are random variables in $L_n^2(X)$ corresponding to orthonormal functions $e_{nj}(*)$, $j=1,2,\cdots,J$, then ξ_{nj} are independent and standard normally distributed.

Lemma 4.1.2 *Suppose that a sequence of families of orthonormal functions* $\{e_j^{(k)}(\cdot);j=1,2,\cdots,J_k<\infty\}$, $k=1,2,\cdots$ *in* $H(R_1)$ *satisfy the following condition:*

$$\sup_{0\leqslant t\leqslant 1}\left|R(t,t)-\sum_{j=1}^{J_k}\{e_j^{(k)}\}^2\right|\to 0\quad\text{as }k\to\infty.\tag{4.1.9}$$

Let $\{\xi_{nj}^{(k)}\}$ *be Gaussian random variables corresponding to* $\{e_{nj}^{(k)}(*)=\theta_n e_j^{(k)}(*)\}$. *Then, for any geometric subsequence of indices* $\{n_r=[C^r],C>1\}$ *and any* $\varepsilon>0$, *there exist for a.s.* ω *some integers* $k_0=k_0(\varepsilon)$ *and* $r_0=r_0(\varepsilon,\omega)$ *such that*

$$\left\|\eta_{n_r}(t,\omega)-(2\sigma^2(1)\mathrm{loglog}n_r)^{-1/2}\sum_{j=1}^{J_k}\xi_{n_rj}^{(k)}(\omega)e_j^{(k)}(t)\right\|<\varepsilon\tag{4.1.10}$$

for all $k\geqslant k_0$ *and all* $r\geqslant r_0$.

Remark 4.1.1 Let $\{e_j(\cdot);j=1,2,\cdots\}$ be any complete orthonormal system in $H(R_1)$. It is known that the partial sums $\sum_{j=1}^{k}e_j^2(t)$ converge to $R(t,t)$ uniformly in $t\in[0,1]$. Hence the

condition (4.1.9) is satisfied for the families $\{e_j(\cdot); j=1,2,\cdots,k\}$, $k=1,2,\cdots$, and we have, for all sufficiently large k and r,

$$\sup_{0\leqslant t\leqslant 1}\left|\eta_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-1/2}\sum_{j=1}^{k}\xi_{n_rj}^{(k)}(\omega)e_j(t)\right|<\varepsilon.$$

Proof Let

$$A_r^{(k)}$$
$$=\left\{\sup_{0\leqslant t\leqslant 1}\left|\eta_{n_r}(t,\omega)-(2\sigma^2(1)\log\log n_r)^{-\frac{1}{2}}\sum_{j=1}^{J_k}\xi_{n_rj}^{(k)}(\omega)e_j^{(k)}(t)\right|\geqslant\varepsilon\right\}$$
$$=\left\{\sup_{0\leqslant t\leqslant 1}\left|v^{-\frac{1}{2}}(n_r)X(n,t)-\sum_{j=1}^{J_k}\xi_{n_rj}^{(k)}e_j^{(k)}(t)\right|\geqslant\varepsilon(2\sigma^2(1)\log\log n_r)^{1/2}\right\}$$

and put

$$Y_{n_r}^{(k)}(t)=v^{-1/2}(n_r)X(n,t)-\sum_{j=1}^{J_k}\xi_{n_rj}^{(k)}e_j^{(k)}(t),\quad 0\leqslant t\leqslant 1.$$

Then $EY_{n_r}^{(k)}(t)=0$ and, noting that

$$E\{X(n,t)\xi_{n_rj}^{(k)}\}=\langle R(*,n,t),e_{n_rj}^{(k)}(*)\rangle_n$$
$$=e_{n_rj}^{(k)}(n,t)=v^{1/2}(n_r)e_j^{(k)}(t),$$

we have

$$\Gamma^{(k)}(s,t):=EY_{n_r}^{(k)}(s)Y_{n_r}^{(k)}(t)=R(s,t)-\sum_{j=1}^{J_k}e_j^{(k)}(s)e_j^{(k)}(t).$$

Since

$$E\{Y_{n_r}^{(k)}(s)-Y_{n_r}^{(k)}(t)\}^2$$
$$=E\{v^{-1/2}(n_r)(X(n,t)-X(n,s))\}^2-\sum_{j=1}^{J_k}\{e_j^{(k)}(t)-e_j^{(k)}(s)\}^2$$
$$\leqslant E\{v^{-1/2}(n_r)\{X(n,t)-X(n,s)\}\}^2$$
$$=R(t,t)-2R(s,t)+R(s,s)$$
$$\leqslant g(|t-s|,1)$$

and

$$|\Gamma^{(k)}(s,t)|\leqslant\{\Gamma^{(k)}(s,s)\}^{1/2}\{\Gamma^{(k)}(t,t)\}^{1/2}$$
$$\leqslant\sup_{0\leqslant t\leqslant 1}\Gamma^{(k)}(t,t),$$

by Fernique's inequality, we have

$$P\{A_r^{(k)}\}\leqslant 4p^2\int_{y_r^{(k)}}^{\infty}e^{-u^2/2}du,$$

where

$$y_r^{(k)}=\varepsilon(2\sigma^2(1)\log\log n_r)^{1/2}\left\{\left(\sup_{0\leqslant t\leqslant 1}\Gamma^{(k)}(t,t)\right)^{1/2}+4\int_1^{\infty}g^{1/2}(p^{-u^2},1)du\right\}^{-1}.$$

We may choose k and p sufficiently large such that

$$\theta:=\varepsilon^2\sigma^2(1)\left\{\left(\sup_{0\leqslant t\leqslant 1}\Gamma^{(k)}(t,t)\right)^{1/2}+4\int_1^{\infty}g^{1/2}(p^{-u^2},1)du\right\}^{-2}>1,$$

since (4.1.9) and

$$\int_1^{\infty}g^{1/2}(p^{-u^2},1)du$$
$$=(\log p)^{-1/2}\int_{(\log p)^{1/2}}^{\infty}g^{1/2}(e^{-u^2},1)du\to 0\quad\text{as}\quad p\to\infty.$$ Then

$$P(A_r^{(k)})\leqslant c(\log C^r)^{-\theta}=cr^{-\theta}$$

and $\sum_r P\{A_r^{(k)}\}<\infty$ for all sufficiently large k. By the Borel-Cantelli lemma we prove Lemma 4.1.2.

Proof of Theorem 4.1.1 Let K^{ε} denote the ε-neighborhood of $\boldsymbol{K}$. To prove that $\boldsymbol{K}$ contains all limit points of $\{\eta_n(t,\omega)\}$ it suffices to show that for arbitrary $\varepsilon>0$ the sequence $\{\eta_n(t,\omega)\}$ ultimately lies in $\boldsymbol{K}^{3\varepsilon}$. Consider a subsequence of indices $\{n_r=[C^r];C>1\}$. Then, for any n, there are n_r and n_{r+1} such that $n_r\leqslant n<n_{r+1}$, and choosing $C=C(\varepsilon)$ sufficiently close to 1, we can make $|1-(n/n_r)|$ arbitrary small. Thus, by Corollary 4.1.1, it is sufficient to show that the subsequence $\{\eta_{n_r}(t,\omega)\}$ ultimately lies

in $\boldsymbol{K}^{2\varepsilon}$. Then, by the remark following Lemma 4.1.2, it suffices to prove that

$$Z(t,\omega,k,n_r) = (2\sigma^2(1)\log\log n_r)^{-1/2}\sum_{j=1}^{k}\xi_{n_r j}(\omega)e_j(t)$$

with a sufficiently large k ultimately lies in $\boldsymbol{K}^{\varepsilon}$. Finally it is enough to show that $\| Z \|_H \leqslant (1+\varepsilon)/\sigma(1)$ ultimately, for then $(1+\varepsilon)^{-1}Z \in \boldsymbol{K}$ and since

$$\begin{aligned}&\| Z(t)-(1+\varepsilon)^{-1}Z(t) \| = \varepsilon(1+\varepsilon)^{-1} \| Z(t) \| \\ &\leqslant \varepsilon(1+\varepsilon)^{-1} \| Z \|_H \sup_{0\leqslant t\leqslant 1} R^{1/2}(t,t) \leqslant \varepsilon,\end{aligned}$$

we have $Z \in \boldsymbol{K}^{\varepsilon}$.

Let

$$\begin{aligned}A_r &= \{ \| Z(\cdot,\omega,k,n_r) \|_H^2 > (1+\varepsilon)^2\sigma^{-1}(1)\} \\ &= \left\{ \| \sum_{j=1}^{k}\xi_{n_r j}(\omega)e_j(\cdot) \|_H^2 > (1+\varepsilon)^2(2\log\log n_r)\right\} \\ &= \left\{ \sum_{j=1}^{k}\{\xi_{n_r j}(\omega)\}^2 > (1+\varepsilon)^2(2\log\log n_r)\right\}.\end{aligned}$$

Let $\chi_k(x)$ denote the distribution function of χ^2-distribution with k degrees of freedom, we have

$$\begin{aligned}P\{A_r\} &= 1-\chi_k((1+\varepsilon)^2(2\log\log n_r)) \\ &\leqslant c\{(1+\varepsilon)^2\log\log n_r\}^{k-1}(\log n_r)^{-(1+\varepsilon)^2} \\ &\leqslant cr^{-(1+\varepsilon)^2},\end{aligned}$$

and hence, by the Borel-Cantelli lemma, $P\{\limsup_r A_r\} = 0$. This concludes the proof of Theorem 4.1.1.

Oodaira (1972) showed that Theorem 4.1.2 holds true for a Gaussian process with conditions (I) and (II') (where $R(s,t) = \int_0^{s\wedge t} Q(s,\lambda)Q(t,\lambda)\,\mathrm{d}\lambda$, $Q(t,\lambda)$ satisfying some conditions), but there is a gap in the proof. Monrad and Rootzen (1995) pointed

out that one can use the Cramer-Wald device (cf. Kuelbs 1976, Theorem 3.1) to prove Theorem 4.1.2 for the fractional Wiener process. The details are omitted here.

4.1.2 Rates of convergence in Strassen's LIL

In this subsection, we summarize the convergence rate to $\boldsymbol{K}$ for various Gaussian processes. First, the best convergence rates to $\boldsymbol{K}$ of Strassen's functional LIL for a standard Wiener process $\{W(t);0\leqslant t<\infty\}$ were due to Grill (1987), which asserted that

$$\lim_{n\to\infty}\sup\inf_{f\in K} \| f(t)-f_n(t) \| (\log\log n)^{\alpha} = \begin{cases}0, & \alpha<2/3,\\ \infty, & \alpha>2/3.\end{cases}$$

The above two statements are equivalent to

$$P\{f_n(t) \in \boldsymbol{K}^{\varepsilon_n} \text{ eventually}\} = 1, \quad \alpha < 2/3$$

and

$$P\{f_n(t) \notin \boldsymbol{K}^{\varepsilon_n}, \text{i.o.}\} = 1, \quad \alpha > 2/3,$$

where $\varepsilon_n = (\log\log n)^{-\alpha}$,

$$\boldsymbol{K}^{\varepsilon_n} = \{g; g\in C[0,1], \inf_{f\in K} \| g(t)-f(t) \| < \varepsilon_n\}.$$

A process $\{X(t);0\leqslant t<\infty\}$ is said to be self-similar with index α, if for each $a>0$ the process $\{X(at);0\leqslant t<\infty\}$ has the same distribution as the process $\{a^{2\alpha}X(t);0\leqslant t<\infty\}$. A self-similar centered Gaussian process $\{Y(t);0\leqslant t<\infty\}$ with stationary increments is a fractional Wiener process. We have

$$Y(t) = V(t) + X(t), \quad t \geqslant 0,$$

where

$$X(t) = \int_0^t (t-s)^{(2\alpha-1)/2}\mathrm{d}W(s), \tag{4.1.11}$$

$$V(t) = \int_{-\infty}^0 \{(t-s)^{(2\alpha-1)/2} - (-s)^{(2\alpha-1)/2}\}\mathrm{d}W(s) \tag{4.1.12}$$

for $0 \leqslant t < \infty$, $0 < \alpha < 1$. When $\alpha = 1/2$, $V = 0$ and hence both $\{X(t); t \geqslant 0\}$ and $\{Y(t); t \geqslant 0\}$ are the Wiener processes. Goodman and Kuelbs (1991a) proved

Theorem 4.1.3 *Let* $\{X(t); t \geqslant 0\}$ *and* $\{Y(t); t \geqslant 0\}$ *be continuous, centered Gaussian processes as in* (4.1.11) *and* (4.1.12), *and set*

$$K = \left\{ f(t) = \int_0^t (t-u)^{(2\alpha-1)/2} g(u) \mathrm{d}u ; 0 \leqslant t \leqslant 1, \int_0^1 g^2(u) \mathrm{d}u \leqslant 1 \right\}, \tag{4.1.13}$$

where $0 < \alpha < 1$.

(A) *If* $\gamma > 0$ *is sufficiently large, then*

$$P\{X(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2} \in K^{\varepsilon_n}\} = 1, \tag{4.1.14}$$

where

$$\varepsilon_n = \begin{cases} \gamma(\log\log\log n/\log\log n)^{2/3} & \text{for} \quad \alpha \geqslant 1/2, \\ \gamma(\log\log\log n/\log\log n)^{(2\alpha+1)/(2\alpha+2)} & \text{for} \quad 0 < \alpha < 1/2, \end{cases}$$

hence

$$\limsup_{n\to\infty} \inf_{f \in K} \| f(t) - \eta_n(t) \| \left(\frac{\log\log\log n}{\log\log n} \right)^{-\theta} = 0,$$

where $\theta > 2/3$, *if* $\alpha \geqslant 1/2$; $\theta > (2\alpha+1)/2(\alpha+1)$, *if* $\alpha < 1/2$.

(B) *If* $0 < \alpha < 1$, $\varepsilon_n = \gamma(\log\log n)^{-1/2}$, *and*

$$K = \left\{ f(t) = T_\alpha g(t) ; 0 \leqslant t \leqslant 1, \int_{-\infty}^1 g^2(u) \mathrm{d}u \leqslant 1 \right\},$$

where

$$T_\alpha g(t) = \int_0^1 (t-u)^{(2\alpha-1)/2} g(u) \mathrm{d}u + \int_{-\infty}^0 ((t-u)^{(2\alpha-1)/2} - (-u)^{(2\alpha-1)/2}) g(u) \mathrm{d}u,$$

then for all $\gamma > 0$

$$P\{Y(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2} \in K^{\varepsilon_n} \text{ eventually}\} = 1, \tag{4.1.15}$$

hence for $\eta_n(\cdot) = Y(n(\cdot))/(2n^{2\alpha}\log\log n)^{1/2}$,

$$\limsup_{n\to\infty} \inf_{f \in K} \| f(t) - \eta_n(t) \| (\log\log n)^\theta = 0$$

for $\theta < 1/2$.

Goodman and Kuelbs (1991a) have also given the convergence rates of Strassen's LIL for the p-parameter Wiener sheet and the p-dimensional Wiener process.

Monrad and Rootzen (1995) have given an exact rate of Strassen's LIL for the fractional Wiener process. Let $H_\alpha \subseteq C[0,1]$ be the r.k. Hilbert space with r.k. function

$$R(s,t) = \{s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}\}/2, \quad 0 \leqslant s,t \leqslant 1$$

and let $\langle f, g \rangle_\alpha$ be the inner product in H_α. If $f \in H_\alpha$, we have

$$|f(t) - f(s)|^2 \leqslant |s-t|^\alpha \langle f, f \rangle_\alpha.$$

Theorem 4.1.4 *Let* $\langle f, f \rangle_\alpha < 1$. *As* $t \downarrow 0$,

$$\liminf (\log\log t)^{(2\alpha+1)/2} \| \eta_t - f \| = \gamma(f), \quad \text{a.s.}$$

where $\gamma(f)$ *is a constant satisfying*

$$2^{-1/2} c^\alpha (1 - \langle f, f \rangle_\alpha)^{-\alpha} \leqslant \gamma(f) \leqslant 2^{-1/2} C^\alpha (1 - \langle f, f \rangle_\alpha)^{-\alpha},$$

for some constants $0 < c < C < \infty$.

Theorem 4.1.5 *As either* $t \downarrow 0$ *or* $t \uparrow \infty$,

$$\liminf (\log\log t)^{(2\alpha+1)/2} \| \eta_t - f \| = \infty, \quad \text{a.s.}$$

if $\langle f, f \rangle_\alpha = 1$, *whereas*

$$\liminf (\log\log t)^{(2\alpha+1)/(2\alpha+2)} \| \eta_t - f \| < \infty, \quad \text{a.s.}$$

if $\langle f, f \rangle_\alpha < 1$.

Kuelbs, Li and Talagrand (1994) investigated lim inf results for Gaussian samples and gave an application to rates of convergence for the functional form of Chung's LIL for Wiener process.

4.1.3 The functional moduli of continuity and the functional limit behavior of the Csörgő-Révész type increments of the Wiener process

Let

$$C_0[0,1] = \{f(x) \in C[0,1]; f(0) = 0\},$$

$$Y_{t,T}(x) = \beta_T(W(t + a_T x) - W(t)),\ 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T - a_T,$$

where $\beta_T = \{2a_T(\log(T/a_T) + \log\log T)\}^{-1/2}$. Let $\boldsymbol{K}$ be defined as in Section 4.1.1. By combining Theorem 0.2 and Theorem S and some precise calculation, Révész (1979) showed the following theorem on the functional limit behavior of the increments.

Theorem R *If a_T is a non-decreasing function of T such that*

(i) $0 < a_T \leqslant T$,

(ii) *T/a_T is non-decreasing,*

then $\{Y_{t,T}; 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T - a_T, T \geqslant 3\}$ is relatively compact in $C_0[0,1]$ with probability one, and the set of its limit points coincides with set $\boldsymbol{K}$.

Furthermore, if we also have

(iii) $\lim_{T\to\infty}(\log(T/a_T))/\log\log T = \infty$,

then

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \inf_{f\in \boldsymbol{K}} \sup_{0\leqslant x\leqslant 1} |Y_{t,T}(x) - f(x)| = 0 \quad \text{a.s.}$$

and for any $f \in \boldsymbol{K}$,

$$\lim_{T\to\infty} \inf_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant x\leqslant 1} |Y_{t,T}(x) - f(x)| = 0 \quad \text{a.s.}$$

Chen B (1998) pointed out that Révész's proof can be shorted by using the large deviations, and he obtained the functional moduli of continuity. Let

$$M_{t,h}(x) = \frac{W(t + hx) - W(t)}{(2h\log h^{-1})^{1/2}}, \quad 0 \leqslant x \leqslant 1.$$

Theorem C *We have*

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \inf_{f\in \boldsymbol{K}} \sup_{0\leqslant x\leqslant 1} |M_{t,h}(x) - f(x)| = 0 \quad \text{a.s.}$$

and for any $f \in \boldsymbol{K}$,

$$\lim_{h\to 0} \inf_{0\leqslant t\leqslant 1-h} \sup_{0\leqslant x\leqslant 1} |M_{t,h}(x) - f(x)| = 0 \quad \text{a.s.}$$

Laterly, by applying the large deviations, Wang Wen-sheng (1999) obtained the rates of convergence on the functional moduli of continuity and the functional limit behavior of the increments. For $f \in C_0[0,1]$, let

$$I(f) = \begin{cases} \int_0^1 (f'(x))^2\,dx, & \text{if } f \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem W.1 *We have*

$$P\left\{\sup_{0\leqslant t\leqslant 1-h} \inf_{f\in \boldsymbol{K}} \sup_{0\leqslant x\leqslant 1} |M_{t,h}(x) - f(x)| \geqslant \gamma\left(\frac{\log\log h^{-1}}{\log h^{-1}}\right)^{2/3} i.o.\right\} = 0$$

for $\gamma > 0$ large enough, and for any $f \in \boldsymbol{K}$,

$$\lim_{h\to 0} \inf_{0\leqslant t\leqslant 1-h} \sup_{0\leqslant x\leqslant 1} |M_{t,h}(x) - f(x)| \log h^{-1} = \begin{cases} \dfrac{\pi}{4\sqrt{1 - I(f)}} & \text{if } I(f) < 1, \\ \infty & \text{if } I(f) = 1, \end{cases} \quad \text{a.s.}$$

Theorem W.2 *Let a_T be defined as in Theorem R. Then for $\gamma > 0$ large enough,*

$$P\left\{\sup_{0\leqslant t\leqslant T-a_T} \inf_{f\in \boldsymbol{K}} \sup_{0\leqslant x\leqslant 1} |Y_{t,T}(x) - f(x)| \geqslant \gamma\left(\frac{\log g(T)}{g(T)}\right)^{2/3} i.o.\right\} = 0,$$

where $g(T) = \log(T/a_T) + \log\log T$, and for any $f \in \boldsymbol{K}$,

$$\liminf_{T\to\infty}\inf_{0\le t\le T-a_T}\sup_{0\le x\le 1}|Y_{t,T}(x)-f(x)|g(T)$$
$$=\begin{cases}\dfrac{\pi}{4\sqrt{1-I(f)}} & \text{if } I(f)<1,\\ \infty & \text{if } I(f)=1,\end{cases}\quad \text{a.s.}$$

Furthermore, if the condition (iii) *in Theorem R is also satisfied, then the* lim inf *can be replaced by* lim.

Wei (1999) obtained results similar to Theorems R and C for l^p-valued Wiener processes under Hölder norm.

4.2 Erdös-Révész's Law of the Iterated Logarithm for Gaussian Processes

Erdös and Révész (1990) established a new type of law of the logarithm for a Wiener process. Later, Shao (1992) extended their results to more general Gaussian processes. We discuss this kind of law of the iterated logarithm in this section.

4.2.1 Erdös-Révész's law of the iterated logarithm for a Wiener process

Let $\{W(t);t\geqslant 0\}$ be a standard Wiener process and define

$$\eta(t)=\sup\{s;0\leqslant s\leqslant t,W(s)\geqslant(2s\log\log s)^{1/2}\},\quad t\geqslant 0,$$
$$\eta_\delta(t)=\sup\{s;0\leqslant s\leqslant t,W(s)\geqslant(2(1-\delta)s\log\log s)^{1/2}\},$$
$$t\geqslant 0,\ 0\leqslant\delta<1,$$
$$\eta_\delta^{(p)}(t)=\sup\{s;0\leqslant s\leqslant t,W(s)\geqslant s^{1/2}\alpha(\delta,p,s)\},\quad t\geqslant 0,$$

where

$$\alpha(\delta,p,s)=\left(2\left(\log_2 s+\frac{3}{2}\log_3 s+\sum_{j=4}^{p}\log_j s-\delta\log_p s\right)\right)^{1/2},\delta\geqslant 0,$$

$p=3,4,\cdots$, $\log_j x=\log_1(\log_{j-1}x)$, $\log_1 x=\ln x$ for $x>0$ and $\log_1 x=1$ if $x\leqslant 0$. Here and in the sequel of this section $\log x=\ln x$. It is clear that by the law of the iterated logarithm

$$\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}\eta_\delta(t)=\lim_{t\to\infty}\eta_\delta^{(p)}(t)=\infty\quad\text{a.s.}$$

and

$$\limsup_{t\to\infty}\frac{\eta(t)}{t}=\limsup_{t\to\infty}\frac{\eta_\delta(t)}{t}=\limsup_{t\to\infty}\frac{\eta_\delta^{(p)}(t)}{t}=1\quad\text{a.s.}$$

Erdös and Révész (1990) considered the lower bound of $\eta(t)$ and obtained a new law of the iterated logarithm:

$$\liminf_{t\to\infty}\frac{(\log_2 t)^{1/2}}{\log_3 t\cdot\log t}\log\frac{\eta(t)}{t}=-C_0\quad\text{a.s.}$$

for some constant C_0 with $1/4\leqslant C_0\leqslant 2^{14}$. At the end of their paper, Erdös and Révész proposed the challenging question of finding the lower bound for the general processes $\eta_\delta(t)$ and $\eta_\delta^{(p)}(t)$.

Shao (1994) found the exact value of C_0 and the exact lower bounds of $\eta_\delta(t)$ and $\eta_\delta^{(p)}(t)$ as well.

Theorem 4.2.1 *We have*

$$\liminf_{t\to\infty}\frac{(\log_2 t)^{1/2}}{\log_3 t\cdot\log t}\log\frac{\eta(t)}{t}=-3\sqrt{\pi}\ \text{a.s.},\qquad(4.2.1)$$

$$\liminf_{t\to\infty}(\log t)^{\delta-1}(\log_2 t)^{-1/2}\log\frac{\eta_\delta(t)}{t}=-2\delta\sqrt{\pi/(1-\delta)}\ \text{a.s.}\qquad(4.2.2)$$

for each $0<\delta\leqslant 1/2$.

Theorem 4.2.2 *We have*

$$\liminf_{t\to\infty}\frac{\log_p\eta_0^{(p)}(t)-\log_p t}{\log_{p+1}t}=-2\sqrt{\pi}\ \text{a.s.},\qquad(4.2.3)$$

$$\liminf_{t\to\infty}\frac{\log_{p-1}\eta_\delta^{(p)}(t)-\log_{p-1}t}{\log_{p+1}t}=-2\delta\sqrt{\pi}\ \text{a.s.}\quad \textit{for } 0<\delta<1, \tag{4.2.4}$$

$$\liminf_{t\to\infty}\frac{\log_{p-2}\eta_\delta^{(p)}-\log_{p-2}t}{\log_{p-2}t(\log_{p-1}t)^{1-\delta}\log_p t}=-2\delta\sqrt{\pi}\ \text{a.s.},\quad \delta>1 \tag{4.2.5}$$

for $p=3,4,\cdots$.

Remark 4.2.1 Theorem 4.2.1 says that for any t big enough, between

$$t^{1-3\sqrt{\pi}\log_3 t\cdot(\log_2 t)^{-1/2}}\quad\text{and}\quad t$$

there exists an s such that $W(s)\geqslant(2s\log\log s)^{1/2}$. The meaning of Theorem 4.2.2 can be interpreted in the same way.

The proofs of the above theorems are omitted here.

Put

$$\bar\eta(t)=\sup\{s;1\leqslant s\leqslant t,W(s)\geqslant(2s\log\log s)^{1/2}\}\quad\text{for } t\geqslant1,$$

$$\hat\eta(t)=\sup\left\{s;0\leqslant s\leqslant t,\frac{W(e^s)}{e^{s/2}}\geqslant(2\log s)^{1/2}\right\}\quad\text{for } t\geqslant0.$$

It is easy to see that

$$\hat\eta(t)=\log\bar\eta(e^t)\quad\text{a.s.}\quad\text{for every } t\geqslant0$$

and hence, by Theorem 4.2.1 we have

$$\liminf_{t\to\infty}\frac{(\log t)^{1/2}}{t\cdot\log_2 t}(\hat\eta(t)-t)=-3\sqrt{\pi}\quad\text{a.s.}$$

Clearly, $\{W(e^s)/e^{s/2};s\geqslant0\}$ is an Ornstein-Uhlenbeck process, a stationary Gaussian process. So, it is natural to study the corresponding problem to $\hat\eta(t)$ for general stationary Gaussian processes.

4.2.2 An Erdös and Révész's type LIL for a Gaussian process

Let $\{X(t);t\geqslant0\}$ be a separable stationary Gaussian process with $EX(t)=0$ and $EX^2(t)=1$ for each $t\geqslant0$. Denote the correlation function

$$r(t)=EX(t+s)X(s)\quad\text{for } s\geqslant0 \text{ and } t\geqslant0.$$

Consider the process

$$\xi(t)=\sup\{s;0\leqslant s\leqslant t,X(s)\geqslant(2\log s)^{1/2}\},\quad t\geqslant0. \tag{4.2.6}$$

The upper class of a law of the iterated logarithm implies

$$P\{X(s)\geqslant(2\log s)^{1/2},\text{i.o.}\}=1$$

under certain condition on $r(t)$ (cf. Qualls and Watanabe 1971). Hence we have

$$\lim_{t\to\infty}\xi(t)=\infty\quad\text{a.s.}$$

and

$$\limsup_{t\to\infty}(\xi(t)-t)=0\quad\text{a.s.}$$

Shao (1992) obtained an Erdös-Révész's type law of the iterated logarithm for the stationary Gaussian process $X(t)$ as follows:

Theorem 4.2.3 *Assume that the following conditions are satisfied:*

$$r(t)=1-C|t|^\alpha+o(|t|^\alpha)\ \textit{as } t\to0\ \textit{for some } C>0,\ 0<\alpha<2, \tag{4.2.7}$$

$$r(t)=O(t^{-2\gamma})\ \textit{as } t\to\infty\ \textit{for some } \gamma>0, \tag{4.2.8}$$

$$\sup_{t\geqslant s}|r(t)|<1\ \textit{for each } s>0. \tag{4.2.9}$$

Then

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t(\log t)^{(\alpha-2)/(2\alpha)}\log_2 t}=-\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_\alpha(2C)^{1/\alpha}}\ \text{a.s.}\ \textit{if } 0<\alpha<2, \tag{4.2.10}$$

$$\liminf_{t\to\infty}\frac{\log(\xi(t)/t)}{\log_2 t}=-\frac{2\sqrt{\pi}}{H_2\sqrt{2C}}\ \text{a.s. } if\ \alpha=2, \quad (4.2.11)$$

where $0<H_\alpha := \lim_{T\to\infty}T^{-1}\int_0^\infty e^s P\{\sup_{0\leqslant t\leqslant T}Y(t)>s\}\,ds<\infty$, *and* $Y(t)$ *is a non-stationary Gaussian process with mean* $EY(t) = -|t|^\alpha$ *and covariance function* $\mathrm{Cov}(Y(s),Y(t))=-|t-s|^\alpha+|s|^\alpha+|t|^\alpha$.

The proof of Theorem 4.2.3 needs the following lemmas. The first one is a version of Lemma 2.4.3.

Lemma 4.2.1 *Suppose* $\xi_1,\cdots,\xi_n$ *are standard normal variables with covariance matrix* $\Lambda^1=(\Lambda_{ij}^1)$ *and* $\eta_1,\cdots,\eta_n$ *similar with covariance matrix* $\Lambda^0=(\Lambda_{ij}^0)$, *and let* $\rho_{ij}=\max(|\Lambda_{ij}^1|,|\Lambda_{ij}^0|)$. *Further, let* $u_1,\cdots,u_n$ *be real numbers. Then*

$$P(\bigcap_{j=1}^n\{\xi_j\leqslant u_j\})-P(\bigcap_{j=1}^n\{\eta_j\leqslant u_j\})$$
$$\leqslant\frac{1}{2\pi}\sum_{1\leqslant i<j\leqslant n}(\Lambda_{ij}^1-\Lambda_{ij}^0)^+(1-\rho_{ij}^2)^{-1/2}\exp\left(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\right).$$

This is Theorem 4.2.1 in the monograph of Leadbetter et al. (1983).

By (1.1.1) and Theorem 1.1.2 we have the following lemma.

Lemma 4.2.2 *If* (4.2.7) *and* (4.2.9) *are satisfied, then*

$$\lim_{x\to\infty}\frac{P\{\sup_{0\leqslant s\leqslant 1}X(s)>x\}}{x^{2/\alpha}\psi(x)}=C^{1/\alpha}H_\alpha, \qquad (4.2.12)$$

$$\lim_{x\to\infty}\frac{P\{\max_{0\leqslant j\leqslant x^{2/\alpha}/\theta}X(j\theta x^{-2/\alpha})>x\}}{x^{2/\alpha}\psi(x)}=C^{1/\alpha}\frac{H_\alpha(\theta)}{\theta} \qquad (4.2.13)$$

for each $\theta>0$, *and*

$$\lim_{\theta\to 0}\frac{H_\alpha(\theta)}{\theta}=H_\alpha, \qquad (4.2.14)$$

where H_α *is defined as in Theorem* 4.2.3 *and*

$$\psi(x)=(2\pi)^{-1/2}x^{-1}e^{-x^2/2}.$$

Lemma 4.2.3 *If the conditions of Theorem* 4.2.3 *are satisfied, then there exist constants* K_0 *and* x_0 *such that*

$$P\{\max_{0\leqslant j\leqslant x^{2/\alpha}/\theta}X(j\theta x^{-2/\alpha})\leqslant x-\theta^{\alpha/4}/x,\ \sup_{0\leqslant s\leqslant 1}X(s)>x\}$$
$$\leqslant K_0x^{2/\alpha}\psi(x)\theta^{\frac{2}{\alpha}-1}\exp(-\theta^{-\alpha/4}/K_0) \qquad (4.2.15)$$

for each $\theta>0$ *and* $x\geqslant x_0$.

This can be obtained from the proof of Lemma 12.2.5 in the monograph of Leadbetter et al. (1983).

Put

$$\tilde{\alpha}=(2+\alpha)/(2\alpha),\ \hat{\alpha}=(2-\alpha)/(2\alpha)\ \text{for } 0<\alpha\leqslant 2. \qquad (4.2.16)$$

Lemma 4.2.4 *Under the conditions of Theorem* 4.2.3, *for each* $0<\varepsilon<1$, *there exist positive constants* N *and* ρ *depending only on* ε, α *and* γ *such that*

$$P\left\{\sup_{a\leqslant s\leqslant b}\frac{X(s)}{(2\log s)^{1/2}}\leqslant 1\right\}$$
$$\leqslant\exp\left(-\frac{(1-\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}}(\log^{\tilde{\alpha}}b-\log^{\tilde{\alpha}}(a+1))\right)+Na^{-\rho} \qquad (4.2.17)$$

for each $b\geqslant a+1\geqslant N$.

Proof By Lemma 4.2.2, there exist $\theta>0$ and x_0 such that

$$P\{\max_{0\leqslant j\leqslant x^{2/\alpha}/\theta}X(j\theta x^{-2/\alpha})\leqslant x\}\geqslant(1-\varepsilon)H_\alpha C^{1/\alpha}x^{2/\alpha}\psi(x) \quad (4.2.18)$$

for each $x\geqslant x_0$. Put

$$b(a,\varepsilon)=[(b-a-1)/(1+\varepsilon)],$$

$$x_i=(2\log(a+1+i(1+\varepsilon)))^{1/2},\ \hat{x}_i=[x_i^{2/\alpha}/\theta],\ i=0,1,\cdots.$$

Then

$$P\left\{\sup_{a\leqslant s\leqslant b}\frac{X(s)}{(2\log s)^{1/2}}\leqslant 1\right\}$$

$$\leqslant P\left\{\max_{0\leqslant i\leqslant b(a,\varepsilon)}\sup_{a+i(1+\varepsilon)\leqslant s\leqslant a+1+i(1+\varepsilon)}\frac{X(s)}{(2\log(a+1+i(1+\varepsilon)))^{1/2}}\leqslant 1\right\}$$

$$\leqslant P\left\{\max_{0\leqslant i\leqslant b(a,\varepsilon)}\max_{0\leqslant j\leqslant \hat{x}_i}\frac{X(a+i(1+\varepsilon)+j\theta x_i^{-2/\alpha})}{x_i}\leqslant 1\right\}.$$

Let

$X_i=\{X(a+i(1+\varepsilon)+j\theta_i x_i^{-2/\alpha});0\leqslant j\leqslant \hat{x}_i\}$, $i=0,1,\cdots,b(a,\varepsilon)$, $\{Y_i;0\leqslant i\leqslant b(a,\varepsilon)\}$ be independent normal random vectors and Y_i and X_i have the same distribution for each $0\leqslant i\leqslant b(a,\varepsilon)$. Applying Lemma 4.2.1 yields

$$P\left\{\max_{0\leqslant i\leqslant b(a,\varepsilon)}\max_{0\leqslant j\leqslant \hat{x}_i}\frac{X(a+i(1+\varepsilon)+j\theta x_i^{-2/\alpha})}{x_i}\leqslant 1\right\}$$

$$\leqslant \prod_{i=0}^{b(a,\varepsilon)}P\{\max_{0\leqslant i\leqslant \hat{x}_i}X(a+i(1+\varepsilon)+j\theta x_i^{-2/\alpha})\leqslant x_i\}$$

$$+\sum_{0\leqslant i<j\leqslant b(a,\varepsilon)}\sum_{u=0}^{\hat{x}_i}\sum_{v=0}^{\hat{x}_j}\frac{|r(i,j,u,v)|}{\sqrt{1-r^2(i,j,u,v)}}$$

$$\times\exp\left(-\frac{(x_i^2+x_j^2)/2}{1+|r(i,j,u,v)|}\right)$$

$$=:I_1+I_2,\tag{4.2.19}$$

where $r(i,j,u,v)=r((j-i)(1+\varepsilon)+v\theta x_j^{-2/\alpha}-u\theta x_i^{-2/\alpha})$. Put

$$r^*(s)=\sup_{t\geqslant s}|r(t)|,\quad s>0.$$

Noting that

$$(j-i)(1+\varepsilon)+v\theta x_j^{-2/\alpha}-u\theta x_i^{-2/\alpha}\geqslant(j-i)\varepsilon\geqslant\varepsilon$$

for every $j>i$, $0\leqslant u\leqslant \hat{x}_i$, $0\leqslant v\leqslant \hat{x}_j$, we have

$$|r(i,j,u,v)|\leqslant r^*((j-i)\varepsilon)\leqslant r^*(\varepsilon)<1\tag{4.2.20}$$

by (4.2.9). From (4.2.8) it follows that there is a t_0 such that

$$r^*(t)\leqslant t^{-\gamma}\leqslant\min(1,\gamma)/4\quad\text{for every } t\geqslant t_0.\tag{4.2.21}$$

We have

$$I_2\leqslant\frac{4}{\theta^2\sqrt{1-r^*(\varepsilon)}}\left(\sum_{0\leqslant i<j\leqslant t_0/\varepsilon}+\sum_{j-i\geqslant t_0/\varepsilon,j\leqslant b(a,\varepsilon)}\right)$$

$$\times x_i^{2/\alpha}x_j^{2/\alpha}r^*((j-i)\varepsilon)\exp\left(-\frac{x_i^2+x_j^2}{2(1+r^*((j-i)\varepsilon))}\right)$$

$$\leqslant c\left\{\sum_{0\leqslant i\leqslant b(a,\varepsilon)}x_i^{4/\alpha}\exp\left(-\frac{x_i^2}{1+r^*(\varepsilon)}\right)\right.$$

$$\left.+\sum_{j-i>t_0/\varepsilon,j\leqslant b(a,\varepsilon)}x_i^{2/\alpha}x_j^{2/\alpha}(j-i)^{-\gamma}\exp\left(-\frac{x_i^2+x_j^2}{2(1+\gamma/4)}\right)\right\}$$

$$\leqslant c\left\{\sum_{i=0}^{\infty}(a+i(1+\varepsilon))^{-2/(1+r^*(\varepsilon))}\log^{2/\alpha}(a+i)\right.$$

$$+\sum_{j-i>t_0/\varepsilon,j\leqslant b(a,\varepsilon)}(a+i)^{-\frac{1}{1+\gamma/4}}(a+j)^{-\frac{1}{1+\gamma/4}}$$

$$\left.\times\log^{1/\alpha}(a+i)\log^{1/\alpha}(a+j)\right\}$$

$$\leqslant c\left\{a^{-\frac{1-r^*(\varepsilon)}{4}}+\sum_{i=0}^{\infty}\frac{\log^{1/\alpha}(a+i)}{(a+i)^{1+\gamma/4}}\right\}$$

$$\leqslant c(a^{-(1-r^*(\varepsilon))/4}+a^{-\gamma/6})$$

$$\leqslant ca^{-\rho},$$

where $\rho=\min\{(1-r^*(\varepsilon))/4,\gamma/6\}$.

Noting that $X(\cdot)$ is a stationary process, we derive from (4.2.18) that

$$I_1=\prod_{i=0}^{b(a,\varepsilon)}P\{\max_{0\leqslant j\leqslant\hat{x}_i}X(j\theta x_i^{-2/\alpha})\leqslant x_i\}$$

$$\leqslant\exp\left\{-\sum_{i=0}^{b(a,\varepsilon)}P(\max_{0\leqslant j\leqslant\hat{x}_i}X(j\theta x_i^{-2/\alpha})>x_i)\right\}$$

$$\leqslant\exp\left(-\sum_{i=0}^{b(a,\varepsilon)}(1-\varepsilon)C^{1/\alpha}H_\alpha x_i^{2/\alpha}\psi(x_i)\right)$$

$$=\exp\left(-(1-\varepsilon)C^{1/\alpha}H_\alpha\sum_{i=0}^{b(a,\varepsilon)}\frac{(2\log(a+1+i(1+\varepsilon)))^{\hat{\alpha}}}{\sqrt{2\pi}(a+1+i(1+\varepsilon))}\right)$$

$$=\exp\left(-\frac{(1-\varepsilon)(2C)^{1/\alpha}H_\alpha}{2\sqrt{\pi}}\sum_{i=0}^{b(a,\varepsilon)}\frac{\log^{\hat{\alpha}}(a+1+i(1+\varepsilon))}{a+1+i(1+\varepsilon)}\right)$$

provided that a is sufficiently large, where $\hat{\alpha}$ is defined as in (4.2.16). An elementary calculation implies

$$\sum_{i=0}^{b(a,\varepsilon)} \frac{\log^{\tilde{a}}(a+1+i(1+\varepsilon))}{a+1+i(1+\varepsilon)}$$
$$\geqslant \int_0^{\frac{b-a-1}{1+\varepsilon}} \frac{\log^{\tilde{a}}(a+1+y(1+\varepsilon))}{a+1+y(1+\varepsilon)}\mathrm{d}y$$
$$= \frac{2\alpha}{(1+\varepsilon)(2+\alpha)}(\log^{\tilde{a}} b - \log^{\tilde{a}}(a+1)).$$

Hence

$$I_1 \leqslant \exp\left(-\frac{(1-\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}}(\log^{\tilde{a}} b - \log^{\tilde{a}}(a+1))\right). \quad (4.2.22)$$

Putting the above inequalities together yields that (4.2.17) holds true.

Lemma 4.2.5 *Under the conditions of Theorem 4.2.3, for each $0<\varepsilon<1$, there exist positive constants N and τ depending only on ε, α and γ such that*

$$P\{\bigcap_{0\leqslant i\leqslant b-a}(\max_{0\leqslant j\leqslant y_i^{2/\alpha}/\theta_i} X(a+i+j\theta_i y_i^{-2/\alpha}) < y_i - \theta_i^{\alpha/4}/y_i)\}$$
$$\geqslant \frac{1}{4}\exp\left(-\frac{(1+\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(2+\alpha)\sqrt{\pi}}(\log^{\tilde{a}} b - \log^{\tilde{a}} a)\right) - Na^{-\tau}$$
$$(4.2.23)$$

for each $b\geqslant a+1\geqslant N$, where

$$y_i = (2\log(a+i))^{1/2},\ \theta_i = \log^{-8/\alpha}(a+i),$$

$\tilde{\alpha}$ is as in (4.2.16).

Proof Put

$$\bar{y}_i = y_i - \theta_i^{\alpha/4}/y_i,\quad \hat{y}_i = [y_i^{2/\alpha}/\theta_i],\quad i=0,1,\cdots.$$

Applying Lemma 4.2.1, we have

$$P\{\bigcap_{0\leqslant i\leqslant b-a}(\max_{0\leqslant j\leqslant y_i^{2/\alpha}/\theta_i} X(a+i+j\theta_i y_i^{-2/\alpha}) < y_i - \theta_i^{\alpha/4}/y_i)\}$$
$$\geqslant \prod_{i=0}^{[b-a]} P\{\max_{0\leqslant j\leqslant \hat{y}_i} X(a+i+j\theta_i y_i^{-2/\alpha}) < \bar{y}_i\}$$

$$-\frac{1}{2\pi}\sum_{0\leqslant i<j\leqslant b-a}\sum_{u=0}^{\hat{y}_i}\sum_{v=0}^{\hat{y}_j}\frac{(\tau(i,j,u,v))^+}{\sqrt{1-\tau^2(i,j,u,v)}}$$
$$\times\exp\left(-\frac{(\bar{y}_i^2+\bar{y}_j^2)/2}{1+|\tau(i,j,u,v)|}\right)$$
$$=: J_1 - J_2, \quad (4.2.24)$$

where $\tau(i,j,u,v) = -r(j-i+v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha})$.

Clearly, for $j\geqslant i+2$, $0\leqslant u\leqslant \hat{y}_i$ and $0\leqslant v\leqslant \hat{y}_i$, by (4.2.9)

$$|\tau(i,j,u,v)| \leqslant r^*(j-i-1) \leqslant r^*(1) < 1.$$

On the other hand, by (4.2.7), there exists a constant $0<t_1<1$ such that

$$r(t) \geqslant 1 - C|t|^\alpha/2 > 0 \quad \text{for every } 0\leqslant t\leqslant t_1. \quad (4.2.25)$$

Hence

$$(\tau(i,j,u,v))^+ = 0,\quad \text{if } j=i+1,\ 1+v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha} \leqslant t_1 \quad (4.2.26)$$

and

$$|\tau(i,j,u,v)| \leqslant r^*(t_1) < 1,$$
$$\text{if } j=i+1,\ 1+v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha} > t_1. \quad (4.2.27)$$

Therefore, by (4.2.25), (4.2.26) and (4.2.27) we obtain

$$J_2 \leqslant \sum_{0\leqslant i\leqslant b-a, j=i+1}\sum_{u=0}^{\hat{y}_i}\sum_{v=0}^{\hat{y}_j}\frac{1}{\sqrt{1-r^*(t_1)}}\exp\left(-\frac{(\bar{y}_i^2+\bar{y}_j^2)/2}{1+r^*(t_1)}\right)$$
$$+\sum_{0\leqslant i+2\leqslant j\leqslant b-a}\sum_{u=0}^{\hat{y}_i}\sum_{v=0}^{\hat{y}_j}\frac{r^*(j-i-1)}{\sqrt{1-r^*(1)}}$$
$$\times\exp\left(-\frac{(\bar{y}_i^2+\bar{y}_j^2)/2}{1+r^*(j-i-1)}\right).$$

Completely similar to the estimation of I_2 in the proof of Lemma 4.2.4, we can arrive that there exist positive constants K and τ such that

$$J_2 \leqslant Ka^{-\tau} \quad (4.2.28)$$

for every a sufficiently large. Using Lemma 4.2.2, we can also obtain that

$$J_1 = \prod_{i=0}^{[b-a]} P\{\max_{0\le j\le \bar{y}_i} X(j\theta_i y_i^{-2/\alpha}) < \bar{y}_i\}$$

$$\ge \prod_{i=0}^{[b-a]} (1 - P\{\sup_{0\le s\le 1} X(s) \ge \bar{y}_i\})$$

$$\ge \frac{1}{4}\exp\left(- \frac{(1+\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(2+\alpha)\sqrt{\pi}}(\log^{\hat{a}} b - \log^{\hat{a}} a)\right) \tag{4.2.29}$$

provided that a is sufficiently large, along the same line of the proof of I_1 in Lemma 4.2.4. This proves (4.2.23), by (4.2.24), (4.2.28) and (4.2.29).

Proof of Theorem 4.2.3 We formulate the proof in three steps.

Step 1 Assume $0<\alpha<2$. Then

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t\log^{-\hat{a}}t\cdot\log\log t} \ge -(1+2\varepsilon)^2 C_1 \quad \text{a.s.} \tag{4.2.30}$$

for every $0<\varepsilon<1/4$, where $C_1=(2+\alpha)\sqrt{\pi}/(\alpha H_\alpha(2C)^{1/\alpha})$, $\hat{\alpha}$ is as in (4.2.16).

Proof Put

$$t_k=\exp(k^{1/\hat{a}}),\quad s_k=t_k-(1+2\varepsilon)^2 C_1 t_k\log^{-\hat{a}}t_k\cdot\log\log t_k,$$
$$k=1,2,\cdots.$$

Then

$$\log^{\hat{a}}t_k-\log^{\hat{a}}s_k$$
$$\sim\log^{\hat{a}}t_k-(\log t_k-(1+2\varepsilon)^2 C_1 t_k(\log t_k)^{-\hat{a}}\cdot\log\log t_k)^{\hat{a}}$$
$$\sim\frac{(1+2\varepsilon)^2(2+\alpha)C_1}{2\alpha}\log\log t_k. \tag{4.2.31}$$

Noting that

$$\{\xi(t)\le a\} = \left\{\sup_{a\le s\le t}\frac{X(s)}{(2\log s)^{1/2}}<1\right\}$$

for every $0<a<t$, and using Lemma 4.2.4 and (4.2.31), we obtain that

$$P\left\{\frac{\xi(t_k)-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}\le-(1+2\varepsilon)^2 C_1\right\}$$
$$=P\left\{\sup_{s_k<s\le t_k}\frac{X(s)}{(2\log s)^{1/2}}<1\right\}$$
$$\le\exp\left(-\frac{(1-\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}}(\log^{-\hat{a}}t_k-\log^{-\hat{a}}(s_k+1))\right)+Ns_k^{-\rho}$$
$$\le\exp\left(-\frac{(1-\varepsilon)(2C)^{1/\alpha}(1+2\varepsilon)^2C_1H_\alpha}{(1+2\varepsilon)2\sqrt{\pi}}\log\log t_k\right)+2Nt_k^{-\rho}$$
$$\le\exp\left(-\frac{(1+\varepsilon/2)(2+\alpha)\log\log t_k}{2\alpha}\right)+2Nt_k^{-\rho}$$
$$\le 2k^{-(1+\varepsilon/2)}$$

for every k sufficiently large. Hence, by the Borel-Cantelli lemma, we have

$$\liminf_{k\to\infty}\frac{\xi(t_k)-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}\ge-(1+2\varepsilon)^2C_1 \quad \text{a.s.} \tag{4.2.32}$$

Since $\xi(t)$ is a non-decreasing random function of t, for every $t_k\le t\le t_{k+1}$, we have

$$\frac{\xi(t)-t}{t\log^{-\hat{a}}t\cdot\log\log t}\ge\frac{\xi(t_k)-t_{k+1}}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_{k+1}}$$
$$=\frac{\xi(t_k)-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}-\frac{t_{k+1}-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}.$$

An elementary calculasion implies

$$\lim_{k\to\infty}\frac{t_{k+1}-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}=0 \tag{4.2.33}$$

which together with (4.2.32) yields

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t\log^{-\hat{a}}t\cdot\log\log t}=\liminf_{k\to\infty}\frac{\xi(t_k)-t_k}{t_k\log^{-\hat{a}}t_k\cdot\log\log t_k}\quad \text{a.s.} \tag{4.2.34}$$

This proves (4.2.30), by (4.2.32) and (4.2.34).

Step 2 Assume $0<\alpha<2$. Then

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t\log^{-\hat{a}}t\cdot\log\log t}\leqslant-(1-\varepsilon)C_1 \quad \text{a.s.} \tag{4.2.35}$$

for every $0<\varepsilon<(2-\alpha)/8<1/4$.

Proof Let

$b_k=\exp(k^{(1+\varepsilon^2)/\hat{a}})$, $a_k=b_k-(1-2\varepsilon)C_1b_k\log^{-\hat{a}}b_k\cdot\log\log b_k$,

$y_{k,i}=(2\log(a_k+i))^{1/2}$, $\theta_{k,i}=\log^{-8/\alpha}(a_k+i)$,

$\bar{y}_{k,i}=y_{k,i}-\theta_{k,i}^{\alpha/4}/y_{k,i}$, $\hat{y}_{k,i}=y_{k,i}^{2/\alpha}/\theta_{k,i}$,

$$E_k=\{\xi(b_k)\leqslant a_k\}=\left\{\sup_{a_k<s\leqslant b_k}\frac{X(s)}{(2\log s)^{1/2}}<1\right\},$$

$$A_k=\bigcap_{0\leqslant i\leqslant b_k-a_k}\{\max_{0\leqslant j\leqslant\hat{y}_{k,i}}X(a_k+i+j\theta_{k,i}y_{k,i}^{-2/\alpha})\leqslant\bar{y}_{k,i}\}.$$

It suffices to show that

$$P\{E_n,\text{i. o.}\}=1. \tag{4.2.36}$$

Clearly, for $m\geqslant1$

$$\begin{aligned}
P\{\bigcup_{k=m}^{\infty}A_k\}&\leqslant P\{\bigcup_{k=m}^{\infty}E_k\}+\sum_{k=m}^{\infty}P\{A_kE_k^c\}\\
&\leqslant P\{\bigcup_{k=m}^{\infty}E_k\}+\sum_{k=m}^{\infty}\sum_{i=0}^{[b_k-a_k]}P\{\max_{0\leqslant j\leqslant\hat{y}_{k,i}}X(a_k+i+j\theta_{k,i}y_{k,i}^{-2/\alpha})\\
&\qquad\leqslant\bar{y}_{k,i},\ \sup_{0\leqslant s\leqslant1}X(a_k+i+s)\geqslant y_{k,i}\}\\
&=P\{\bigcup_{k=m}^{\infty}E_k\}+\sum_{k=m}^{\infty}\sum_{i=0}^{[b_k-a_k]}P\{\max_{0\leqslant j\leqslant\hat{y}_{k,i}}X(j\theta_{k,i}y_{k,i}^{-2/\alpha}\leqslant\bar{y}_{k,i},\\
&\qquad\sup_{0\leqslant s\leqslant1}X(s)\geqslant y_{k,i}\}\\
&=:P\{\bigcup_{k=m}^{\infty}E_k\}+L_m.
\end{aligned}\tag{4.2.37}$$

Using Lemma 4.2.3, we have, for some $K_0>0$

$$\begin{aligned}
L_m&\leqslant K_0\sum_{k=m}^{\infty}\sum_{i=0}^{[b_k-a_k]}y_{k,i}^{2/\alpha}\psi(y_{k,i})\theta_{k,i}^{\alpha/2-1}\exp\left(-\frac{\theta_{k,i}^{-\alpha/4}}{K_0}\right)\\
&\leqslant c\sum_{k=m}^{\infty}\sum_{i=0}^{[b_k-a_k]}\log^{9\hat{a}}(a_k+i)\cdot(a_k+i)^{-1}\exp\left(-\frac{\log^2(a_k+i)}{K_0}\right)\\
&\leqslant c\sum_{k=m}^{\infty}\sum_{i=0}^{\infty}\log^{9\hat{a}}(a_k+i)\cdot(a_k+i)^{-3}\\
&\leqslant c\sum_{k=m}^{\infty}a_k^{-1}\leqslant cm^{-4}
\end{aligned}$$

provided m is large enough. Therefore

$$\lim_{m\to\infty}L_m=0$$

and

$$\lim_{m\to\infty}P\{\bigcup_{k=m}^{\infty}E_k\}\geqslant\lim_{m\to\infty}P\{\bigcup_{k=m}^{\infty}A_k\}.$$

To finish the proof of (4.2.36), we need only to show that

$$P\{A_n,\text{i. o.}\}=1. \tag{4.2.38}$$

Similar to (4.2.31), we have

$$\log^{\hat{a}}b_k-\log^{\hat{a}}a_k\sim\frac{(1-2\varepsilon)(2+\alpha)C_1}{2\alpha}\log\log b_k.$$

Now from Lemma 4.2.5 it follows that

$$\begin{aligned}
P\{A_k\}&\geqslant\frac{1}{4}\exp\left(-\frac{(1+\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(2+\alpha)\sqrt{\pi}}(\log^{\hat{a}}b_k-\log^{\hat{a}}a_k)\right)-Na_k^{-\tau}\\
&\geqslant\frac{1}{4}\exp\left(-\frac{(1+2\varepsilon)(2C)^{1/\alpha}(1-2\varepsilon)C_1H_\alpha}{2\sqrt{\pi}}\log\log b_k\right)-2Nb_k^{-\tau}\\
&\geqslant k^{-(1-\varepsilon^4)}/8
\end{aligned}$$

for every k sufficiently large. Hence

$$\sum_{k=1}^{\infty}P\{A_k\}=\infty. \tag{4.2.39}$$

If we can prove

$$\liminf_{n\to\infty}\frac{\sum_{1\leqslant k\neq l\leqslant n}P(A_kA_l)}{(\sum_{k=1}^{n}P(A_k))^2}\leqslant 1, \tag{4.2.40}$$

(4.2.38) will be implied by the general form of the Borel-Cantelli lemma (cf. Lemma 2.1.1). We below verify that (4.2.40) is satisfied. It is easy to see that

$$(a_{k+1}-b_k)/(b_{k+1}-b_k)\sim 1. \tag{4.2.41}$$

Applying Lemma 4.2.1, we get that for $k<l$

$$\begin{aligned}P(A_kA_l)&\leqslant P(A_k)P(A_l)\\&\quad+\sum_{i=0}^{[b_k-a_k]}\sum_{j=0}^{[\hat{y}_{k,i}]}\sum_{u=0}^{[b_l-a_l]}\sum_{v=0}^{[\hat{y}_{l,u}]}\frac{\bar{\tau}(i,j,u,v)}{\sqrt{1-\bar{\tau}(i,j,u,v)}}\\&\quad\times\exp\left(-\frac{(\bar{y}_{k,i}^2+\bar{y}_{l,u}^2)/2}{1+\bar{\tau}(i,j,u,v)}\right)\\&=:P(A_k)P(A_l)+C_{k,l},\end{aligned} \tag{4.2.42}$$

where

$$\begin{aligned}\bar{\tau}(i,j,u,v)&=|r(a_l+u+v\theta_{l,u}y_{l,u}^{-2/\alpha}-a_k-i-j\theta_{k,i}y_{k,i}^{-2/\alpha})|\\&\leqslant r^*(a_l-b_k-1)\\&\leqslant r^*((b_l-b_{l-1})/2)\\&\leqslant(b_l-b_{l-1})^{-\gamma}\leqslant\min(1,\gamma)/4\end{aligned}$$

by (4.2.41) and (4.2.8), for every $k,l(k<l)$ sufficiently large. Therefore

$$\begin{aligned}C_{k,l}&\leqslant c\sum_{i=0}^{[b_k-a_k]}\sum_{u=0}^{[b_l-a_l]}\hat{y}_{k,i}\hat{y}_{l,u}(b_l-b_{l-1})^{-\gamma}\\&\quad\times\exp\left(-\frac{\log(a_k+i)+\log(a_l+u)}{1+\gamma/4}\right)\\&\leqslant cb_k^{\gamma/4}\log^{9/\alpha}b_k\cdot b_l^{\gamma/4}\log^{9/\alpha}b_l\cdot(b_l-b_{l-1})^{-\gamma}\\&\leqslant c\exp(-\gamma l^{\alpha/4}/8).\end{aligned}$$

Hence we have

$$\sum_{0\leqslant k<l<\infty}C_{k,l}<\infty. \tag{4.2.43}$$

Now (4.2.40) follows from (4.2.42), (4.2.43) and (4.2.39). This proves (4.2.36) and so does (4.2.35).

Step 3 If $\alpha=2$, then

$$\liminf_{t\to\infty}\frac{\log(\xi(t)/t)}{\log\log t}\geqslant-(1+2\varepsilon)^2C_2\quad\text{a.s.} \tag{4.2.44}$$

and

$$\liminf_{t\to\infty}\frac{\log(\xi(t)/t)}{\log\log t}\leqslant-(1-2\varepsilon)^2C_2\quad\text{a.s.} \tag{4.2.45}$$

for every $0<\varepsilon<1/4$, where $C_2=2\sqrt{\pi}/(H_2\sqrt{2C})=2\pi/\sqrt{2C}$.

Proof Put

$$t_k=e^k,\quad s_k=t_k\exp(-(1+2\varepsilon)^2C_2\log\log t_k),\quad k=1,2,\cdots.$$

Proceeding the same way as that of the proof of (4.2.32), one can obtain that

$$\liminf_{k\to\infty}\frac{\log(\xi(t_k)/t_k)}{\log\log t_k}\geqslant-(1+2\varepsilon)^2C_2\quad\text{a.s.}$$

On the other hand, it is clear that

$$\liminf_{k\to\infty}\frac{\log(\xi(t)/t)}{\log\log t}=\liminf_{k\to\infty}\frac{\log(\xi(t_k)/t_k)}{\log\log t_k}\quad\text{a.s.}$$

since $\lim_{k\to\infty}\log(t_{k+1}/t_k)/\log\log t_k=0$. This proves (4.2.44).

Let

$$b_k=\exp(k^{1+\varepsilon^2}),\quad a_k=b_k\exp(-(1-2\varepsilon)C_2\log\log b_k),$$
$$k=1,2,\cdots.$$

Noting that

$$\frac{a_{k+1}-b_k}{a_{k+1}}\sim 1,$$

along the same line of the proof of (4.2.35), we also have

$$\liminf_{k\to\infty}\frac{\log(\xi(b_k)/b_k)}{\log\log b_k}\leqslant-(1-2\varepsilon)C_2 \quad \text{a. s.}$$

This proves (4. 2. 45).

Now the proof of Theorem 4. 2. 3 is completed.

4. 2. 3 Applications to two special Gaussian processes

In this subsection, we apply Theorem 4. 2. 3 to two special Gaussian processes, infinite series of independent Ornstein-Uhlenbeck processes and the fractional Wiener process (cf. Shao 1992).

Let $Y(t)=(X_1(t),X_2(t),\cdots)$, where $\{X_i(t);-\infty<t<\infty\}$ ($i=1,2,\cdots$) are independent Ornstein-Uhlenbeck processes with coefficients γ_i and λ_i. Assume

$$0<\Gamma_0^2=\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i}<\infty.$$

Define

$$X(t)=\frac{1}{\Gamma_0}\sum_{i=1}^{\infty}X_i(t), \tag{4.2.46}$$

$$\sigma^2(t)=\frac{1}{\Gamma_0^2}\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i t}), \quad t\geqslant 0. \tag{4.2.47}$$

It is well-known that $\{X(t);t\geqslant 0\}$ is a stationary Gaussian process with $EX(t)=0$, $EX^2(t)=1$ and covariance function

$$r(t)=EX(t+s)X(s)=1-\sigma^2(t) \quad \text{for } s,t\geqslant 0.$$

Theorem 4. 2. 4 *Let* $\{X(t);t\geqslant 0\}$ *be defined as in* (4. 2. 46). *Put* $\xi(t)=\sup\{s;0\leqslant s\leqslant t,X(s)\geqslant(2\log s)^{1/2}\}$, $t\geqslant 0$. *Let* $\sigma^2(t)$ *be as in* (4. 2. 47). *Assume that there exist* $0<\alpha\leqslant 1$, $C>0$ *and* $\delta>0$ *such that*

$$\sum_{i=1}^{\infty}\frac{\gamma_i}{\min(\lambda_i,\lambda_i^{1+\delta})}<\infty, \tag{4.2.48}$$

$$\lim_{t\downarrow 0}\frac{\sigma^2(t)}{t^\alpha}=C. \tag{4.2.49}$$

Then (4. 2. 10) *holds.*

Proof It is easy to find that (4. 2. 9) is satisfied since $\sigma^2(t)$ is a positive non-decreasing function for $t>0$. By Theorem 4. 2. 3, it suffices to verify that (4. 2. 8) is satisfied. Notice that

$$\begin{aligned}r(t)&=\frac{1}{\Gamma_0^2}\Big(\sum_{i;\lambda_i\geqslant t^{-1/2}}+\sum_{i;\lambda_i<t^{-1/2}}\Big)\frac{\gamma_i}{\lambda_i}e^{-\lambda_i t}\\&\leqslant\frac{1}{\Gamma_0^2}\Big(\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i}e^{-t^{1/2}}+\sum_{i;1/\lambda_i>t^{1/2}}\frac{\gamma_i}{\lambda_i}\Big)\\&\leqslant\exp(-t^{1/2})+\frac{1}{\Gamma_0^2}\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i^{1+\delta}}\cdot t^{-\delta/2}\\&\leqslant ct^{-\delta/2}\end{aligned}$$

for every $t\geqslant 2$. This proves (4. 2. 8). The proof of Theorem 4. 2. 4 is completed.

Corollary 4. 2. 1 *Let* $\{X(t);t\geqslant 0\}$ *and* $\{\xi(t);t\geqslant 0\}$ *be defined as in Theorem* 4. 2. 4. *Assume for some* $\delta>0$

$$\sum_{i=1}^{\infty}\frac{\gamma_i}{\min(1,\lambda_i^{1+\delta})}<\infty. \tag{4.2.50}$$

Then

$$\liminf_{t\to\infty}\frac{\xi(t)-t}{t\log^{-1/2}t\cdot\log\log t}=-\frac{3\sqrt{\pi}}{2\Gamma_1} \quad \text{a. s.}, \tag{4.2.51}$$

where $\Gamma_1=\Gamma_0^{-2}\sum_{i=1}^{\infty}\gamma_i$.

Proof Clearly, (4. 2. 50) implies (4. 2. 48) and

$$\sum_{i=1}^{\infty}\gamma_i<\infty.$$

The latter yields

$$\lim_{t\downarrow 0}\frac{\sigma^2(t)}{t}=\Gamma_1.$$

Now (4.2.51) follows from Theorem 4.2.4 with $\alpha=1$, $C=\Gamma_1$, and $H_1=1$.

Let $\{Z(t);t\geqslant 0\}$ be a fractional Wiener process of order α. Consider

$$\eta(t)=\sup\{s;0\leqslant s\leqslant t,Z(s)\geqslant(2s^{2\alpha}\log\log s)^{1/2}\},\quad t\geqslant 0.$$

By the upper class of increments for $Z(t)$ (cf. Grill 1991), one has

$$\lim_{t\to\infty}\eta(t)=\infty\quad\text{a.s.}$$

and

$$\limsup_{t\to\infty}(\eta(t)-t)=0\quad\text{a.s.}$$

The following theorem presents the lower bound of $\eta(\cdot)$.

Theorem 4.2.5 *We have*

$$\liminf_{t\to\infty}\frac{(\log\log t)^{(1-\alpha)/(2\alpha)}\cdot\log(\eta(t)/t)}{\log t\cdot\log\log\log t}=-\frac{(1+\alpha)\sqrt{\pi}}{\alpha H_{2\alpha}}\quad\text{a.s.}\tag{4.2.52}$$

Proof Let

$$X(t)=Z(e^t)/e^{\alpha t},\quad t\geqslant 0.$$

Then, $EX(t)=0$, $EX^2(t)=1$ and the correlation function

$$r(t)=(e^{\alpha t}+e^{-\alpha t}-(e^{t/2}-e^{-t/2})^{2\alpha})/2\quad\text{for } t\geqslant 0.$$

It is easy to find that

$$r(t)-1\sim t^{2\alpha}/2\quad\text{as}\quad t\downarrow 0,$$
$$r(t)\sim e^{-\alpha t}+2\alpha e^{-(1-\alpha)t}\quad\text{as}\quad t\to\infty$$

and

$$\sup_{t\geqslant s}|r(t)|<1\quad\text{for every } s>0.$$

Therefore, by Theorem 4.2.3

$$\liminf_{t\to\infty}\frac{(\log t)^{(1-\alpha)/(2\alpha)}(\xi(t)-t)}{t\log\log t}=-\frac{(1+\alpha)\sqrt{\pi}}{\alpha H_{2\alpha}}\quad\text{a.s.}\tag{4.2.53}$$

where $\xi(t)$ is defined as in (4.2.6). Let

$$\eta(t)=\sup\{s;1\leqslant s\leqslant t,Z(s)\geqslant(2s^{2\alpha}\log\log s)^{1/2}\}\quad\text{for } t\geqslant 1.$$

Then

$$\xi(t)=\log\bar{\eta}(e^t)\quad\text{a.s. for every } t\geqslant 0.\tag{4.2.54}$$

Consequently, we have, by (4.2.53)

$$\liminf_{t\to\infty}\frac{(\log\log)^{(1-\alpha)/(2\alpha)}\log(\bar{\eta}(t)/t)}{(\log t)\log\log\log t}=-\frac{(1+\alpha)\sqrt{\pi}}{\alpha H_{2\alpha}}\quad\text{a.s.}\tag{4.2.55}$$

This proves (4.2.52) by (4.2.55) and the fact that $|\bar{\eta}(t)-\eta(t)|\leqslant 1$ for every $t\geqslant 1$.

Remark 4.2.2 When $\alpha=1/2$, (4.2.52) is just (4.2.1).

4.3 The Small Ball Probability and Chung's Law of the Iterated Logarithm of Gaussian Processes

The small ball probability is a key step in establishing a Chung type law of the iterated logarithm. In this section, we first discuss the small ball probability of a Gaussian process, then apply it to obtain Chung's LIL for a Gaussian process, especially we estimate the small ball probability and show the Chung's LIL for a fractional Wiener process and the infinite se-

ries of Orenstein-Uhlenbeck processes.

4.3.1 The small ball probability of a Gaussian process

Shao (1993) and Monrad and Rootzen (1995) established small ball probabilities for Gaussian processes with stationary increments. We introduce Shao's results here, which has explicit constants.

Theorem 4.3.1 *Let $\{X(t);0\leqslant t\leqslant 1\}$ be a Gaussian process with stationary increments and $EX(t)=0$, $X(0)=0$ a.s. If*

$$\sigma^2(h)=E(X(t+h)-X(t))^2,\quad 0\leqslant t\leqslant t+h\leqslant 1 \tag{4.3.1}$$

is non-decreasing and concave on $(0,1)$, then

$$P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)\}\leqslant 2\exp(-0.17/x) \tag{4.3.2}$$

and

$$P\left\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)+6\mathrm{e}\int_0^\infty\sigma(xe^{-y^2})\mathrm{d}y\right\}\geqslant\exp(-1.87/x) \tag{4.3.3}$$

for every $0<x<1$.

Particularly, if $\sigma^2(x)$ is concave and $\sigma(x)/x^\alpha$ is non-decreasing on $(0,1)$ for some $\alpha>0$, then

$$P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant c_\alpha\sigma(x)\}\geqslant\exp(-1.87/x), \tag{4.3.4}$$

where $c_\alpha=1+3\mathrm{e}\sqrt{\pi/\alpha}$.

Corollary 4.3.1 *Let $\{Z(t);0\leqslant t\leqslant 1\}$ be a fractional Wiener process of order $0<\alpha<1/2$. Then we have*

$$\exp(-\theta_\alpha x^{-1/\alpha})\leqslant P\{\sup_{0\leqslant t\leqslant 1}|Z(t)|\leqslant x\}\leqslant 2\exp(-0.17x^{-1/\alpha}) \tag{4.3.5}$$

for every $0<x<1$, where $\theta_\alpha=2(1+3\mathrm{e}\sqrt{\pi/\alpha})^{1/\alpha}$.

Remark 4.3.1 Monrad and Rootzen (1995) showed that the inequalities similar to (4.3.5) hold for all cases, i.e. for a fractional Wiener process of order $0<\alpha<1$, there are constants $0<c\leqslant C<\infty$ not dependent on x, such that

$$\exp(-Cx^{-1/\alpha})\leqslant P\{\sup_{0\leqslant t\leqslant 1}|Z(t)|\leqslant x\}\leqslant\exp(-cx^{-1/\alpha}). \tag{4.3.6}$$

See also Theorem 4.4.3 in the next section.

The proof of Theorem 4.3.1 needs the following lemmas.

Lemma 4.3.1 *Let $\{Y(t);t\geqslant 0\}$ be a Gaussian process with mean zero and finite variance. Assume that there exists a non-decreasing function $u(h)$ on $[0,1]$ such that*

$$E(Y(t+h)-Y(t))^2\leqslant u^2(h)\quad \textit{for all } 0\leqslant t\leqslant t+h\leqslant 1. \tag{4.3.7}$$

Then

$$\begin{aligned}&P\left\{\sup_{0\leqslant t\leqslant 1}|Y(t)|\leqslant x+2\mathrm{e}\int_0^\infty u\left(\frac{\mathrm{e}\cdot\mathrm{e}^{-y^2}}{R}\right)\mathrm{d}y\right\}\\&\geqslant\mathrm{e}^{-R}P\left\{\max_{0\leqslant i\leqslant R}\left|Y\left(\frac{i}{R}\right)\right|\leqslant x\right\}\end{aligned} \tag{4.3.8}$$

for every $R\geqslant 1$, $x>0$.

Proof If $\int_0^\infty u(\mathrm{e}\cdot\mathrm{e}^{-y^2}/R)\mathrm{d}y=\infty$, then (4.3.8) is trivial. So we only need to consider the case

$$\int_0^\infty u\left(\frac{\mathrm{e}\cdot\mathrm{e}^{-y^2}}{R}\right)\mathrm{d}y<\infty, \tag{4.3.9}$$

which, in turn, implies that $Y(\cdot)$ is almost surely continuous (cf. Theorem 2.1.3). Let

$$t_j=\left[\frac{tR\cdot\mathrm{e}^{\mathrm{e}^j}}{\mathrm{e}}\right]\frac{\mathrm{e}}{R\cdot\mathrm{e}^{\mathrm{e}^j}},\ j=0,1,2,\cdots.$$

Noting that $Y(\cdot)$ is almost surely continuous, we can write

$$|Y(t)| \leqslant |Y(t_0)| + \sum_{j=0}^{\infty} |Y(t_{j+1}) - Y(t_j)|. \quad (4.3.10)$$

Put

$$x_j = e^{(j+1)/2} u\left(\frac{e \cdot e^{-e^j}}{R}\right), \quad j = 0,1,2,\cdots.$$

We have

$$\begin{aligned}
\sum_{j=0}^{\infty} x_j &\leqslant e^{1/2} u\left(\frac{1}{R}\right) + \sum_{j=1}^{\infty} \int_{j-1}^{j} e^{(y+2)/2} u\left(\frac{e}{R} e^{-e^y}\right) dy \\
&= e^{1/2} u\left(\frac{1}{R}\right) + e\int_0^{\infty} e^{y/2} u\left(\frac{e \cdot e^{-e^y}}{R}\right) dy \\
&= e^{1/2} u\left(\frac{1}{R}\right) + 2e\int_1^{\infty} u\left(\frac{e \cdot e^{-y^2}}{R}\right) dy \\
&\leqslant e^{1/2} \int_0^1 u\left(\frac{e \cdot e^{-y^2}}{R}\right) dy + 2e\int_1^{\infty} u\left(\frac{e \cdot e^{-y^2}}{R}\right) dy \\
&\leqslant 2e\int_0^{\infty} u\left(\frac{e \cdot e^{-y^2}}{R}\right) dy. \qquad (4.3.11)
\end{aligned}$$

Noting that

$$|t_{j+1} - t_j| \leqslant \frac{e \cdot e^{-e^j}}{R} \quad \text{for } j = 0,1,\cdots, \qquad (4.3.12)$$

we obtain from (4.3.10)—(4.3.12) and Theorem 1.2.4′ that

$$\begin{aligned}
&P\left\{\sup_{0\leqslant t\leqslant 1} |Y(t)| \leqslant x + 2e\int_0^{\infty} u\left(\frac{e \cdot e^{-y^2}}{R}\right) dy\right\} \\
&\geqslant P\{\sup_{0\leqslant t\leqslant 1} |Y(t_0)| \leqslant x, \sup_{0\leqslant t\leqslant 1} |Y(t_{j+1}) - Y(t_j)| \leqslant x_j, j = 0,1,\cdots\} \\
&\geqslant P\{\sup_{0\leqslant t\leqslant 1} |Y(t_0)| \leqslant x\} \prod_{j=0}^{\infty} \prod_{0\leqslant t\leqslant 1} P\{|Y(t_{j+1}) - Y(t_j)| \leqslant x_j\} \\
&\geqslant P\{\sup_{0\leqslant t\leqslant 1} |Y(t_0)| \leqslant x\} \\
&\quad \times \prod_{j=0}^{\infty} \left(P\left\{|N(0,1)| \leqslant \frac{x_j}{u(e \cdot e^{-e^j}/R)}\right\}\right)^{(1+Re^{e^{j+1}})/e} \\
&= P\left\{\max_{0\leqslant i\leqslant R} \left|Y\left(\frac{i}{R}\right)\right| \leqslant x\right\} \\
&\quad \times \prod_{j=0}^{\infty} (P\{|N(0,1)| \leqslant e^{(j+1)/2}\})^{(1+Re^{e^{j+1}})/e} \\
&\geqslant P\left\{\max_{0\leqslant i\leqslant R} \left|Y\left(\frac{i}{R}\right)\right| \leqslant x\right\} \prod_{j=0}^{\infty} \left(1 - \frac{4e^{-e^{j+1}}}{3(1+e^{(j+1)/2})}\right)^{\frac{1+Re^{e^{j+1}}}{e}} \\
&\geqslant P\left\{\max_{0\leqslant i\leqslant R} \left|Y\left(\frac{i}{R}\right)\right| \leqslant x\right\} \exp\left(-2\sum_{j=0}^{\infty} \frac{e^{-e^{j+1}}}{1+e^{(j+1)/2}} \frac{1+Re^{e^{j+1}}}{e}\right) \\
&\geqslant P\left\{\max_{0\leqslant i\leqslant R} \left|Y\left(\frac{i}{R}\right)\right| \leqslant x\right\} \exp(-R),
\end{aligned}$$

as desired.

Lemma 4.3.2 *Let* $\{\xi_i; 1\leqslant i\leqslant n\}$ *be Gaussian random variables with mean zero and finite variances. Then for every* $x>0$

$$P\left\{\max_{1\leqslant i\leqslant n} \left|\sum_{j=1}^{i} \xi_j\right| \leqslant x\right\} \geqslant \prod_{i=1}^{n} \sqrt{\frac{1}{2\pi}} \int_0^{2x/\rho_i} e^{-y^2/2} dy,$$

where $\rho_i^2 = \sum_{j=1}^{n} |E\xi_i\xi_j|$.

Proof Let $\{\eta_i; 1\leqslant i\leqslant n\}$ be a sequence of independent random variables with $\eta_i \overset{d}{=} N(0,\rho_i^2)$. Let Σ_1, Σ_1^*, Σ_2 and Σ_2^* be the covariance matrices of $\{\sum_{j=1}^{i} \xi_j; 1\leqslant i\leqslant n\}$, $\{\xi_i; 1\leqslant i\leqslant n\}$, $\{\sum_{j=1}^{i} \eta_i; 1\leqslant i\leqslant n\}$ and $\{\eta_i; 1\leqslant i\leqslant n\}$, respectively. Clearly, $\Sigma_2^* - \Sigma_1^*$ is a real valued matrix with positive diagonal elements and the sum of the absolute values of all the off-diagonal elements in a given row is less than or equal to the diagonal element in that row, that is, $\Sigma_2^* - \Sigma_1^*$ is a matrix with dominant principal diagonal and hence is positive semidefinite. Consequently, $\Sigma_2 - \Sigma_1$ is also positive semidefinite. From Corollary 1.2.5 it follows that

$$P\left\{\max_{1\leqslant i\leqslant n} \left|\sum_{j=1}^{i} \xi_j\right| \leqslant x\right\} \geqslant P\left\{\max_{1\leqslant i\leqslant n} \left|\sum_{j=1}^{i} \eta_j\right| \leqslant x\right\}.$$

Since $\{\eta_i; 1\leqslant i\leqslant n\}$ are independent, we can write

$$P\Big\{\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x\Big\}$$

$$=\int_{-x}^{x}P\{|\eta_n+y|\leqslant x\}\mathrm{d}P\Big\{\max_{1\leqslant i\leqslant n-1}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x, \sum_{j=1}^{n-1}\eta_j<y\Big\}$$

$$\geqslant \inf_{|y|\leqslant x}P\{|\eta_n+y|\leqslant x\}P\Big\{\max_{1\leqslant i\leqslant n-1}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x\Big\}$$

$$=P\{|\eta_n+x|\leqslant x\}P\Big\{\max_{1\leqslant i\leqslant n-1}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x\Big\}$$

$$=P\Big\{\max_{1\leqslant i\leqslant n-1}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x\Big\}\frac{1}{\sqrt{2\pi}}\int_0^{2x/\rho_n}\mathrm{e}^{-y^2/2}\mathrm{d}y$$

and hence, by recurrence

$$P\Big\{\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x\Big\}\geqslant\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\int_0^{2x/\rho_i}\mathrm{e}^{-y^2/2}\mathrm{d}y.$$

This proves Lemma 4. 3. 2.

Proof of Theorem 4. 3. 1 Using Corollary 1. 2. 6 and $\log(2\Phi(\sqrt{2})-1)\leqslant -0.17$, we have

$$P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)\}\leqslant P\{\max_{0\leqslant i\leqslant 1/x}|X(ix)|\leqslant\sigma(x)\}$$

$$\leqslant\prod_{i=1}^{[1/x]}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}}\mathrm{e}^{-t^2/2}\mathrm{d}t$$

$$=(2\Phi(\sqrt{2})-1)^{[1/x]}$$

$$\leqslant 2\exp(-0.17/x).$$

This proves (4. 3. 2). We next prove (4. 3. 3) is true.

Take $R=1/x$ in Lemma 4. 3. 1. We have

$$P\Big\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)+2\mathrm{e}\int_0^{\infty}\sigma(\mathrm{e}\mathrm{e}^{-y^2}x)\mathrm{d}y\Big\}$$

$$\geqslant \mathrm{e}^{-1/x}P\{\max_{1\leqslant i\leqslant 1/x}|X(ix)|\leqslant\sigma(x)\}. \tag{4.3.13}$$

By Lemma 4. 3. 2, we get

$$P\{\max_{1\leqslant i\leqslant 1/x}|X(ix)|\leqslant\sigma(x)\}$$

$$\geqslant\prod_{i=1}^{[1/x]}\frac{1}{\sqrt{2\pi}}\int_0^{2\sigma(x)/\rho_i}\mathrm{e}^{-y^2/2}\mathrm{d}y, \tag{4.3.14}$$

where

$$\rho_i^2=\sum_{j=1}^{[1/x]}|E(X(jx)-X((j-1)x))(X(ix)-X((i-1)x))|,$$

$1\leqslant i\leqslant[1/x]$. Since $\sigma^2(h)$ is concave on $(0,1)$, we obtain that

$$\rho_i^2\leqslant 2E(X(ix)-X((i-1)x))^2=2\sigma^2(x)$$

for $1\leqslant i\leqslant 1/x$. Hence, by $\log((2\Phi(\sqrt{2})-1)/2)\geqslant -0.87$, we have

$$P\{\max_{1\leqslant i\leqslant 1/x}|X(ix)|\leqslant\sigma(x)\}\geqslant\prod_{i=1}^{[1/x]}\Big(\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}}\mathrm{e}^{-y^2/2}\mathrm{d}y\Big)$$

$$\geqslant\exp(-0.87/x).$$

A combination of the above inequalities yields

$$P\Big\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)+2\mathrm{e}\int_0^{\infty}\sigma(\mathrm{e}\cdot x\cdot\mathrm{e}^{-y^2})\mathrm{d}y\Big\}$$

$$\geqslant\exp(-1.87/x). \tag{4.3.15}$$

Noting that for every $0\leqslant h\leqslant 1/3$

$$\sigma(3h)=(EX^2(3h))^{1/2}$$

$$\leqslant(E(X(3h)-X(2h))^2)^{1/2}+(E(X(2h)-X(h))^2)^{1/2}$$

$$+(EX^2(h))^{1/2}$$

$$=3\sigma(h)$$

by Minkowski's inequality, we have

$$\sigma(x\mathrm{e}\cdot\mathrm{e}^{-y^2})\leqslant\sigma(x\cdot 3\mathrm{e}^{-y^2})\leqslant 3\sigma(x\cdot\mathrm{e}^{-y^2})\quad\text{for } y\geqslant 0. \tag{4.3.16}$$

Now (4. 3. 3) follows from (4. 3. 15) and (4. 3. 16) immediately.

4.3.2 Chung's LIL for fractional Wiener processes

By using the small ball probabilities of fractional Wiener processes, i.e. (4.3.6), Monrad and Rootzen (1995) established Chung's LIL for fractional Wiener processes as follows.

Theorem 4.3.2 *Let $\{Z(t);t\geqslant 0\}$ be a fractional Wiener process of order $0<\alpha<1$. Then there are positive constants c_α, c_α' such that*

$$\liminf_{t\to 0}\sup_{0\leqslant s\leqslant t}\frac{|Z(s)|}{t^\alpha(\log\log t^{-1})^{-\alpha}}=c_\alpha \quad \text{a.s.},$$

$$\liminf_{t\to\infty}\sup_{0\leqslant s\leqslant t}\frac{|Z(s)|}{t^\alpha(\log\log t)^{-\alpha}}=c_\alpha' \quad \text{a.s.}$$

The proof of Theorem 4.3.2 needs the following lemmas.

Lemma 4.3.3 *Let $\{X(t);t\geqslant 0\}$ be a separable, centered, real valued Gaussian process with incremental variance $\sigma_t^2(h)=\mathrm{Var}(X(t+h)-X(t))$. Assume that*

$$\sigma_t(h)\leqslant\varphi(h),\ t>0,\ h>0$$

for some continuous non-decreasing function φ with $\varphi(0)=0$. Put $M(t)=\sup_{0\leqslant s\leqslant t}|X(s)-X(0)|$. For any positive integer $k>1$ and any positive constants t, x and $\theta(p)$, $p=1,2,3\cdots$,

$$P\Big\{M(t)>x\varphi(t)+\sum_{p=1}^\infty\theta(p)\varphi(tk^{-2^p})\Big\}$$

$$\leqslant k^2\mathrm{e}^{-x^2/2}+\sum_{p=1}^\infty k^{2^{p+1}}\mathrm{e}^{-\theta(p)^2/2}.$$

This lemma is a version of Fernique's inequality (Theorem 1.1.3).

Let $\{Z(t);t\in\mathbf{R}^d\}$ be a mean zero real or complex valued random field with $Z(0)=0$. We assume that $Z(t)$ has stationary increments and a continuous covariance function

$$R(t,s)=EZ(t)Z(s),\quad t,s\in\mathbf{R}^d.$$

It is known that $R(t,s)$ has a unique Fourier representation

$$\mathbf{R}(t,s)=\int_{\mathbf{R}^d}(\mathrm{e}^{\mathrm{i}t\cdot\lambda}-1)(\mathrm{e}^{\mathrm{i}s\cdot\lambda}-1)\Delta(\mathrm{d}\lambda)+\langle t,Bs\rangle,$$

where $\boldsymbol{B}=(b_{ij})$ is a positive semiindefinite matrix and $\Delta(\mathrm{d}\lambda)$ is a non-negative measure on $\mathbf{R}^d-\{0\}$ satisfying $\int_{\mathbf{R}^d}\frac{|\lambda|^2}{1+|\lambda|^2}\Delta(\mathrm{d}\lambda)<\infty$. For $T>0$, let $\mathscr{H}(T)$ be the Hilbert space of random variables obtained by closing up the space of all finite combinations

$$Z=a_1Z(t_1)+\cdots+a_nZ(t_n),\quad |t_j|\leqslant T$$

in the L^2 norm $\|Z\|=(E|Z|^2)^{1/2}$. Set

$$\mathscr{H}(0)=\bigcap\{\mathscr{H}(T);T>0\}.$$

We say that $X(t)$ satisfies the (wide sense) **zero-one law** at $t=0$ iff $\mathscr{H}(0)=\{0\}$. If $F(T)$ is the σ-field generated by $\mathscr{H}(T)$ and if $\{X(t);t\in\mathbf{R}^d\}$ is a Gaussian field then the wide sense zero-one law at $t=0$ is equivalent to the zero-one law for the "tail" σ-field $\mathscr{F}(0)=\bigcap\{\mathscr{F}(T);T>0\}$.

For $\lambda\in\mathbf{R}^d$ and $h>0$ we let

$$B(\lambda,h)=\{x\in\mathbf{R}^d;|\lambda-x|\leqslant h\}$$

be the closed ball centered at λ of radius h.

Lemma 4.3.4 *If for some $h>0$*

$$\liminf_{|\lambda|\to\infty}|\lambda|^{d+2}\Delta(B(\lambda,h))>0$$

then $Z(t)$ satisfies the zero-one law at $t=0$.

This lemma is due to Pitt and Tran (1979).

Proof of Theorem 4.3.2

Let $\mathbf{Z}=\{Z(t);t\geqslant 0\}$ be a fractional Wiener process of order α, $0<\alpha<1$. Its covariance function has the representation

$$R(s,t)=\frac{1}{2}\{|s|^{2\alpha}+|t|^{2\alpha}-|s-t|^{2\alpha}\}$$

$$= c(\alpha)\int_{-\infty}^{\infty}(e^{is\lambda}-1)(e^{-it\lambda}-1)|\lambda|^{-(2\alpha+1)}d\lambda$$

$$=:\int_{-\infty}^{\infty}(e^{is\lambda}-1)(e^{-it\lambda}-1)\Delta(d\lambda) \qquad (4.3.17)$$

if denote $\Delta(d\lambda)=c(\alpha)|\lambda|^{-(2\alpha+1)}d\lambda$. Thus the symmetric spectral measure Δ satisfies

$$\int_{-\infty}^{\infty}\frac{\lambda^2}{1+\lambda^2}\Delta(d\lambda) = c(\alpha)\int_{-\infty}^{\infty}\frac{d\lambda}{(1+\lambda^2)|\lambda|^{2\alpha-1}} < \infty.$$

There exists a centered, complex valued, Gaussian random measure $W(d\lambda)$ such that

$$Z(t) = \int_{-\infty}^{\infty}(e^{it\lambda}-1)W(d\lambda). \qquad (4.3.18)$$

The measures W and Δ are related by

$$EW(A)\overline{W(B)} = \Delta(A\cap B) = c(\alpha)\int_{A\cap B}|\lambda|^{-(2\alpha+1)}d\lambda$$

for all real Borel sets A and B. Furthermore

$$W(-A) = \overline{W(A)}.$$

We first show that

$$\liminf_{t\downarrow 0} M(t)/\psi(t) \geqslant c^{\alpha} > 0 \quad \text{a.s.} \qquad (4.3.19)$$

for $\psi(t)=t^{\alpha}(\log\log t^{-1})^{-\alpha}$ and c given by (4.3.6). Let $\varepsilon>0$ and $\gamma>1$, and for $k=1,2,\cdots$, put $t_k=\gamma^{-k}$, $\beta=(c/(1+\varepsilon))^{\alpha}$. Then, by (4.3.6)

$$\sum^{\infty}P\{M(t_k)/\psi(t_k)\leqslant\beta\}\leqslant\sum^{\infty}(\log\gamma^k)^{-(1+\varepsilon)}<\infty,$$

where the sums are over all k large enough to make $k\log\gamma>1$ and $\beta(\log\log\gamma^k)^{-\alpha}<1$. Hence, by the Borel-Cantelli lemma, $M(t_k)\geqslant\beta\psi(t_k)$ for all k greater than some $k_0=k_0(\omega)$. Further, for $k\geqslant k_0$ and $t_{k+1}\leqslant t<t_k$,

$$M(t)\geqslant M(t_{k+1})\geqslant\beta\psi(t_{k+1})\geqslant\beta\psi(t)\psi(t_{k+1})/\psi(t_k).$$

Hence

$$\liminf_{t\downarrow 0} M(t)/\psi(t) \geqslant \beta\gamma^{-\alpha} \quad \text{a.s.} \qquad (4.3.20)$$

Since ε and γ may be chosen arbitrarily close to 0 and 1, respectively, this proves (4.3.19).

Next, we prove that

$$\liminf_{t\downarrow 0} M(t)/\psi(t) \leqslant C^{\alpha} < \infty \quad \text{a.s.} \qquad (4.3.21)$$

for C given by (4.3.6). We choose

$$\beta = C^{\alpha}, \quad t_k = k^{-k}, \quad d_k = k^{k+(1-\alpha)}.$$

It follows from (4.3.6) that

$$\sum^{\infty}P\{M(t_{k+1})/\psi(t_k)\leqslant\beta\}$$

$$\geqslant\sum^{\infty}(k\log k)^{-1}=\infty, \qquad (4.3.22)$$

where the sums are over all $k\geqslant 2$ large enough to make $\beta(\log\log t_k)^{-\alpha}<1$. If the events in the first sum were independent, this would conclude the proof. However, they are not. We shall use the spectral representation (4.3.18) to get the necessary independnece. It follows from (4.3.17) that

$$\sigma_t^2(h) = 2\int_{-\infty}^{\infty}(1-\cos(h\lambda))\Delta(d\lambda).$$

Since $\sigma_t^2(h)=\mathrm{Var}(Z(t+h)-Z(t))=h^{2\alpha}$, there exists a constant $K>0$ such that for all $t>1$,

$$\int_{|\lambda|\geqslant t}\Delta(d\lambda)\leqslant Kt^{-2\alpha}$$

and

$$\int_{|\lambda|\leqslant t}\lambda^2\Delta(d\lambda)\leqslant Kt^{2(1-\alpha)}.$$

Define for $k=1,2,\cdots$ and $-\infty<t<\infty$,

$$Z_k(t) = \int_{|\lambda|\in(d_{k-1},d_k]}(e^{it\lambda}-1)W(d\lambda), \qquad (4.3.23)$$

$$\widetilde{Z}_k(t) = Z(t) - Z_k(t). \tag{4.3.24}$$

By standard Borel-Cantelli's arguments, (4.3.21) follows if we prove that

$$\sum^{\infty} P\{\max_{0 \leqslant t \leqslant t_k} |Z_k(t)| \psi(t_k) \leqslant \beta\} = \infty \tag{4.3.25}$$

and

$$\sum^{\infty} P\{\max_{0 \leqslant t \leqslant t_k} |\widetilde{Z}_k(t)| / \psi(t_k) > \varepsilon\} < \infty \quad \text{for any } \varepsilon > 0, \tag{4.3.26}$$

since the events in (4.3.25) are independent. Here (4.3.25) follows from (4.3.22) since

$$P\{\max_{0 \leqslant t \leqslant t_k} |Z_k(t)| / \psi(t_k) \leqslant \beta\} \geqslant P\{M(t_k)/\psi(t_k) \leqslant \beta\},$$

according to Anderson's inequality (Theorem 1.2.2). For $0 \leqslant h \leqslant t_k$

$$\begin{aligned} \mathrm{Var}(\widetilde{Z}_k(h)) &= 2\int_{|\lambda| \in (d_{k-1}, d_k]} (1 - \cos(h\lambda))\Delta(\mathrm{d}\lambda) \\ &\leqslant t_k^2 \int_{|\lambda| \leqslant d_{k-1}} \lambda^2 \Delta(\mathrm{d}\lambda) + 4\int_{|\lambda| \geqslant d_k} \Delta(\mathrm{d}\lambda) \\ &\leqslant ck^{-2\alpha k - 2\alpha(1-\alpha)}. \end{aligned}$$

Put $\delta = 2\alpha(1-\alpha)$. For a suitable constant K (that does not depend on k)

$$\varphi_k(h)^2 = K \min\{h^{2\alpha}, k^{-2\alpha k - \delta}\} \geqslant \mathrm{Var}(\widetilde{Z}_k(h)) \tag{4.3.27}$$

for $0 \leqslant h \leqslant t_k$. We shall now apply Lemma 4.3.3 to the process $\widetilde{Z}_k$. Put $x_k = (8 \log k)^{1/2}$. Given $\varepsilon > 0$ define

$$\theta_k(p) = \varepsilon(p+1)^{-2} \psi(t_k) / \varphi_k(t_k k^{-2^p})$$

for $p = 1, 2, \cdots$. For large enough k,

$$\theta_k(p) > 4(\log k)^{1/2} 2^{p/2} \quad \text{for all } p \geqslant 1,$$

in addition to

$$x_k \varphi_k(t_k) + \sum_{p=1}^{\infty} \theta_k(p) \varphi_k(t_k k^{-2^p}) < \varepsilon \psi(t_k).$$

Since

$$\sum_{k=1}^{\infty} k^2 \mathrm{e}^{-x_k^2/2} + \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} k^{2^{p+1}} \mathrm{e}^{-8(\log k) 2^p} < \infty,$$

it follows from Lemma 4.3.3 that

$$\sum_{k=1}^{\infty} P\{\sup_{0 \leqslant s \leqslant t_k} |\widetilde{Z}_k(s)| > \varepsilon \psi(t_k)\} < \infty.$$

We have established that

$$c^{\alpha} \leqslant \liminf_{t \downarrow 0} M(t)/\psi(t) \leqslant C^{\alpha} \quad \text{a.s.}$$

By Lemma 4.3.4 that $Z(t)$ satisfies the zero-one law at $t=0$, it guarantees that the liminf is a constant c_α, $c^\alpha \leqslant c_\alpha \leqslant C^\alpha$.

The proof of Chung's law at time $t = \infty$ is identical to the proof of the law at $t=0$. We change the definition of ψ to $\psi(t) = t^\alpha (\log\log t)^{-\alpha}$, invert the expressions defining t_k, and change $k+1$ to $k-1$ in all expressions. To establish that the tail σ-algebra at time $t = \infty$ is trivial we use the fact that $\{Z(t); t > 0\}$ and $\{t^{2\alpha} Z(t^{-1}); t > 0\}$ are equivalent processes. The zero-one law at time $t = \infty$ therefore follows from the zero-one law at time $t = 0$.

Remark 4.3.2 Monrad and Rootzen (1995) showed Chung's type LIL for a more large class of Gaussian processes. Let $\{X(t); -\infty < t < \infty\}$ be a real-valued, centered continuous Gaussian process with stationary increments. Assume that $X(0) = 0$ and with continuous covariance function

$$R(s,t) = \int_{-\infty}^{\infty} (\mathrm{e}^{is\lambda} - 1)(\mathrm{e}^{it\lambda} - 1)\Delta(\mathrm{d}\lambda),$$

where the symmetric spectral measure Δ satisfies

$$\int_{-\infty}^{\infty} \frac{\lambda^2}{1 + \lambda^2} \Delta(\mathrm{d}\lambda) < \infty.$$

Theorem 4.3.3 *If*

$$\sigma^2(h)=\mathrm{Var}(X(t+h)-X(t))\leqslant c_1h^\alpha,\quad 0\leqslant h\leqslant\delta,\ 0\leqslant t\leqslant\delta-h,$$

$$\mathrm{Var}(X(t+h)\,|\,X(s),0\leqslant s\leqslant t)\geqslant c_2h^\alpha,\quad 0\leqslant h\leqslant\delta,\ 0\leqslant t\leqslant\delta-h,$$

and for some $l>0$

$$\liminf_{|\lambda|\to 0}|\lambda|^3\Delta([\lambda,\lambda+l])>0,$$

then there exists a positive constants c_0 *such that*

$$\liminf_{t\downarrow 0}\frac{M(t)}{t^{\alpha/2}(\log\log t)^{-\alpha/2}}=c_0\quad \text{a.s.}$$

Remark 4.3.3 Theorem 6.9 of Li and Shao (2000) (c.f. also Li and Linde 1998, Shao 1999) tells us that the exact small ball constant exists for the fractional Wiener process $\{Z(t);t\geqslant 0\}$ of order $0<\alpha<1$, i.e., there exists a positive constant C_α such that

$$\lim_{x\to 0}x\log P\{\sup_{0\leqslant t\leqslant 1}|Z(t)|\leqslant x^\alpha\}=-C_\alpha.$$

And hence, the following Chung's laws of the iterated logarithm hold (c.f. Theorem 4.4.5 in the next section):

$$\liminf_{t\to\infty}\frac{\sup_{0\leqslant s\leqslant t}|Z(s)|}{t^\alpha(\log\log t)^{-\alpha}}=C_\alpha\quad \text{a.s.},$$

$$\liminf_{t\to 0}\frac{\sup_{0\leqslant s\leqslant t}|Z(s)|}{t^\alpha(\log\log(1/t))^{-\alpha}}=C_\alpha\quad \text{a.s.}$$

4.3.3 Chung's LIL for infinite series of O-U processes

Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^\infty$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k. Define

$$X(t)=\sum_{k=1}^\infty X_k(t),\quad -\infty<t<\infty. \tag{4.3.28}$$

Shao and Wang (1995) proved Chung's LIL for $\{X(t)\}$.

Theorem 4.3.4 *Assume that*

$$0<\sum_{k=1}^\infty\frac{\gamma_k}{\lambda_k}<\infty \tag{4.3.29}$$

and that $\sigma(h)/h^\alpha$ *is non-decreasing on* $[0,1]$ *for some* $\alpha>0$, *where*

$$\sigma^2(h)=E(X(h)-X(0))^2$$
$$=2\sum_{k=1}^\infty\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h}). \tag{4.3.30}$$

Then

$$\liminf_{h\to 0}\sup_{0\leqslant t\leqslant h}\frac{|X(t)|}{\sigma(ch/\log\log(1/h))}=1\quad \text{a.s.}$$

for some $0<c<\infty$.

The proof of Theorem 4.3.4 is based on the small ball probability estimates and will be given in subsection 4.4.3 as a consequence of Theorem 4.4.5.

Zhang (1995) gave a Chung's type law of the iterated logarithm for $X(\cdot)$ as follows.

Theorem 4.3.5 *Assume that* (4.3.29) *is satisfied and*

$$\Gamma_1=2\sum_{k=1}^\infty\gamma_k<\infty, \tag{4.3.31}$$

then

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|\geqslant 1\quad \text{a.s.} \tag{4.3.32}$$

Moreover, if

$$\sum_{k=1}^\infty\gamma_k(1\vee\log\log\lambda_k)^{2+\delta}<\infty \tag{4.3.33}$$

for some $\delta>0$, *then we have*

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|=1\quad \text{a.s.}$$

(4.3.34)

The proof of Theorem 4.3.5 needs the following lemma.

Lemma 4.3.5 *Let $\{W(t);t\geqslant 0\}$ be a standard Wiener process. If (4.3.31) is satisfied, then, for any $\delta>0$, there exists $h_0=h_0(\delta)$ small enough, such that*

$$\begin{aligned}&\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8(1-\delta)x^2}\right)\\ &\leqslant P\{\sup_{0\leqslant t\leqslant 1}|W(t)|\leqslant x\sqrt{1-\delta}\}\\ &\leqslant P\{\sup_{0\leqslant t\leqslant h}|X(t)-X(0)|\leqslant x(\Gamma_1 h)^{1/2}\}\\ &\leqslant P\{\sup_{0\leqslant t\leqslant 1}|W(t)|\leqslant x\sqrt{1+\delta}\}\\ &\leqslant\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8(1+\delta)x^2}\right)\end{aligned}\qquad (4.3.35)$$

for any $x>0$, $0<h<h_0$.

Proof It is easy to see that for any $x_1\leqslant x_2\leqslant x_3\leqslant x_4$, we have

$$E(X(x_4)-X(x_3))((X(x_2)-X(x_1))\leqslant 0$$

and for all $0\leqslant t\leqslant t+s\leqslant T$ we have

$$\begin{aligned}&-E(X(T)-X(t+s))(X(t+s)-X(t))\\ &\quad -E(X(t+s)-X(t))(X(t)-X(0))\\ &=\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-\exp(-\lambda_k s))\\ &\quad\times(2-\exp(-\lambda_k(T-t-s))-\exp(-\lambda_k t))\\ &\leqslant 2s\sum_{k=1}^{\infty}\gamma_k\left(1-\exp\left(-\lambda_k\frac{T}{2}\right)\right)=:sH(T),\end{aligned}$$

where

$$H(T)=2\sum_{k=1}^{\infty}\gamma_k\left(1-\exp\left(-\lambda_k\frac{T}{2}\right)\right)\to 0\quad(T\to 0).$$

For any fixed $\delta>0$, choose an $\varepsilon>0$ small enough and for any fixed positive integer n, set

$$\begin{aligned}&\xi_i=X\left(\frac{i}{n}h\right)-X\left(\frac{i-1}{n}h\right),\\ &\eta_i=W\left(\Gamma_1(1+\varepsilon)\frac{i}{n}h\right)-W\left(\Gamma_1(1+\varepsilon)\frac{i-1}{n}h\right),\\ &\eta_i^*=W\left(\Gamma_1(1-\varepsilon)\frac{i}{n}h\right)-W\left(\Gamma_1(1-\varepsilon)\frac{i-1}{n}h\right)\end{aligned}\qquad(4.3.36)$$

for $i=1,2,\cdots,n$. Let Σ_ξ, Σ_η, Σ_{η^*} be the covariance matrices of $(\xi_1,\cdots,\xi_n)'$, $(\eta_1,\cdots,\eta_n)'$, $(\xi_1^*,\cdots,\xi_n^*)$ respectively. Then we have

$$\begin{aligned}\sum_{\substack{j=1\\ i\neq j}}^{n}E\xi_i\xi_j&=E\left(X(h)-X\left(\frac{i}{n}h\right)\right)\left(X\left(\frac{i}{n}h\right)-X\left(\frac{i-1}{n}h\right)\right)\\ &\quad+E\left(X\left(\frac{i}{n}h\right)-X\left(\frac{i-1}{n}h\right)\right)\left(X\left(\frac{i-1}{n}h\right)-X(0)\right)\\ &\geqslant-\frac{1}{n}hH(h).\end{aligned}\qquad(4.3.37)$$

Since $E\xi_i\xi_j\leqslant 0(i\neq j)$, we have

$$\rho_i:=\sum\nolimits_{\substack{j=1\\ i\neq j}}^{n}|E\xi_i\xi_j|=\left|\sum\nolimits_{\substack{j=1\\ i\neq j}}^{n}E\xi_i\xi_j\right|\leqslant\frac{1}{n}hH(h).$$

Note that $H(h)\to 0$, $\sigma^2(h)/h\to\Gamma_1(h\to 0)$. There exists h_0 small enough such that for $0<h\leqslant h_0$,

$$(1+\varepsilon)\Gamma_1\frac{h}{n}-\sigma^2\left(\frac{h}{n}\right)-\rho_i>0,$$

$$\sigma^2\left(\frac{h}{n}\right)-(1-\varepsilon)\Gamma_1\frac{h}{n}-\rho_i>0.$$

Hence, $\Sigma_\eta-\Sigma_\xi$ and $\Sigma_\xi-\Sigma_{\eta^*}$ are two matrices with positive diagonal elements and the sum of absolute values of all the off diagonal elements in a given row is less than or equal to the diagonal

element in that row, i. e. $\Sigma_\eta - \Sigma_\xi$, $\Sigma_\xi - \Sigma_{\eta^*}$ are positive semi-definite. By Corollary 1. 2. 4 (Anderson's inequality) it follows that

$$P\{\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{i}\eta_j\Big|\leqslant x(\Gamma_1 h)^{1/2}\}\leqslant P\{\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{i}\xi_j\Big|\leqslant x(\Gamma_1 h)^{1/2}\}$$
$$\leqslant P\{\max_{1\leqslant i\leqslant n}\Big|\sum_{j=1}^{i}\eta_j^*\Big|\leqslant x(\Gamma_1 h)^{1/2}\},$$

that is

$$P\left\{\max_{1\leqslant i\leqslant n}\left|W\left((1+\varepsilon)\Gamma_1\frac{i}{n}h\right)\right|\leqslant x(\Gamma_1 h)^{1/2}\right\}$$
$$\leqslant P\left\{\max_{1\leqslant i\leqslant n}\left|X\left(\frac{i}{n}h\right)-X(0)\right|\leqslant x(\Gamma_1 h)^{1/2}\right\}$$
$$\leqslant P\left\{\max_{1\leqslant i\leqslant n}\left|W\left((1-\varepsilon)\Gamma_1\frac{i}{n}h\right)\right|\leqslant x(\Gamma_1 h)^{1/2}\right\}. \qquad (4.3.38)$$

Letting $n=2^m\to\infty$ $(m\to\infty)$ in (4. 3. 38), we obtain (4. 3. 35) immediately.

Remark 4. 3. 4 From the proof of Lemma 4. 3. 5 we can see that if $\{\xi(t);0\leqslant t\leqslant T\}$ is a mean zero almost surely continuous Gaussian process with

(a) For all $0\leqslant x_1\leqslant x_2\leqslant x_3\leqslant x_4\leqslant T$,

$$E(\xi(x_4)-\xi(x_3))(\xi(x_2)-\xi(x_1))\leqslant 0$$

or

$$E(\xi(x_4)-\xi(x_3))(\xi(x_2)-\xi(x_1))\geqslant 0;$$

(b) For some $0<A<\infty$ and all $0\leqslant t\leqslant t+s\leqslant T$

$$E(\xi(t+s)-\xi(t))^2\leqslant As;$$

(c) For all $x\in[0,T]$ and any integer m, $1\leqslant i\leqslant m$,

$$|E(\xi(x)-\xi(ix/m))(\xi(ix/m)-\xi((i-1)x/m))$$
$$+E(\xi(ix/m)-\xi((i-1)x/m))(\xi((i-1)x/m)-\xi(0))|$$
$$\leqslant f(x)x/m,$$

where $f(x)$ is a real function on $(0,\infty)$ with $f(x)\to 0(x\to 0)$. Then, for any $\varepsilon>0$, there exists h_0 small enough such that

$$P\{\sup_{0\leqslant t\leqslant h}|\xi(t)-\xi(0)|\leqslant y(As)^{1/2}\}$$
$$\geqslant P\{\sup_{0\leqslant t\leqslant 1}|W(t)|\leqslant y\sqrt{1-\varepsilon}\}\geqslant\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8(1-\varepsilon)y^2}\right)$$

for all $y>0$, $0<h\leqslant h_0$.

Proof of Theorem 4. 3. 5

Step 1 First we prove that (4. 3. 32) holds true. By Lemma 4. 3. 5, we can choose h_0 small enough such that for $0<h\leqslant h_0$

$$P\left\{\sup_{0\leqslant h\leqslant s}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|\leqslant 1-\varepsilon\right\}$$
$$\leqslant\frac{4}{\pi}\exp\left(-\frac{1}{(1+\delta)(1-\varepsilon)^2}\log\log h^{-1}\right)$$
$$=\frac{4}{\pi}(\log h^{-1})^{-(1+\theta)},$$

where $\theta=(1+\delta)^{-1}(1-\varepsilon)^{-2}-1>0$ (for $\delta>0$ small enough). Let $h_n=v^{-n}(v>1)$. By the Borel-Cantelli lemma

$$\liminf_{n\to\infty}\sup_{0\leqslant s\leqslant h_n}\left(\frac{8\log\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}|X(s)-X(0)|\geqslant 1-\varepsilon\quad\text{a.s.} \qquad (4.3.39)$$

For $h_{n+1}<h\leqslant h_n$, we have

$$\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|$$
$$\geqslant\sup_{0\leqslant s\leqslant h_{n+1}}\left(\frac{8\log\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}|X(s)-X(0)|.$$

Hence

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|\geqslant(1-\varepsilon)/v\quad\text{a.s.}, \qquad (4.3.40)$$

which implies (4. 3. 32) immediately.

Step 2 Next we prove that if (4.3.33) is satisfied, then

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|\leqslant 1+\varepsilon \quad \text{a.s.} \tag{4.3.41}$$

Let $\rho_k=\mathrm{e}^{-k\log k}$, $k=1,2,\cdots$. Then

$$\begin{aligned}
&\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(s)-X(0)|\\
&\leqslant\liminf_{k\to\infty}\sup_{0\leqslant s\leqslant\rho_k}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|X(s)-X(0)|\\
&\leqslant\liminf_{k\to\infty}\sup_{\rho_{k+1}\leqslant s\leqslant\rho_k}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|X(s)-X(0)|\\
&\quad+\limsup_{k\to\infty}\sup_{0\leqslant s\leqslant\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|X(s)-X(0)|\\
&=:J_1+J_2.
\end{aligned}$$

Noting that $\sigma^2(h)\leqslant\Gamma_1/h$, by Fernique's inequality (Theorem 1.1.3), we have

$$P\left\{\sup_{0\leqslant s\leqslant\rho_{k+1}}|X(s)-X(0)|\geqslant x\Gamma_1^{1/2}\rho_{k+1}^{1/2}(1+(2+\sqrt{2})\sqrt{\pi})\right\}\leqslant\mathrm{e}^{-x^2/2}.$$

It follows that

$$P\left\{\sup_{0\leqslant s\leqslant\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|X(s)-X(0)|\geqslant\varepsilon\right\}\leqslant\exp\left(-c\varepsilon\frac{\rho_k}{\rho_{k+1}\log\log\rho_k^{-1}}\right)\leqslant\exp(-c\varepsilon k/\log k),$$

which together with the Borel-Cantelli lemma implies that

$$J_2=0\quad\text{a.s.} \tag{4.3.42}$$

Denote

$$\xi_k(s)=\sum_{j=1}^{\infty}\left(\frac{\gamma_j}{\lambda_j}\right)^{1/2}\frac{W_j(\mathrm{e}^{2\lambda_j(s+\rho_{k+1})})-W_j(\mathrm{e}^{2\lambda_j\rho_{k+1}})}{\mathrm{e}^{\lambda_j(s+\rho_{k+1})}},$$

$$\eta_k(s)=\sum_{j=1}^{\infty}\left(\frac{\gamma_j}{\lambda_j}\right)^{1/2}\frac{W_j(\mathrm{e}^{2\lambda_j\rho_{k+1}})}{\mathrm{e}^{\lambda_j\rho_{k+1}}}(1-\mathrm{e}^{-\lambda_j s}),\ 0\leqslant s\leqslant\rho_k-\rho_{k+1},\ k\geqslant 1.$$

Then we have

$$\begin{aligned}
J_1&\leqslant\liminf_{k\to\infty}\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|\xi_k(s)|\\
&\quad+\limsup_{k\to\infty}\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\Gamma_1\rho_k}\right)^{1/2}|\eta_k(s)|\\
&=:J_{11}+J_{12}.
\end{aligned}$$

Consider J_{11}. Noting that for $0\leqslant t\leqslant t+s\leqslant\rho_k-\rho_{k+1}$,

$$\begin{aligned}
\hat\sigma_k^2(s)&:=E(\xi_k(t+s)-\xi_k(t))^2\\
&=\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(1-\mathrm{e}^{-\lambda_j s})(2-(1-\mathrm{e}^{-\lambda_j s})\mathrm{e}^{-2\lambda_j t}),
\end{aligned}$$

we have

$$\begin{aligned}
s\Gamma_1&\geqslant\sigma^2(s)\geqslant\hat\sigma_k^2(s)\\
&\geqslant\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(1-\mathrm{e}^{-\lambda_j s})(1+\mathrm{e}^{-\lambda_j s})=\sigma^2(2s)/2.
\end{aligned}$$

Hence

$$\hat\sigma_k^2(s)/s\to\Gamma_1\quad(s\to 0)\quad\text{uniformly for } k\geqslant 1.$$

For any $0\leqslant x_1\leqslant x_2\leqslant x_3\leqslant x_4\leqslant\rho_k-\rho_{k+1}$, we have

$$E(\xi_k(x_4)-\xi_k(x_3))(\xi_k(x_2)-\xi_k(x_1))\leqslant 0.$$

For any integer m, $1\leqslant i\leqslant m$ and any $0<x\leqslant\rho_k-\rho_{k+1}$, we have

$$\begin{aligned}
&\left|E\left(\xi_k\left(\frac{i}{m}x\right)-\xi_k\left(\frac{i-1}{m}x\right)\right)\left(\xi_k\left(\frac{i-1}{m}x\right)-\xi_k(0)\right)\right|\\
&\quad+\left|E\left(\xi_k(x)-\xi_k\left(\frac{i}{m}x\right)\right)\left(\xi_k\left(\frac{i}{m}x\right)-\xi_k\left(\frac{i-1}{m}x\right)\right)\right|\\
&=\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}\{\mathrm{e}^{-\lambda_j x(i-1)/m}(1-\mathrm{e}^{-\lambda_j x/m})(1-\mathrm{e}^{-\lambda_j x(i-1)/m})(1+\mathrm{e}^{\lambda_j x(i-1)/m})\\
&\quad+\mathrm{e}^{-\lambda_j xi/m}(1-\mathrm{e}^{-\lambda_j x/m})(1-\mathrm{e}^{-\lambda_j x(1-i/m)})(\mathrm{e}^{\lambda_j xi/m}+\mathrm{e}^{\lambda_j x(i-1)/m})\}
\end{aligned}$$

$$\leqslant 2\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(1-e^{-\lambda_j x/m})(2-e^{\lambda_j x(i-1)/m}-e^{-\lambda_j x(1-i/m)})$$
$$\leqslant 4\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(1-e^{-\lambda_j x/m})(1-e^{-\lambda_j x/2})$$
$$\leqslant 2xH(x)/m,$$

where $H(x)=2\sum_{j=1}^{\infty}\gamma_j(1-e^{-\lambda_j x/2})\to 0\ (x\to 0)$. Then by Remark 4. 3. 4 for any $\varepsilon'>0$ we can choose k_0 large enough such that for $k\geqslant k_0$, $x>0$

$$P\{\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}|\xi_k(s)|\leqslant x(\Gamma_1(\rho_k-\rho_{k+1}))^{1/2}\}$$
$$\geqslant\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8(1-\varepsilon')x^2}\right).$$

Noting that $\rho_{k+1}/\rho_k\to 0\ (k\to\infty)$, we have

$$P\left\{\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\rho_k\Gamma_1}\right)^{1/2}|\xi_k(s)|\leqslant 1+\varepsilon\right\}$$
$$\geqslant\frac{2}{\pi}\exp\left(-\frac{1}{(1-\varepsilon')(1+\varepsilon/2)^2}\log\log\rho_k^{-1}\right)$$
$$=O((k\log\log k)^{-(1-\varepsilon'')}),\tag{4.3.43}$$

where $\varepsilon''=1-\frac{1}{(1-\varepsilon')(1+\varepsilon/2)^2}>0$ (for ε' small enough). By the Borel-Cantelli lemma, (4. 3. 43) and independence of $\{\xi_k(s)\}_{k=1}^{\infty}$ we obtain

$$J_{11}\leqslant 1+\varepsilon\quad\text{a.s.}\tag{4.3.44}$$

Finally, we show that

$$J_{12}\leqslant\varepsilon\quad\text{a.s.}$$

For $0\leqslant s\leqslant s+h\leqslant\rho_k-\rho_{k+1}$, we have

$$E(\eta_k(s+h)-\eta_k(s))^2$$
$$=\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(e^{-\lambda_j s}-e^{-\lambda_j(s+h)})^2$$

$$\leqslant\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}(1-e^{-\lambda_j h})^2$$
$$\leqslant hH(2h)/2\leqslant hH(2\rho_k)/2.$$

Let $h_k=\rho_k-\rho_{k+1}$. Since

$$\int_1^{\infty}\left(\frac{1}{2}H(2\rho_k)h_k e^{-y^2}\right)^{1/2}dy\leqslant\sqrt{\pi}(h_kH(2\rho_k))^{1/2},$$
$$E\eta_k^2(s)\leqslant sH(2s)/2\leqslant h_kH(2\rho_k)/2,$$

we have by Fernique's inequality (Theorem 1. 1. 3) that for any $x\geqslant 0$

$$P\{\sup_{0\leqslant s\leqslant h_k}|\eta_k(s)|\geqslant x(1/2+\sqrt{\pi})(h_kH(2\rho_k))^{1/2}\}\leqslant ce^{-x^2/2}.$$

Hence

$$P\left\{\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\rho_k\Gamma_1}\right)^{1/2}|\eta_k(s)|>2\varepsilon\right\}$$
$$\leqslant P\left\{\sup_{0\leqslant s\leqslant h_k}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2h_k\Gamma_1}\right)^{1/2}|\eta_k(s)|>2\varepsilon\right\}$$
$$\leqslant c\exp\left(-\frac{\varepsilon'}{H(2\rho_k)\log\log\rho_k^{-1}}\right),\tag{4.3.45}$$

where ε' is a positive constant dependent on ε. It is easy to show that for $0<h\leqslant 1/e$

$$(\log\log h^{-1})^{2+\delta}(1-e^{-\lambda_j h})\leqslant c(1\vee\log\log\lambda_j)^{2+\delta},$$

which implies

$$H(2h)(\log\log h^{-1})^{2+\delta}\leqslant c\sum_{j=1}^{\infty}\gamma_j(1\vee\log\log\lambda_j)^{2+\delta}\leqslant c.\tag{4.3.46}$$

Combining it with (4. 3. 45) we have

$$P\left\{\sup_{0\leqslant s\leqslant\rho_k-\rho_{k+1}}\left(\frac{8\log\log\rho_k^{-1}}{\pi^2\rho_k\Gamma_1}\right)^{1/2}|\eta_k(s)|>2\varepsilon\right\}$$
$$\leqslant c\exp(-\varepsilon''(\log\log\rho_k^{-1})^{1+\delta})$$
$$\leqslant c\exp(-2\log\log\rho_k^{-1})$$

$$=c(k\log k)^{-2}.$$

By the Borel-Cantelli lemma we get

$$J_{12} \leqslant 2\varepsilon \quad \text{a.s.}$$

The proof of Theorem 4.3.5 is completed.

Remark 4.3.5 Using the arguments in the proof of Theorem 4.3.4 (c.f. Theorem 4.4.5 in the next section) one can show that (4.3.32) holds true under condition $\Gamma_1<\infty$ only.

Remark 4.3.6 In Kuelbs, Li and Shao (2000), the small ball probabilities are estimated for Gaussian processes with stationary increments when the small balls are given by various Hölder norms. As applications they established results related to Chung's functional LIL for fractional Wiener processes under Hölder norms. In particular, they identified the points approached slowest in the functional LIL. In Li and Shao (1999), a sharp small ball estimate under Sobolev type norms is obtained for certain Gaussian processes in general and for fractional Wiener processes in particular. A new method using the techniques in large deviation theory is developed for small ball estimates. As an application the Chung's LIL for fractional Wiener processes was given.

4.4 The Small Ball Probability and Chung's Law of the Iterated Logarithm of Gaussian Fields

A lower bound of small ball probability for d-parameter Gaussian fields was given by Shao and Wang (1995), where a

sharp bound was found for a fractional Lévy-Wiener field, and an exact lower and upper bound of small ball probability for Wiener sheet were given by Talagrand (1994). The estimates are then used to study the Chung LIL.

4.4.1 The small ball probabilities for Gaussian fields

Let $d\geqslant 1$, $\mathbf{X}=\{X(t);t\in \mathbf{R}^d\}$ be a real Gaussian field with mean zero and $a\leqslant t\leqslant b$ denotes $a\leqslant t_i\leqslant b$ for each $i=1,2,\cdots,d$. Throughout this section for $t=(t_1,\cdots,t_d)\in\mathbf{R}^d$, $\|t\|$ denotes the Euclidean norm. Shao and Wang (1995) gave the lower bounds of small ball probability for Gaussian fields.

Theorem 4.4.1 *Assume that there is a non-decreasing function $\sigma(x)$ on $[0,1]$ such that*

$$E|X(t)-X(s)|^2\leqslant \sigma^2(\|t-s\|) \tag{4.4.1}$$

for every $s,t\in[0,1]^d$. Suppose that $\sigma(x)/x^{\alpha}$ is non-decreasing on $[0,1]$ for some $\alpha>0$ and that

$$\sigma(kh)\leqslant k^2\sigma(h) \tag{4.4.2}$$

for every $0\leqslant h\leqslant 1$ and integer k with $1\leqslant k\leqslant 1/h$. Then there exists a positive constant c depending only on α and d such that for any $0<x<1$

$$P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant c\sigma(x)\}\geqslant \exp\left(-\frac{c}{x^d}\right). \tag{4.4.3}$$

Put

$$\mathscr{A}=\{(\boldsymbol{a},\boldsymbol{b}];\boldsymbol{a},\boldsymbol{b}\in[0,1]^d,\boldsymbol{a}\leqslant\boldsymbol{b}\}.$$

For any $A\in\mathscr{A}$, $|A|$ denotes the Lebesgue measure of A in $\mathbf{R}^d$, and define formly

$$X(A)=\int_a^b \mathrm{d}X(t),$$

where $\int_a^b$ can also be understood as the difference operator of X.

Theorem 4.4.2 *Assume that there is a non-decreasing function $\sigma(x)$ on $[0,1]$ such that for any $A\in\mathscr{A}$*

$$E(X(A))^2\leqslant\sigma^2(|A|). \tag{4.4.4}$$

Suppose that $\sigma(x)/x^a$ is non-decreasing on $[0,1]$ for some $\alpha>0$ and that (4.4.2) is satisfied. Then there is a positive constant c depending only on α and d such that for any $0<x<1/2$

$$P\left\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant c\left(\log\frac{1}{x}\right)^{d-1}\sigma(x)\right\}$$

$$\overset{\text{KS}}{\geqslant}\exp\left(-\frac{c(\log(1/x))^{d-1}}{x}\right). \tag{4.4.5}$$

Proof of Theorem 4.4.1 We prove that $\exp(-c/x^d)$ is actually a KSLB of $P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant c\sigma(x)\}$. For any $t=(t^{(1)},\cdots,t^{(d)})\in[0,1]^d$ and integer $k\geqslant 0$, denote

$$t_k=\frac{[t2^k]}{2^k}=\left(\frac{[t^{(1)}2^k]}{2^k},\cdots,\frac{[t^{(d)}2^k]}{2^k}\right). \tag{4.4.6}$$

$t_{-1}=\mathbf{0}$. By the condition (4.4.1), the monotonousness of $\sigma(x)/x^a$ and Theorem 2.1.3, it is easy to see that $\{X(t);t\in[0,1]^d\}$ is a.s. continuous and $\sigma(0)=0$. Also it is clear that $X(\mathbf{0})=0$ a.s. by (4.4.1). Hence for $\mathbf{0}\leqslant t<\mathbf{1}$,

$$|X(t)|=\lim_{k\to\infty}|X(t_k)-X(\mathbf{0})|$$

$$\leqslant\sum_{k=1}^{\infty}|X(t_k)-X(t_{k-1})|.$$

It follows that

$$\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sum_{k=1}^{\infty}\sup_{0\leqslant t\leqslant 1}|X(t_k-X(t_{k-1})|, \tag{4.4.7}$$

and

$$\sup_{0\leqslant t\leqslant 1}\|t_k-t_{k-1}\|\leqslant\sqrt{d}\,2^{-k}. \tag{4.4.8}$$

Without loss of generality, we can assume that $0<x<1/d$. Otherwise there is nothing to prove. Let n_0 be an integer such that

$$1/x\leqslant 2^{n_0}\leqslant 2/x. \tag{4.4.9}$$

Let $N=N(0,1)$ be a standard normal variables. Define

$$C:=C_a=(1-2^{-a/2})/(2d^2), \tag{4.4.10}$$

$$x_k=C\sigma\left(\left(\frac{3}{2}\right)^{-|k-n_0|}\sqrt{d}\,x\right),\quad k=1,2,\cdots. \tag{4.4.11}$$

Since $\sigma(x)/x^a$ is non-decreasing, by (4.4.2) we have

$$\sum_{k=1}^{\infty}x_k\leqslant\frac{(1-2^{-a/2})}{2d^2}\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^{-a|k-n_0|}\sigma(\sqrt{d}\,x)$$

$$\leqslant\frac{(1-2^{-a/2})}{d^2}\sigma(\sqrt{d}\,x)\sum_{j=1}^{\infty}\left(\frac{3}{2}\right)^{-ak}$$

$$\leqslant(1-2^{-a/2})\sigma(x)\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^{-ak}$$

$$\leqslant\sigma(x). \tag{4.4.12}$$

Since $\mathrm{Card}\{t_k;\mathbf{0}\leqslant t\leqslant\mathbf{1}\}\leqslant 2^{kd}$, by (4.4.7), (4.4.8), (4.4.12) and Khatri-Šidák's inequality (Theorem 1.2.4) we have

$$P\{\sup_{0\leqslant t\leqslant 1}|X(t)|\leqslant\sigma(x)\}$$

$$\geqslant P\left\{\sum_{k=1}^{\infty}\sup_{0\leqslant t\leqslant 1}|X(t_k)-X(t_{k-1})|\leqslant\sigma(x)\right\}$$

$$\geqslant P\{\sup_{0\leqslant t\leqslant 1}|X(t_k)-X(t_{k-1})|\leqslant x_k,k=1,2,\cdots\}$$

$$\geqslant\prod_{k\geqslant 1}\left\{P\left(|N|\leqslant\frac{x_k}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}}$$

$$=\prod_{k\geqslant 1}\left\{P\left(|N|\leqslant\frac{C\sigma((3/2)^{-|k-n_0|}\sqrt{d}\,x)}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}}$$

$$=:I_1\cdot I_2, \tag{4.4.13}$$

where

$$I_1 = \prod_{k=1}^{n_0}\left\{P\left(|N| \leqslant \frac{C\sigma((3/2)^{k-n_0}\sqrt{d}\,x)}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}},$$

$$I_2 = \prod_{k=n_0+1}^{\infty}\left\{P\left(|N| \leqslant \frac{C\sigma((3/2)^{n_0-k}\sqrt{d}\,x)}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}}.$$

Note that

$$P\{|N| \leqslant t\} \geqslant t/2, \quad \text{if } 0 < t \leqslant 1, \tag{4.4.14}$$

$$P\{|N| \leqslant st\} \geqslant \exp\left(-\frac{1}{1-e^{-s^2/2}}e^{-(st)^2/2}\right), \text{ if } s>0,\ t \geqslant 1. \tag{4.4.15}$$

Recalling (4.4.9) and that $\sigma(x)$ is non-decreasing, we get

$$\begin{aligned}
I_1 &\geqslant \prod_{k=1}^{n_0}\left\{P\left(|N| \leqslant C\frac{\sigma((3/2)^{k-n_0}\sqrt{d}\,2^{-n_0})}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}} \\
&= \prod_{k=1}^{n_0}\left\{P\left(|N| \leqslant C\frac{\sigma(3^{k-n_0}\sqrt{d}\,2^{-k})}{\sigma(\sqrt{d}\,2^{-k})}\right)\right\}^{2^{kd}} \\
&\geqslant \prod_{k=1}^{n_0}\{P(|N| \leqslant C3^{2(k-n_0)})\}^{2^{dk}} \quad \text{(by (4.4.2))} \\
&\geqslant \prod_{k=1}^{n_0}\left(\frac{C}{2}3^{2(k-n_0)}\right)^{2^{kd}} \quad \text{(by (4.4.14))} \\
&= \exp\left(-\sum_{k=1}^{n_0}2^{kd}\left(\log\frac{2}{C} + 2(n_0-k)\log 3\right)\right) \\
&\geqslant \exp\left(-2^{n_0 d}\sum_{l=0}^{\infty}2^{-ld}\left(\log\frac{2}{C} + 2l\log 3\right)\right) \\
&= \exp(-2^{n_0 d}D_\alpha) \\
&\geqslant \exp(-2^d D_\alpha/x^d),
\end{aligned} \tag{4.4.16}$$

where D_α denotes a constant depending only on α and d.

We now estimate I_2. Since $2^{-n_0} \leqslant x \leqslant 2^{-n_0+1}$ and $\sigma(x)/x^\alpha$ is non-decreasing, we have

$$\begin{aligned}
I_2 &\geqslant \prod_{k=n_0+1}^{\infty}\left\{P\left(|N| \leqslant C\frac{\sigma((3/2)^{n_0-k}\sqrt{d}\,x)}{\sigma(2^{-(k-n_0)}\sqrt{d}\,x)}\right)\right\}^{2^{kd}} \\
&\geqslant \prod_{k=n_0+1}^{\infty}\left\{P\left(|N| \leqslant C\left(\frac{4}{3}\right)^{(k-n_0)\alpha}\right)\right\}^{2^{kd}} \\
&\geqslant \prod_{k=n_0+1}^{\infty}\exp\left(-\frac{2^{kd}}{1-e^{-C^2/2}}\exp\left(-\frac{C^2}{2}\left(\frac{4}{3}\right)^{2(k-n_0)\alpha}\right)\right) \\
&\qquad \text{(by (4.4.15))} \\
&= \exp\left(\frac{-2^{n_0 d}}{1-e^{-C^2/2}}\sum_{k=n_0+1}^{\infty}2^{(k-n_0)d}\exp\left(-\frac{C^2}{2}\left(\frac{4}{3}\right)^{2(k-n_0)\alpha}\right)\right) \\
&= \exp\left(\frac{-2^{n_0 d}}{1-e^{-C^2/2}}\sum_{l=1}^{\infty}2^{ld}\exp\left(-\frac{C^2}{2}\left(\frac{4}{3}\right)^{2l\alpha}\right)\right) \\
&= \exp(-2^{n_0}D_\alpha) \\
&\geqslant \exp(-2^d D_\alpha/x^d).
\end{aligned} \tag{4.4.17}$$

This proves (4.4.3) by (4.4.13), (4.4.16) and (4.4.17).

Remark 4.4.1 Talagrand (1993) established a small ball probability by the entropy number. Let $\{Y(t); t \in S\}$ be a real valued Gaussian process, where $S \subset \mathbf{R}^k (k \geqslant 1)$ is a given set equipped with the canonical metric $d(s,t) = (E(Y(s)-Y(t))^2)^{1/2}$. Denote $N(S,d;\varepsilon)$ be the entropy number, that is the smallest number of open balls of radius ε in metric d which form a covering of S. If there exists a positive function $\psi(\varepsilon)$ and a constant $A>0$ such that $N(S,d;\varepsilon) \leqslant \psi(\varepsilon)$ and $\psi(\varepsilon)/A \leqslant \psi(\varepsilon/2) \leqslant A\psi(\varepsilon)$ for any $\varepsilon > 0$, then there exists a constant $C>0$ such that for all $u \geqslant 0$,

$$P\left(\sup_{s,t \in S}|Y(t) - Y(s)| \leqslant u\right) \geqslant \exp(-C\psi(u)).$$

Proof of Theorem 4.4.2 Since the condition of the theorem implies that $\sigma(0) = 0$, $X(t) = 0$ a.s. if one of t_i is zero. Therefore, we can write

$$X(t) = \int_0^t \mathrm{d}X(s).$$

Put

$$X_{i,k} = \int_{(i-1)/2^k}^{i/2^k} \mathrm{d}X(s), \quad \mathbf{1} \leqslant \boldsymbol{i} \leqslant 2^k, \boldsymbol{i}, \boldsymbol{k} \in \mathbf{Z}^d.$$

From (4. 4. 4) it follows that

$$\operatorname{Var}(X_{i,k}) \leqslant \sigma^2(2^{-k_1-\cdots-k_d}). \tag{4. 4. 18}$$

For $0 \leqslant a \leqslant 1$, let

$$a_k = [a2^k]/2^k. \quad k = 0, 1, 2, \cdots.$$

By Theorem 2. 1. 3, the sample function of $X(\cdot)$ is a. s. continuous. Thus, for any $\mathbf{0} \leqslant \boldsymbol{t} = (t^{(1)}, \cdots, t^{(d)}) < \mathbf{1}$,

$$\begin{aligned}
|X(t)| &= \left| \int_0^{t^{(d)}} \cdots \int_0^{t^{(1)}} \mathrm{d}X(s) \right| \\
&= \lim_{k \to \infty} \left| \int_0^{t^{(1)}_{k_d}} \cdots \int_0^{t^{(d)}_{k_d}} \mathrm{d}X(s) \right| \\
&= \left| \sum_{k_d=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} \int_{t^{(d)}_{k_d-1}}^{t^{(d)}_{k_d}} \cdots \int_{t^{(1)}_{k_1-1}}^{t^{(1)}_{k_1}} \mathrm{d}X(s) \right| \\
&\leqslant \sum_{k_d=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} \max_{1 \leqslant i \leqslant 2^k} \left| \int_{(i_d-1)/2^{k_d}}^{i_d/2^{k_d}} \cdots \int_{(i_1-1)/2^{k_1}}^{i_1/2^{k_1}} \mathrm{d}X(s) \right| \\
&\leqslant \sum_{1 \leqslant k < \infty} \max_{1 \leqslant i \leqslant 2^k} |X_{i,k}| \quad \text{a. s.}
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{0 \leqslant t \leqslant 1} |X(t)| &\leqslant \sum_{1 \leqslant k < \infty} \max_{1 \leqslant i \leqslant 2^k} |X_{i,k}| \\
&\leqslant \sum_{n=d}^{\infty} \sum_{1 \leqslant k < \infty, \hat{k} = n} \max_{1 \leqslant i \leqslant 2^k} |X_{i,k}|,
\end{aligned} \tag{4. 4. 19}$$

where $\hat{k} = k_1 + \cdots + k_d$. Let n_0 be an integer such that

$$1/2^{n_0} \leqslant x \leqslant 2/2^{n_0}$$

and let

$$x_n = \sigma(x(2/3)^{|n-n_0|}), \quad n \geqslant 1.$$

Clearly,

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{1 \leqslant k < \infty, \hat{k} = n} x_n &\leqslant \sum_{n=1}^{\infty} n^{d-1} \sigma(x(2/3)^{|n-n_0|}) \\
&\leqslant \sum_{n=1}^{\infty} n^{d-1} \sigma(x)(2/3)^{|n-n_0|\alpha} \\
&\leqslant c n_0^{d-1} \sigma(x) \\
&\leqslant c(\log(1/x))^{d-1} \sigma(x).
\end{aligned}$$

Hence, by Theorem 1. 2. 4 and (4. 4. 18)

$$\begin{aligned}
&P\left\{ \sup_{0 \leqslant t \leqslant 1} |X(t)| \leqslant c \left(\log \frac{1}{x} \right)^{d-1} \sigma(x) \right\} \\
&\overset{\text{KS}}{\geqslant} \prod_{n=d}^{\infty} \prod_{1 \leqslant k < \infty, \hat{k} = n} \prod_{1 \leqslant i \leqslant 2^k} P\{|X_{i,k}| \leqslant x_n\} \\
&\geqslant \prod_{n=d}^{\infty} \prod_{1 \leqslant k < \infty, \hat{k} = n} \prod_{1 \leqslant i \leqslant 2^k} P\{|N(0,1)| \leqslant x_n / \sigma(2^{-n})\} \\
&\geqslant \prod_{n=d}^{\infty} P\{|N(0,1)| \leqslant x_n / \sigma(2^{-n})\}^{n^{d-1}2^n} \\
&=: K_1 \cdot K_2,
\end{aligned} \tag{4. 4. 20}$$

where

$$K_1 = \prod_{n=d}^{n_0-1} P\{|N(0,1)| \leqslant \sigma(x(2/3)^{n_0-n}) / \sigma(2^{-n})\}^{n^{d-1}2^n},$$

$$K_2 = \prod_{n=n_0}^{\infty} P\{|N(0,1)| \leqslant \sigma(x(2/3)^{n-n_0}) / \sigma(2^{-n})\}^{n^{d-1}2^n}.$$

By (4. 4. 2) and (4. 4. 14), we get

$$\begin{aligned}
K_1 &\geqslant \prod_{n=d}^{n_0-1} P\{|N(0,1)| \leqslant \sigma(2^{-n}3^{n-n_0}) / \sigma(2^{-n})\}^{n^{d-1}2^n} \\
&\geqslant \prod_{n=d}^{n_0-1} P\{|N(0,1)| \leqslant 3^{2(n-n_0)}\}^{n^{d-1}2^n} \\
&\geqslant \prod_{n=d}^{n_0-1} \left(\frac{3^{n-n_0}}{2}\right)^{n^{d-1}2^n}
\end{aligned}$$

$$\geqslant \exp\{c2^{n_0}\sum_{n=d}^{n_0-1}(n-n_0)n^{d-1}2^{n-n_0}\}$$
$$\geqslant \exp\{-cn_0^{d-1}2^{n_0}\}$$
$$\geqslant \exp\left\{-\frac{c\log^{d-1}(1/x)}{x}\right\}. \tag{4.4.21}$$

By (4.4.15)

$$K_2 \geqslant \prod_{n=n_0}^{\infty} P\{|N(0,1)| \leqslant \sigma(2^{-n}(4/3)^{n-n_0})/\sigma(2^{-n})\}^{n^{d-1}2^n}$$
$$\geqslant \prod_{n=n_0}^{\infty} P\{|N(0,1)| \leqslant (4/3)^{(n-n_0)\alpha}\}^{n^{d-1}2^n}$$
$$\geqslant \exp\{-3\sum_{n=n_0}^{\infty} n^{d-1}2^n\exp(-(4/3)^{2(n-n_0)\alpha}/2)\}$$
$$\geqslant \exp\{-cn_0^{d-1}2^{n_0}\}$$
$$\geqslant \exp\left\{-\frac{c\log^{d-1}(1/x)}{x}\right\}. \tag{4.4.22}$$

This proves (4.4.5) by (4.4.20)−(4.4.22).

Now, let us apply Theorems 4.4.1 and 4.4.2 to get the small ball probability estimates for the fractional Lévy-Wiener fields and Wiener sheet. We have

Theorem 4.4.3 *Let $d\geqslant 1$, $\{Z(t);t\in \mathbf{R}^d\}$ be a fractional Lévy Wiener field of order α, $0<\alpha<1$, i.e. $E(Z(s)-Z(t))^2=\|s-t\|^{2\alpha}$ for all $s,t\geqslant 0$. Then there are $0<c_1\leqslant c_2<\infty$ depending only on α and d such that*

$$\exp\left\{-\frac{c_2}{x^{d/\alpha}}\right\}\overset{\mathrm{KS}}{\leqslant} P\{\sup_{0\leqslant t\leqslant 1}|Z(t)|\leqslant x\}\leqslant \exp\left(-\frac{c_1}{x^{d/\alpha}}\right) \tag{4.4.23}$$

for any $0<x<1$.

Proof The lower bound of (4.4.23) follows from Theorem 4.4.1 immediately. We next establish the upper bound. For $i=(i_1,\cdots,i_d)\in \mathbf{Z}^d$, write $t_i=ix^{1/\alpha}$. Clearly

$$P\{\sup_{0\leqslant t\leqslant 1}|Z(t)|\leqslant x\}\leqslant P\{\max_{1\leqslant i\leqslant x^{-1/\alpha}}|Z(t_i)|\leqslant x\}$$

and for $1\leqslant j\leqslant x^{-1/\alpha}$

$$P\{\max_{1\leqslant i\leqslant x^{-1/\alpha}}|Z(t_i)|\leqslant x\}$$
$$=EI\{\max_{1\leqslant i\leqslant x^{-1/\alpha},i\neq j}|Z(t_i)|\leqslant x\}$$
$$\times P\{|Z(t_j)|\leqslant x\,|\,Z(t_i),1\leqslant i\leqslant x^{-1/\alpha},i\neq j\}.$$

In terms of Lemma 7.1 of Pitt (1978), there is a positive constant $C=C(\alpha,d)$ such that

$$\mathrm{Var}(Z(t_j)|Z(t_i),1\leqslant i\leqslant x^{-1/\alpha},i\neq j)$$
$$\geqslant \mathrm{Var}(Z(t_j)|Z(s),\ \|s-t_j\|\geqslant x^{1/\alpha})=Cx^2.$$

Hence,

$$P\{|Z(t_j)|\leqslant x\,|\,Z(t_i),1\leqslant i\leqslant x^{-1/\alpha},i\neq j\}$$
$$\leqslant P\{|N(0,1)|\leqslant 1/\sqrt{C}\}<1$$

and

$$P\{\max_{1\leqslant i\leqslant x^{-1/\alpha}}|Z(t_i)|\leqslant x\}$$
$$\leqslant P\{|N(0,1)|\leqslant 1/\sqrt{C}\}P\{\max_{1\leqslant i\leqslant x^{-1/\alpha},i\neq j}|Z(t_i)|\leqslant x\}.$$

Thus, by recurrence

$$P\{\max_{1\leqslant i\leqslant x^{-1/\alpha}}|Z(t_i)|\leqslant x\}\leqslant P\{|N(0,1)|\leqslant 1/\sqrt{C}\}^{(x^{-1/\alpha}-1)^d}$$
$$\leqslant \exp(-c/x^{d/\alpha}),$$

This proves Theorem 4.4.3.

Corollary 4.4.1 *Let $\{W(t);t\in \mathbf{R}^d\}$ be a standard Wiener sheet. Then*

$$P\{\sup_{0\leqslant t\leqslant 1}|W(t)|\leqslant x\}\overset{\mathrm{KS}}{\geqslant}\exp\left(-\frac{c(\log(1/x))^{3(d-1)}}{x^2}\right). \tag{4.4.24}$$

Particularly, if $d=2$, for some constant $C>0$ and any $0<x<1/2$ we have

$$\exp\left(-\frac{C(\log(1/x))^3}{x^2}\right) \overset{\text{KS}}{\leqslant} P\{\sup_{0\leqslant t\leqslant 1} |W(t)| \leqslant x\}$$
$$\leqslant \exp\left(-\frac{(\log(1/x))^3}{Cx^2}\right). \tag{4.4.25}$$

Proof (4.4.24) of Corollary 4.4.1 follows from Theorem 4.4.2 immediately, the proof of (4.4.25) will be omitted here (cf. Talagrand 1994).

Remark 4.4.2 The lower bound given in (4.4.24) coincides with that of Bass (1988). (4.4.25) is due to Talagrand (1994) that is sharp. Shao and Wang (1995) conjecture that lower bound (4.4.24) is also sharp for $d>2$.

Remark 4.4.3 To our surprise we discover from Theorem 4.4.3 and Corollary 4.4.1 that the small ball probability of a fractional Lévy Wiener field is totally different from that of a Wiener sheet. This indicates that their limit inferior behavior must be quite different, as we will see in next subsection.

Remark 4.4.4 An upper bound of the small ball probability for some Gaussian fields was claimed in Kôno (1976). But, the upper bound for general Gaussian fields with stationary increments i.e., $E(X(s)-X(t))^2=\sigma^2(\|s-t\|)$, is still not obtained. We conjecture that if $\sigma^2(\cdot)$ is concave on $(0,1)$, then, the bound given in (4.4.3) is sharp.

4.4.2 Chung's LIL for Gaussian fields

Let $d\geqslant 1$ and $\{X(t);t\in[0,1]^d\}$ be a centered Gaussian field with stationary increments in the following sense: for any $0\leqslant s,t\leqslant 1$.

$$E|X(s)-X(t)|^2=\sigma^2(\|s-t\|) \tag{4.4.26}$$

for some non-decreasing continuous function σ on $[0,1]$. Here we are interested in the liminf behavior of $\sup_{0\leqslant t\leqslant h}|X(t)|$. Using an upper bound of small ball probability, one can easily obtain a lower bound of the liminf. However, we have some trouble in deducing an upper bound of the liminf from the small ball estimate. Shao and Wang (1995) found that the method used by Monrad and Rootzen (1995) and Theorem 4.4.1 can be used to establish a very general result.

Theorem 4.4.4 *Let $\{X(t);t\in[0,1]^d\}$ be a centered Gaussian field with stationary increments. Assume that $X(0)=0$, $\sigma(x)/x^\beta$ is non-decreasing on $[0,1]$ for some $\beta>0$ and that there is $0<\theta<2$ such that for any $0<h<1/2$*
$$\sigma(2h)\leqslant\theta\sigma(h). \tag{4.4.27}$$
Then there exists a positive c such that
$$\liminf_{h\to 0}\frac{\sup_{0\leqslant t\leqslant h}|X(t)|}{\sigma(h(c/\log\log(1/h))^{1/d})}\leqslant 1 \quad \text{a.s.} \tag{4.4.28}$$

Theorem 4.4.5 *Let $\{X(t);t\in[0,1]^d\}$ be a centered Gaussian field satisfying the conditions in Theorem 4.4.4. If, in addition, there exist $0<c_1\leqslant c_2<\infty$ such that*
$$\exp(-c_2(h/x)^d)\leqslant P\{\sup_{0\leqslant t\leqslant h}|X(t)|\leqslant\sigma(x)\}$$
$$\leqslant\exp(-c_1(h/x)^d) \tag{4.4.29}$$
for some $0<h_0<1$ and for any $0\leqslant x\leqslant h_0 h\leqslant h_0^2$. Then
$$1\leqslant\liminf_{h\to 0}\frac{\sup_{0\leqslant t\leqslant h}|X(t)|}{\sigma(h(c_1/\log\log(1/h))^{1/d})} \quad \text{a.s.} \tag{4.4.30}$$
and
$$\liminf_{h\to 0}\frac{\sup_{0\leqslant t\leqslant h}|X(t)|}{\sigma(h(c_2/\log\log(1/h))^{1/d})}\leqslant 1. \quad \text{a.s.} \tag{4.4.31}$$

In particular, from Theorem 4.4.4 we have the following Chung's LIL for a fractional Lévy Wiener field.

Theorem 4.4.6 *Let $\{Z(t);t\in[0,1]^d\}$ be a fractional Lévy Wiener field of order α, $0<\alpha<1$. Then we have*

$$\liminf_{h\to 0}\frac{(\log\log(1/h))^{\alpha/d}}{h^{\alpha}}\sup_{0\leqslant t\leqslant h}|Z(t)|=c\quad \text{a.s.}$$

for some $0<c<\infty$.

Proof of Theorem 4.4.4 Let

$$M(h)=\sup_{0\leqslant t\leqslant h}|X(t)|.$$

From (4.4.27) it follows that there is $0<\delta:=\delta_{\theta,\beta}\leqslant\beta$ such that

$$\sigma(kh)\leqslant 2k^{1-\delta}\sigma(h) \tag{4.4.32}$$

for every $0<h\leqslant 1$ and integer k with $1\leqslant k\leqslant 1/h$. By (4.4.26) and Minkowski's inequality, we have

$$\sigma(ah)\leqslant\sigma(h)+\sigma((a-1)h)\leqslant(1+(a-1)^{\beta})\sigma(h) \tag{4.4.33}$$

for every $1\leqslant a<2$ and $0<h<1/2$. Thus, the conditions of Theorem 4.4.1 are satisfied and whence there is $c_0>0$ such that

$$P\{M(h)\leqslant\sigma(x)\}\geqslant\exp(-c_0(h/x)^d) \tag{4.4.34}$$

for any $0<x\leqslant h\leqslant 1$. We shall prove that

$$\liminf_{h\to 0}M(h)/\sigma(h(c_0/\log\log(1/h))^{1/d})\leqslant 1\quad \text{a.s.}$$

For arbitrary $0<\epsilon<1$, put

$$t_k=\mathrm{e}^{-k^{1+\epsilon}},\quad d_k=\mathrm{e}^{k^{1+\epsilon}+k^{\epsilon}}\text{ and }\sigma_k=\sigma(t_k(c_0/\log\log(1/t_k))^{1/d}). \tag{4.4.35}$$

It suffices to show that

$$\liminf_{k\to\infty}M(t_k)/\sigma_k\leqslant 1+\epsilon^{\beta}\quad \text{a.s.} \tag{4.4.36}$$

To prove (4.4.36), we use the spectral representation of X, as Monrad and Rootzen (1995) did (cf. proof of Theorem 4.3.2). In what follows $s\cdot v$ or $\langle s,v\rangle$ denotes $\sum_{i=1}^d s_iv_i$. It is

known that $EX(s)X(t)$ has a unique Fourier representation of the form

$$E\{X(s)X(t)\}=\int_{\mathbf{R}^d}(\mathrm{e}^{is\cdot v}-1)(\mathrm{e}^{it\cdot v}-1)\Delta(\mathrm{d}v)+\langle s,Bt\rangle. \tag{4.4.37}$$

Here $B=(b_{ij})$ is a positive semidefinite matrix and $\Delta(\cdot)$ is a nonnegative measure on $\mathbf{R}^d-\{0\}$ satisfying

$$\int_{\mathbf{R}^d}\frac{\|v\|^2}{1+\|v\|^2}\Delta(\mathrm{d}v)<\infty.$$

Moreover, there exist a centered, complex-valued Gaussian random measure $W(\cdot)$ and a Gaussian random vector $\boldsymbol{Y}$ which is independent of W such that

$$X(t)=\int_{\mathbf{R}^d}(\mathrm{e}^{it\cdot v}-1)W(\mathrm{d}v)+\boldsymbol{Y}\cdot t. \tag{4.4.38}$$

The measure W and Δ are related by the identity

$$E\{W(A)\overline{W(B)}\}=\Delta(\boldsymbol{A}\cap\boldsymbol{B})$$

for all Borel sets $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbf{R}^d$. Furthermore

$$W(-\boldsymbol{A})=\overline{W(\boldsymbol{A})}.$$

It follows from (4.4.37) and (4.4.26) that

$$\sigma^2(\|t-s\|)=2\int_{\mathbf{R}^d}(1-\cos((t-s)\cdot v))\Delta(\mathrm{d}v)+\langle t,Bt\rangle.$$

In particular, for $0<h<1$ and for every $i=1,\cdots,d$

$$\begin{aligned}\sigma^2(h)&=2\int_{\mathbf{R}^d}(1-\cos(hv_i))\Delta(\mathrm{d}v)+t_i^2b_{ii}\\&\geqslant 2\int_{\mathbf{R}^d}(1-\cos(hv_i))\Delta(\mathrm{d}v).\end{aligned} \tag{4.4.39}$$

For $0<h<1$ and $1\leqslant i\leqslant d$, we have

$$\int_{\mathbf{R}^d,|v_i|\geqslant 1/h}\Delta(\mathrm{d}v)\leqslant\frac{1}{1-\sin 1}\int_{\mathbf{R}^d,|v_i|\geqslant 1/h}\left(1-\frac{\sin(hv_i)}{hv_i}\right)\Delta(\mathrm{d}v)$$

$$= \frac{1}{(1-\sin 1)h} \int_{\mathbf{R}^d, |v_i| \geqslant 1/h} \int_0^h (1-\cos(uv_i)) \mathrm{d}u \Delta(\mathrm{d}\boldsymbol{v})$$
$$= \frac{1}{(1-\sin 1)h} \int_0^h \int_{\mathbf{R}^d, |v_i| \geqslant 1/h} (1-\cos(uv_i)) \Delta(\mathrm{d}\boldsymbol{v}) \mathrm{d}u$$
$$\leqslant 4\sigma^2(h)$$

and hence

$$\int_{\|\boldsymbol{v}\| \geqslant 1/h} \Delta(\mathrm{d}\boldsymbol{v}) \leqslant 4d\sigma^2(dh) \leqslant 4d^3\sigma(h). \qquad (4.4.40)$$

Similarly, by (4.4.39)

$$\int_{\|\boldsymbol{v}\| \Delta(\mathrm{d}\boldsymbol{v}) \leqslant 1/h} \leqslant dh^{-2} \sum_{i=1}^d \int_{\mathbf{R}^d, |v_i| \leqslant 1/h} (hv_i)^2 \Delta(\mathrm{d}\boldsymbol{v})$$
$$\leqslant 4dh^{-2} \sum_{i=1}^d \int_{\mathbf{R}^d, |v_i| \leqslant 1/h} (1-\cos(hv_i)) \Delta(\mathrm{d}\boldsymbol{v})$$
$$\leqslant 4d^2h^{-2}\sigma^2(h). \qquad (4.4.41)$$

Define for $k=1,2,\cdots$ and $\mathbf{0} \leqslant \boldsymbol{t} \leqslant \mathbf{1}$

$$X_k(\boldsymbol{t}) = \int_{\|\boldsymbol{v}\| \in (d_{k-1}, d_k]} (\mathrm{e}^{i\boldsymbol{t}\cdot\boldsymbol{v}-1)}) W(\mathrm{d}\boldsymbol{v}),$$

$$\widetilde{X}_k(\boldsymbol{t}) = \int_{\|\boldsymbol{v}\| \notin (d_{k-1}, d_k]} (\mathrm{e}^{i\boldsymbol{t}\cdot\boldsymbol{v}} - 1) W(\mathrm{d}\boldsymbol{v}).$$

Clearly,

$$\liminf_{k\to\infty} \frac{M(t_k)}{\sigma_k} \leqslant \liminf_{k\to\infty} \frac{\sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |X_k(\boldsymbol{t})|}{\sigma_k}$$
$$+ \liminf_{k\to\infty} \frac{\sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |\widetilde{X}_k(\boldsymbol{t})|}{\sigma_k}$$
$$+ d \limsup_{k\to\infty} \frac{t_k \|\boldsymbol{Y}\|}{\sigma_k}. \qquad (4.4.42)$$

It is easy to see from (4.4.35) and (4.4.32) that

$$\limsup_{k\to\infty} \frac{t_k \|\boldsymbol{Y}\|}{\sigma_k}$$
$$\leqslant \|Y\| \limsup_{k\to\infty} t_k \big((\log\log(1/t_k))^{1/d} / (t_k c_0^{1/d}) \big)^{1-\delta}$$

$$= 0 \text{ a.s.} \qquad (4.4.43)$$

By Anderson's inequality (Corollary 1.2.4)

$$P\{\sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |X_k(\boldsymbol{t})| \leqslant (1+\varepsilon^\beta)\sigma_k\} \geqslant P\{M(t_k) \leqslant (1+\varepsilon^\beta)\sigma_k\}.$$

Therefore, by (4.4.33), (4.4.34) and (4.4.35)

$$\sum_{k=1}^\infty P\{\sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |X_k(\boldsymbol{t})| \leqslant (1+\varepsilon^\beta)\sigma_k\}$$
$$\geqslant \sum_{k=1}^\infty P\{M(t_k) \leqslant \sigma(t_k(1+\varepsilon)(c_0/\log\log(1/t_k))^{1/d})\}$$
$$\geqslant \sum_{k=1}^\infty \exp\{-(1+\varepsilon)^{-d} \log\log(1/t_i)\} = \infty. \qquad (4.4.44)$$

Since $\{\sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |X_k(\boldsymbol{t})|; k\geqslant 1\}$ are independent, by the Borel-Cantelli lemma, it follows from (4.4.44) that

$$\liminf_{k\to\infty} \sup_{\mathbf{0}\leqslant \boldsymbol{t}\leqslant t_k} |X_k(\boldsymbol{t})|/\sigma_k \leqslant 1 + \varepsilon^\beta \quad \text{a.s.} \qquad (4.4.45)$$

We next estimate the second term on the right hand side of (4.4.42). From (4.4.40), (4.4.41), (4.4.35) and (4.4.32) we obtain, for $\mathbf{0} \leqslant \boldsymbol{t} \leqslant t_k$

$$\mathrm{Var}(\widetilde{X}_k(\boldsymbol{t})) = 2\int_{\|\boldsymbol{v}\| \notin (d_{k-1}, d_k]} (1-\cos(\boldsymbol{t}\cdot\boldsymbol{v})) \Delta(\mathrm{d}\boldsymbol{v})$$
$$\leqslant \int_{\|\boldsymbol{v}\| \leqslant d_{k-1}} \|\boldsymbol{t}\|^2 \|\boldsymbol{v}\|^2 \Delta(\mathrm{d}\boldsymbol{v}) + 4\int_{\|\boldsymbol{v}\| \geqslant d_k} \Delta(\mathrm{d}\boldsymbol{v})$$
$$\leqslant 4d^3 t_k^2 d_{k-1}^2 \sigma^2(t_k/(t_k d_{k-1})) + d^3\sigma^2(t_k/(t_k d_k))$$
$$\leqslant 4d^3(t_k d_{k-1})^{2\delta}\sigma^2(t_k) + 4d^3(t_k d_k)^{-2\beta}\sigma^2(t_k)$$
$$\leqslant 8d^4 \mathrm{e}^{-\varepsilon k^\varepsilon}\sigma^2(t_k).$$

Therefore

$$\mathrm{Var}(\widetilde{X}_k(\boldsymbol{s}) - \widetilde{X}_k(\boldsymbol{t})) \leqslant \widetilde{\sigma}_k^2(h) \qquad (4.4.46)$$

for every $\mathbf{0} < \boldsymbol{s}, \boldsymbol{t} \leqslant t_k$, $\|\boldsymbol{s}-\boldsymbol{t}\| \leqslant h \leqslant t_k$, where $\widetilde{\sigma}_k^2(h) = \min(\sigma^2(h), 16d^4\mathrm{e}^{-\varepsilon k^\varepsilon}\sigma^2(t_k))$. Note that

$$\int_1^{\infty} \tilde{\sigma}_k(t_k e^{-y^2})dy \leqslant \int_1^{\infty} \min(4d^2 e^{-\epsilon k^\epsilon/2}\sigma(t_k),\ \sigma(t_k e^{-y^2}))dy$$
$$\leqslant \int_1^{\infty} \min(4d^2 e^{-\epsilon k^\epsilon/2}\sigma(t_k),\ \sigma(t_k)e^{-\beta y^2})dy$$
$$\leqslant Kk e^{-\epsilon k^\epsilon/2}\sigma(t_k),$$

where K denotes the constant depending only on d and β. Applying Fernique's inequality (Theorem 1.1.3) yields

$$P\{\sup_{0\leqslant t\leqslant t_k} |\tilde{X}_k(t)| > \eta\sigma_k\}$$
$$\leqslant K \exp\left(-\frac{(\eta\sigma_k)^2}{K\ k \exp(-\epsilon k^\epsilon)\sigma(t_k)}\right)$$
$$\leqslant K \exp\left(-\frac{\eta^2(\log\log(1/t_k))^{-2/d}}{K\ k \exp(\epsilon k^\epsilon)c_0^2}\right)$$
$$\leqslant K \exp\left(-\frac{\eta^2 \exp(\epsilon k^\epsilon)}{K\ k^2 c_0^2}\right)$$

for every $\eta > 0$. Thus, by the Borel-Cantelli lemma

$$\limsup_{k\to\infty} \sup_{0\leqslant t\leqslant t_k} |\tilde{X}_k(t)|/\sigma_k = 0 \quad \text{a.s.} \tag{4.4.47}$$

This proves (4.4.36) by (4.4.42), (4.4.43), (4.4.45) and (4.4.47).

Proof of Theorem 4.4.5 Using (4.4.33) and the subsequence method, one can easily obtain (4.4.30). (4.4.31) follows from the proof of Theorem 4.4.4.

Proof of Theorem 4.4.6 The results follow immediately from Theorems 4.4.3 and 4.4.5 and the zero-one law of Pitt and Tran (1979) (cf. Lemma 4.3.3).

4.4.3 An application to infinite series of O-U processes

The Chung's law of the iterated logarithm for the infinite series $X(t) = \sum_{k=1}^{\infty} X_k(t)$ of Ornstein-Uhlenbeck processes have

been given in Theorem 4.3.4. The proof is as follows:

Proof of Theorem 4.3.4

Actually, $0 < \sum_{k=1}^{\infty} \gamma_k/\lambda_k < \infty$ implies that $X(\cdot)$ is a stationary Gaussian process and that $\sigma^2(h)$ is concave on $(0,\infty)$. Let $f(v)$ be the spectral density of Δ in (4.4.37). It is easy to see that

$$f(v) = \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} \cdot \frac{\lambda_k}{\pi(\lambda_k^2 + v^2)}.$$

So

$$\liminf_{|v|\to\infty} |v|^3 \Delta(C_a(v)) > 0,$$

where $C_a(v) = \{t;\ |t-v| \leqslant a\}$. On the other hand, by Theorem 4.4.1 and Theorem 4.3.1, (4.4.29) holds. Then $X(\cdot)$ satisfies the zero-one law at $t=0$. Therefore, by Theorem 4.4.5, we have Theorem 4.3.4.

4.5 Liminfs for Increments of Gaussian Processes

In this section, we introduce the results of Csörgő and Shao (1994), in which they discuss a criterion on the limit inferior for $\inf_{0\leqslant t\leqslant b_T} \sup_{0\leqslant s\leqslant a_T} \| \Gamma(t+s) - \Gamma(t) \|$. The latter is related to the notion of non-differentiability of $\Gamma(\cdot)$. We state and prove this criterion in the subsection 4.5.1. Applications to Gaussian processes with stationary increments are given in subsection 4.5.2. In subsection 4.5.3, we establish an exact bound for the limit inferior for an infinite series of independent Ornstein-Uhlenbeck

processes. This, in turn, shows that the bound obtained via the general criterion is sharp.

4.5.1 Criteria for limit inferior

Let B be a separable Banach space with norm $\|\cdot\|$, $\{\Gamma(t); -\infty<t<\infty\}$ be a stochastic process with value in B, and P be the probability measure generated by $\Gamma(\cdot)$.

Theorem 4.5.1 *Let a_T and b_T be non-negative continuous functions and $v(t)$ be a non-negative monotonically non-decreasing function. Assume that there exist positive constants c and d such that for each $t>0$*

$$\frac{1+b_T}{a_T}+a_T\to\infty \quad as \quad T\to\infty, \tag{4.5.1}$$

$$P\{\sup_{0\leqslant s\leqslant a_T/2}\|\Gamma(t+s)-\Gamma(t)\|\leqslant v(t)\}$$
$$\leqslant c\exp(-da_T/x) \tag{4.5.2}$$

for each $0\leqslant t\leqslant 2b_T$ and

$$\frac{da_T}{4(\log(b_T/a_T)+\log\log\tilde{a}_T)}\leqslant x\leqslant\frac{4da_T}{(\log(b_T/a_T)+\log\log\tilde{a}_T)},$$

where $\tilde{x}=x+1/x$. Then, we have

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(da_T/2(\log(b_T/a_T)+\log\log\tilde{a}_T))}\geqslant\frac{1}{2}\ \text{a.s.} \tag{4.5.3}$$

It is easy to see that if $a_T\to 0$ or $a_T\to\infty$, or $b_T\to\infty$ as $T\to\infty$, then (4.5.1) is satisfied. Consequently, (4.5.3) includes the usual large and small increments as special cases.

The following example shows the generality of the above theorem. Let $\{W(t); t\geqslant 0\}$ be a standard Wiener process. It is

well-known that (cf. Csörgő and Révész 1981)

$$P\{\sup_{0\leqslant s\leqslant a_T/2}|W(t+s)-W(t)|\leqslant x^{1/2}\}\leqslant 2\exp\left(-\frac{\pi^2 a_T}{16x}\right)$$

for any $t\geqslant 0$, $a_T>0$, $0<x\leqslant a_T$. Therefore, by Theorem 4.5.1

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|W(t+s)-W(t)\|}{(\pi^2 a_T/32(\log(b_T/a_T)+\log\log\tilde{a}_T))^{1/2}}$$
$$\geqslant\frac{1}{2}\ \text{a.s.} \tag{4.5.4}$$

provided that (4.5.1) is satisfied. In particular,

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\frac{|W(s)|}{(h/\log\log h^{-1})^{1/2}}$$
$$\geqslant\frac{\pi}{8\sqrt{2}}\ \text{a.s.}\quad (b_T=0, a_T=1/T),$$

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{|W(t+s)-W(t)|}{(h/\log h^{-1})^{1/2}}$$
$$\geqslant\frac{\pi}{8\sqrt{2}}\ \text{a.s.}\quad (b_T=1, a_T=1/T),$$

$$\liminf_{T\to\infty}\sup_{0\leqslant s\leqslant T}\frac{|W(s)|}{(T/\log\log T)^{1/2}}$$
$$\geqslant\frac{\pi}{\sqrt{24}}\ \text{a.s.}\quad (b_T=0, a_T=T),$$

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\frac{|W(t+s)-W(t)|}{(a_T/(\log(T/a_T)+\log\log\tilde{a}_T))^{1/2}}$$
$$\geqslant\frac{\pi}{8\sqrt{2}}\ \text{a.s.}\quad (b_T=T),$$

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}(\log T)^{1/2}|W(t+s)-W(t)|$$
$$\geqslant\frac{\pi}{8\sqrt{2}}\ \text{a.s.}\quad (b_T=T, a_T=1).$$

It is well-known that in all the above results, the liminf, up to a constant, is sharp (cf. Csörgő and Révész 1981). These important examples for illustrating the sharpness in rate of Theorem

4.5.1 do not, of course, imply that our general rate function in (4.5.3) can necessarily be made sharp in all possible situations.

The following lemma is crucial for the proof of Theorem 4.5.1.

Lemma 4.5.1 *For any $a>0$, $b>0$, we have*

$$\inf_{0\leqslant t\leqslant b}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|$$

$$\geqslant\frac{1}{2}\min_{0\leqslant i\leqslant[2b/a]+1}\sup_{0\leqslant s\leqslant a/2}\left\|\Gamma\left(\frac{ia}{2}+S\right)-\Gamma\left(\frac{ia}{2}\right)\right\|. \tag{4.5.5}$$

Proof Clearly,

$$\inf_{0\leqslant t\leqslant b}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|$$

$$\geqslant\min_{0\leqslant i\leqslant[2b/a]}\inf_{ia/2\leqslant t\leqslant(i+1)a/2}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|.$$

It suffices to show that for each $0\leqslant i\leqslant[2b/a]$

$$\sup_{0\leqslant s\leqslant a/2}\left\|\Gamma\left(\frac{(i+1)a}{2}+s\right)-\Gamma\left(\frac{(i+1)a}{2}\right)\right\|$$

$$\leqslant 2\inf_{ia/2\leqslant t\leqslant(i+1)a/2}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|. \tag{4.5.6}$$

Assume $\inf_{ia/2\leqslant t\leqslant(i+1)a/2}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|<\infty$. Otherwise, (4.5.6) is trivial. For any fixed $0<\varepsilon<1$, let $ia/2\leqslant t_i\leqslant(i+1)a/2$ such that

$$\sup_{0\leqslant s\leqslant a}\|\Gamma(t_i+s)-\Gamma(t_i)\|$$

$$\leqslant\varepsilon+\inf_{ia/2\leqslant t\leqslant(i+1)a/2}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\|. \tag{4.5.7}$$

Then

$$\sup_{0\leqslant s\leqslant a/2}\|\Gamma((i+1)a/2+s)-\Gamma((i+1)a/2)\|$$

$$\leqslant\sup_{0\leqslant s\leqslant a/2}\|\Gamma((i+1)a/2+s)-\Gamma(t_i)\|+\|\Gamma((i+1)a/2)-\Gamma(t_i)\|$$

$$\leqslant 2\sup_{0\leqslant s\leqslant a}\|\Gamma(t_i+s)-\Gamma(t_i)\|$$

$$\leqslant 2\varepsilon+2\inf_{ia/2\leqslant t\leqslant(i+1)a/2}\sup_{0\leqslant s\leqslant a}\|\Gamma(t+s)-\Gamma(t)\| \tag{4.5.8}$$

by (4.5.7). This proves (4.5.6), by (4.5.8) and the arbitrariness of ε, as desired.

Proof of Theorem 4.5.1 Set

$$c_T=da_T/2(\log(b_T/a_T)+\log\log\widetilde{a}_T).$$

Let $1<\theta<5/4$, and define

$$A_k=\{T;\theta^k\leqslant a_T\leqslant\theta^{k+1}\},\ -\infty<k<\infty,$$

$$A_{k,j}=\{T;\theta^j\leqslant\frac{b_T}{a_T}+1\leqslant\theta^{j+1},T\in A_k\},\ j=0,1,2,\cdots,$$

$$G=\{(k,j);-\infty<k<\infty,j\geqslant0,A_{k,j}\neq\varnothing\}.$$

Noting that for $T\in A_{k,j}$, if $|k|+j\to\infty$, $(k,j)\in G$, we have

$$c_T\leqslant\frac{d\theta^{k+1}}{2(\log\theta^j+\log\log\theta^{|k|})}\leqslant\frac{2}{3}\frac{d\theta^k}{(\log\theta^j+\log\log\theta^{|k|})}=:c_{k,j},$$

and using (4.5.1) and Lemma 4.5.1, we obtain

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_T)}$$

$$\geqslant\liminf_{|k|+l\to\infty}\inf_{j\geqslant l,(k,j)\in G}\inf_{T\in A_{k,j}}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_T)}$$

$$\geqslant\liminf_{|k|+l\to\infty}\inf_{j\geqslant l,(k,j)\in G}\inf_{T\in A_{k,j}}\inf_{0\leqslant t\leqslant\theta^{j+k+2}}\sup_{0\leqslant s\leqslant\theta^k}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_{k,j})}$$

$$=\liminf_{|k|+l\to\infty}\inf_{j\geqslant l,(k,j)\in G}\inf_{0\leqslant t\leqslant\theta^{j+k+2}}\sup_{0\leqslant s\leqslant\theta^k}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_{k,j})}$$

$$\geqslant\frac{1}{2}\liminf_{|k|+l\to\infty}\inf_{j\geqslant l,(k,j)\in G}\min_{0\leqslant i\leqslant2\theta^{j+2}+1}\sup_{0\leqslant s\leqslant\frac{\theta}{2}^k}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_{k,j})}. \tag{4.5.9}$$

By (4.5.2), we have

$$P\left\{\inf_{j\geqslant l,(k,j)\in G}\min_{0\leqslant i\leqslant2\theta^{j+2}+1}\sup_{0\leqslant s\leqslant\theta^k/2}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(c_{k,j})}\leqslant1\right\}$$

$$\leqslant\sum_{j\geqslant l}\sum_{0\leqslant i\leqslant2\theta^{j+2}+1}P\{\sup_{0\leqslant s\leqslant\theta^k/2}\|\Gamma(t+s)-\Gamma(t)\|<v(c_{k,j})\}$$

$$\leqslant c\sum_{j\geqslant l}\sum_{0\leqslant i\leqslant2\theta^{j+2}+1}\exp\left(-d\frac{\theta^k}{c_{k,j}}\right)$$

$$=c\sum_{j\geqslant l}\sum_{0\leqslant i\leqslant2\theta^{j+2}+1}\exp\left(-\frac{3}{2}(\log\theta^j+\log\log\theta^{|k|})\right)$$

$$\leqslant c \sum_{j\geqslant l}(|k|+1)^{-3/2}\theta^{-j/2}$$
$$\leqslant c(|k|+1)^{-3/2}\theta^{-l/2}. \tag{4.5.10}$$

Now (4.5.3) follows from (4.5.9), (4.5.10) and the Borel-Cantelli lemma.

4.5.2 Gaussian processes with stationary increments

As an application of Theorem 4.5.1, we now study a general limit inferior problem for real valued Gaussian processes. Let $\{G(t);t\geqslant 0\}$ be a real valued Gaussian process with mean zero and stationary increments. Put

$$\sigma^2(h)=E(G(t+h)-G(t))^2,\ t,h\geqslant 0. \tag{4.5.11}$$

Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,1)$. Then, by Theorem 4.3.1, we have

$$P\{\sup_{0\leqslant t\leqslant 1}|G(t)-G(0)|\leqslant\sigma(x)\}\leqslant 2\exp(-0.17/x) \tag{4.5.12}$$

for all $0<x<1$. A combination of (4.5.12) and Theorem 4.5.1 yields the following theorem.

Theorem 4.5.2 *Let $\{a_T;t\geqslant 0\}$ and $\{b_T;T\geqslant 0\}$ be non-negative continuous functions satisfying (4.5.1), and $\{G(t);t\geqslant 0\}$ be a real valued Gaussian process with mean zero and stationary increments. Put $a^*=\sup_{T>0}a_T$. Assume that $\sigma^2(h)$ is non-decreasing and concave on $(0,a^*)$. Then, we have*

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\frac{|G(t+s)-G(t)|}{\sigma(a_T/24(\log(b_T/a_T)+\log\log\widetilde{a}_T))}\geqslant\frac{1}{2}\ \text{a.s.} \tag{4.5.13}$$

Proof By Theorem 4.5.1 it suffices to show that

$$P\{\sup_{0\leqslant s\leqslant a/2}|G(t+s)-G(t)|\leqslant\sigma(x)\}\leqslant 2\exp\left(-\frac{a}{12x}\right) \tag{4.5.14}$$

for each $0<x<a/2$, $0<a<a^*$. Put

$$G^*(s)=G(t+sa/2)-G(t),\quad s\geqslant 0,$$
$$\sigma^{*2}(h)=E(G^*(s+h)-G^*(s))^2=\sigma^2(ah/2).$$

By (4.5.12), we have

$$P\{\sup_{0\leqslant s\leqslant a/2}|G(t+s)-G(t)|\leqslant\sigma(x)\}$$
$$=P\left\{\sup_{0\leqslant s\leqslant 1}|G(t+sa/2)-G(t)|\leqslant\sigma\left(\frac{a}{2}\frac{x}{(a/2)}\right)\right\}$$
$$=P\left\{\sup_{0\leqslant s\leqslant 1}|G^*(s)|\leqslant\sigma^*\left(x/(\frac{a}{2})\right)\right\}$$
$$\leqslant 2\exp\{-0.17a/(2x)\}\leqslant 2\exp(-a/(12x)),$$

as desired.

An immediate consequence of Theorem 4.5.2 for a fractional Wiener process reads as follows.

Theorem 4.5.3 *Let $\{Z(t);t\geqslant 0\}$ be a fractional Wiener process of order α, $0<\alpha\leqslant 1/2$, i.e., $\sigma^2(h)=h^{2\alpha}$. Then*

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant a_T}\left(\frac{\log(b_T/a_T)+\log\log\widetilde{a}_T}{a_T}\right)^\alpha|Z(t+s)-Z(t)|$$
$$\geqslant 0.1\ \text{a.s.}, \tag{4.5.15}$$

provided (4.5.1) *is satisfied. In particular, we have*

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\left(\frac{\log\log(1/h)}{h}\right)^\alpha|Z(s)|\geqslant 0.1\quad\text{a.s.}, \tag{4.5.16}$$

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\left(\frac{\log(1/h)}{h}\right)^\alpha|Z(t+s)-Z(t)|\geqslant 0.1\ \text{a.s.}, \tag{4.5.17}$$

$$\liminf_{T\to\infty}\sup_{0\leqslant s\leqslant T}\left(\frac{\log\log T}{T}\right)^\alpha|Z(s)|\geqslant 0.1\ \text{a.s.}, \tag{4.5.18}$$

$$\liminf_{T\to\infty}\inf_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant a_T}\left(\frac{\log(T/a_T)+\log\log\widetilde{a}_T}{a_T}\right)^\alpha|Z(t+s)-Z(t)|$$
$$\geqslant 0.1\ \text{a.s.} \tag{4.5.19}$$

4.5.3 Infinite series of independent O-U processes

In this subsection, we will prove that the bound in (4.5.13) is best possible for the infinite series of Ornstein-Uhlenbeck processes, which, in turn, shows that the bound of our limit inferior criteria is quite sharp.

The infinite series of the independent Ornstein-Uhlenbeck coordinate processes of $Y(\cdot)$ is defined by

$$\{X(t);-\infty<t<\infty\}=\left\{\sum_{k=1}^{\infty}X_k(t);-\infty<t<\infty\right\} \tag{4.5.20}$$

and the Lévy exact moduli of continuity for $\{X(\cdot)\}$ have been established by Csáki et al. (1991) (cf. also Theorem 2.2.5 and Remark 2.2.4).

The main aim of this subsection is to present limit inferior results for $\{X(\cdot)\}$. Throughtout this subsection we assume that

$$0<\Gamma_0=\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}<\infty, \tag{4.5.21}$$

which, in turn, implies that $X(\cdot)$ is a stationary Gaussian process, and

$$\begin{aligned}\sigma^2(h)&=E(X(t+h)-X(t))^2\\&=2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h}),\quad h\geqslant 0, t\geqslant 0.\end{aligned} \tag{4.5.22}$$

On account of $\Gamma_0<\infty$, it is easy to find that $\sigma^2(h)$ is concave on $(0,\infty)$. Therefore, as a direct corollary of Theorem 4.5.2, we obtain

Theorem 4.5.4 *Let b_h be a non-negative continuous function of h. Then*

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant b_h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h/(24(\log(b_h/h)+\log\log(1/h))))}\geqslant 0.5\quad \text{a.s.} \tag{4.5.23}$$

In particular, we have

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h/(24\log(1/h)))}\geqslant 0.5\quad \text{a.s.} \tag{4.5.24}$$

and

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h/(24\log\log(1/h)))}\geqslant 0.5\quad \text{a.s.} \tag{4.5.25}$$

In next theorem, we give the exact upper bound for limit inferior.

Theorem 4.5.5 *Let b_h be a non-negative continuous function on $(0,1)$, satisfying*

$$\lim_{h\to 0}\frac{\log(b_h/h)}{\log\log(1/h)}=\infty. \tag{4.5.26}$$

Assume that there exist constants $\theta>1$, $\alpha>0$ such that

$$\sigma(ah)\leqslant\theta a^{\alpha}\sigma(h)\quad \textit{for all } 0\leqslant a,\ h\leqslant 1. \tag{4.5.27}$$

Then, there is a $d>0$ depending only on α such that

$$\limsup_{h\to 0}\inf_{0\leqslant t\leqslant b_h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(db_h\log(b_h/h))}\leqslant 4\theta\quad \text{a.s.} \tag{4.5.28}$$

A combination of Theorem 4.5.4 and Theorem 4.5.5 yields

Corollary 4.5.1 *Let b_h be a non-negative continuous function satisfying (4.5.26). Assume that (4.5.27) is satisfied. Then*

$$\begin{aligned}\frac{1}{2}&\leqslant\liminf_{h\to 0}\inf_{0\leqslant t\leqslant b_h}\sup_{0\leqslant s\leqslant h}\frac{1}{\sigma(h/(24\log(b_h/h)))}|X(t+s)-X(t)|\\&\leqslant\liminf_{h\to 0}\inf_{0\leqslant t\leqslant b_h}\sup_{0\leqslant s\leqslant h}\frac{1}{\sigma(dh/\log(b_h/h))}|X(t+s)-X(t)|\leqslant 4\theta\quad \text{a.s.}\end{aligned}$$

In particular

$$\frac{1}{2}\leqslant\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\left(\frac{1}{\sigma(h/(24\log(1/h)))}\right)|X(t+s)-X(t)|$$

$$\leqslant\limsup_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\left(\frac{1}{\sigma(dh/\log(1/h))}\right)|X(t+s)-X(t)|$$

$$\leqslant 4\theta \quad \text{a. s.}$$

Corollary 4. 5. 2 *Assume that*

$$\lim_{h\to 0}\frac{\sigma^2(h)}{h^{2\alpha}}=\theta_0 \tag{4. 5. 29}$$

for some $0<\alpha\leqslant 1/2$, $\theta_0>0$. *Then*

$$\frac{1}{2(24)^{\alpha}}\leqslant\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|$$

$$\leqslant\limsup_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|$$

$$\leqslant d \quad \text{a. s.} \tag{4. 5. 30}$$

for some $d=d(\alpha)<\infty$.

From (4. 5. 21) it follows that $\sigma^2(h)/h$ is non-increasing on $(0,\infty)$. Hence we have

$$\sigma(ah)\geqslant a^{1/2}\sigma(h) \tag{4. 5. 31}$$

for each $0<a\leqslant 1$, $h>0$, which together with Theorem 4. 5. 4 implies

Corollary 4. 5. 3 *We have*

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|\geqslant 0.1 \quad \text{a. s.}$$

Remark 4. 5. 1 A combination of (4. 5. 24), (4. 5. 25), (4. 5. 30) with the zero-one law of Pitt and Tran (1979) (cf. Lemma 4. 3. 3) yields that, under the condition (4. 5. 29), there are constants $0<c_1<\infty$, $0<c_2$, $c_3<\infty$ such that

$$\liminf_{h\to 0}\sup_{0\leqslant s\leqslant h}\frac{(\log\log h^{-1})^{\alpha}}{\sigma(h)}|X(s)-X(0)|=c_1 \quad \text{a. s.},$$

$$\liminf_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=c_2 \quad \text{a. s.},$$

$$\limsup_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=c_3 \quad \text{a. s.}$$

Remark 4. 5. 2 From (4. 5. 31) it follows that $\liminf_{h\to 0}\sigma^2(h)/h>0$ and hence

$$\lim_{h\to 0}h(\log h^{-1})^{1/2}/\sigma(h)=0.$$

Therefore by Corollary 4. 5. 3, almost all sample functions of the process $X(\cdot)$ are non-differentiable.

Remark 4. 5. 3 It should be mentioned that almost all of the known results on the limit superior for $X(\cdot)$ or l^p-valued Gaussian processes (cf. Section 3. 3) parallel the corresponding ones for a standard Wiener process (cf. e. g. Chapter 1 of Csörgő and Révész 1981). The above results show that the situation is quite different for limit inferior. It seems that the limit inferior is more sensitive to deviations from a standard Wiener process than the limit superior.

Remark 4. 5. 4 If $\sigma(h)/h^{\alpha}$ is non-decreasing, then (4. 5. 27) is satisfied. Actually, (4. 5. 27) is equivalent to $\sigma(h)/h^{\alpha}$ being quasi-increasing.

Zhang (1995) established the following exact moduli of non-diferentiability for $X(\cdot)$.

Theorem 4. 5. 6 *Assume that*

$$\Gamma_1:=2\sum_{k=1}^{\infty}\gamma_k<\infty. \tag{4. 5. 32}$$

Then, we have

$$\lim_{h\to 0}\inf_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(t+s)-X(t)|=1 \quad \text{a. s.} \tag{4. 5. 33}$$

Csörgő and Shao (1994) pointed out that there should exist a constant $C(\alpha)$ such that (4.5.30) holds with equality instead of inequalities. They proposed the following conjecture.

Conjecture *Assume that* (4.5.29) *is satisfied. Then, there exists a positive constant* $C(\alpha)$ *depending only on* α, *such that*

$$\liminf_{h\to 0}\sup_{0\le s\le h}\frac{(\log\log h^{-1})^{\alpha}}{\sigma(h)}|X(s)| = C(\alpha)\quad \text{a.s.},$$

$$\liminf_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\frac{(\log h^{-1})^{\alpha}}{\sigma(h)}|X(t+s)-X(t)|=C(\alpha)\quad \text{a.s.}$$

The exact value of $C(\alpha)$ seems to be not easy to obtain.

Proof of Theorem 4.5.5

Let $\{W_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent standard Wiener processes. Then

$$\left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}};-\infty<t<\infty\right\}_{k=1}^{\infty} \text{ and } \{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$$

have the same distribution. Hence we can rewrite $\{X(t);-\infty<t<\infty\}$ as

$$X(t)=\sum_{k=1}^{\infty}\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}},\tag{4.5.34}$$

and remain the almost sure path properties of $X(\cdot)$ without change. Define

$$A_k=\{0<h<1;e^k\le b_h/h\le e^{k+1}\},\ k=0,1,2,\cdots,$$
$$A_{k,j}=\{h;e^{-j-1}\le h\le e^{-j},h\in A_k\},\quad j=0,1,2,\cdots,$$
$$h_{k,j}=e^{-j-1}k^8.$$

From (4.5.26) it follows that

$$A_{k,j}=\varnothing\quad \text{if } j\ge e^{k/4} \text{ and } k \text{ is sufficiently large.}$$

Thus, we have

$$\limsup_{h\to 0}\inf_{0\le t\le b_h}\sup_{0\le s\le h}\frac{|X(t+s)-X(t)|}{\sigma(dh/\log(b_h/h))}$$
$$\le\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\sup_{h\in A_{k,j}}\inf_{0\le t\le b_h}\sup_{0\le s\le h}\frac{|X(t+s)-X(t)|}{\sigma(dh/\log(b_h/h))}$$
$$\le\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\sup_{h\in A_{k,j}}\inf_{0\le t\le e^{-j+k-1}}\sup_{0\le s\le e^{-j}}\frac{|X(t+s)-X(t)|}{\sigma(de^{-j-1}/\log e^{k+1})}$$
$$=\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\inf_{0\le t\le e^{-j+k-1}}\sup_{0\le s\le e^{-j}}\frac{|X(t+s)-X(t)|}{\sigma(de^{-j-1}/(k+1))}$$
$$\le\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\inf_{0\le i\le e^{-j+k-1}/h_{k,j}}\sup_{0\le s\le e^{-j}}\frac{|X(2ih_{k,j}+s)-X(2ih_{k,j})|}{\sigma(de^{-j-1}/(k+1))}$$
$$\le\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\min_{0\le i\le e^{k}/k^8}\sup_{0\le s\le e^{-j}}\frac{|X(2ih_{k,j}+s)-X(2ih_{k,j})|}{\sigma(de^{-j-1}/(k+1))}.\tag{4.5.35}$$

By (4.5.34), we can write

$$\sup_{0\le s\le e^{-j}}|X(2mh_{k,j}+s)-X(2mh_{k,j})|$$
$$=\sup_{0\le s\le e^{-j}}\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}\left(\frac{W_i(e^{2\lambda_i(s+2mh_{k,j})})}{e^{\lambda_i(s+2mh_{k,j})}}-\frac{W_i(e^{2\lambda_i 2mh_{k,j}})}{e^{\lambda_i 2mh_{k,j}}}\right)\right|$$
$$\le\sup_{0\le s\le e^{-j}}\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}\frac{W_i(e^{2\lambda_i(s+2mh_{k,j})})-W_i(e^{2\lambda_i(2m-1)h_{k,j}})}{e^{\lambda_i(s+2mh_{k,j})}}\right|$$
$$+\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}\frac{W_i(e^{2\lambda_i 2mh_{k,j}})-W_i(e^{2\lambda_i(2m-1)h_{k,j}})}{e^{\lambda_i 2mh_{k,j}}}\right|$$
$$+\sup_{0\le s\le e^{-j}}\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}W_i(e^{2\lambda_i(2m-1)h_{k,j}})(e^{-\lambda_i(s+2mh_{k,j})}-e^{-\lambda_i 2mh_{k,j}})\right|$$
$$\le 2\sup_{0\le s\le e^{-j}}\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}\frac{W_i(e^{2\lambda_i(s+2mh_{k,j})})-W_i(e^{2\lambda_i(2m-1)h_{k,j}})}{e^{\lambda_i(s+2mh_{k,j})}}\right|$$
$$+\sup_{0\le s\le e^{-j}}\left|\sum_{i=1}^{\infty}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}\frac{W_i(e^{2\lambda_i(2m-1)h_{k,j}})}{e^{\lambda_i 2mh_{k,j}}}(1-e^{-\lambda_i s})\right|$$
$$=:2\sup_{0\le s\le e^{-j}}|\xi_{m,j}(s)|+\sup_{0\le s\le e^{-j}}|\eta_{m,j}(s)|.\tag{4.5.36}$$

Therefore

$$\limsup_{k\to\infty}\max_{2\le j\le e^{k/4}}\min_{0\le i\le e^{k}/k^8}\sup_{0\le s\le e^{-j}}\frac{|X(2ih_{k,j}+s)-X(2ih_{k,j})|}{\sigma(de^{-j-1}/(k+1))}$$

$$\leqslant 2\limsup_{k\to\infty}\max_{2\leqslant j\leqslant e^{k/4}}\min_{0\leqslant i\leqslant e^k/k^8}\sup_{0\leqslant s\leqslant e^{-j}}\frac{|\xi_{i,j}(s)|}{\sigma(de^{-j-1}/(k+1))}$$
$$+\limsup_{k\to\infty}\max_{2\leqslant j\leqslant e^{k/4}}\max_{0\leqslant i\leqslant e^k/k^8}\sup_{0\leqslant s\leqslant e^{-j}}\frac{|\eta_{i,j}(s)|}{\sigma(de^{-j-1}/(k+1))}$$
$$=: 2J_1+J_2. \tag{4.5.37}$$

We first prove that

$$\sum_{k=1}^{\infty}\sum_{2\leqslant j\leqslant e^{k/4}}P\left\{\min_{0\leqslant i\leqslant e^k/k^8}\sup_{0\leqslant s\leqslant e^{-j}}\frac{|\xi_{i,j}(s)|}{\sigma(de^{-j-1}/(k+1))}\geqslant 2\right\}$$
$$=\sum_{k=1}^{\infty}\sum_{2\leqslant j\leqslant e^{k/4}}\prod_{0\leqslant i\leqslant e^k/k^8}\left(1-P\left\{\sup_{0\leqslant s\leqslant e^{-j}}\frac{|\xi_{i,j}(s)|}{\sigma(de^{-j-1}/(k+1))}<2\right\}\right)$$
$$<\infty \tag{4.5.38}$$

for some $d>0$, which, in turn, will imply immediately

$$2J_1\leqslant 4 \quad \text{a.s.} \tag{4.5.39}$$

by the Borel-Cantelli lemma.

Clearly, $\{\xi_{i,j}(s);0\leqslant s\leqslant e^{-j}\}$ is a Gaussian process with mean zero. For $0\leqslant s\leqslant s+h\leqslant e^{-j}$, we have

$$E(\xi_{i,j}(s+h)-\xi_{i,j}(s))^2$$
$$=E\left\{\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}\left(\frac{W_l(e^{2\lambda_l(s+h+2ih_{k,j})})-W_l(e^{2\lambda_l(2i-1)h_{k,j}})}{e^{\lambda_l(s+h+2ih_{k,j})}}-\frac{W_l(e^{2\lambda_l(s+2ih_{k,j})})-W_l(e^{2\lambda_l(2i-1)h_{k,j}})}{e^{\lambda_l(s+2ih_{k,j})}}\right)\right\}^2$$
$$=\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}\left(\frac{e^{2\lambda_l(s+h+2ih_{k,j})}-e^{2\lambda_l(s+2ih_{k,j})}}{e^{2\lambda_l(s+h+2ih_{k,j})}}+(1-e^{-\lambda_l h})^2\frac{e^{2\lambda_l(s+2ih_{k,j})}-e^{2\lambda_l(2i-1)h_{k,j}}}{e^{2\lambda_l(s+2ih_{k,j})}}\right)$$
$$=\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}(1-e^{-2\lambda_l h}+(1-e^{-\lambda_l h})^2(1-e^{-2\lambda_l(s+h_{k,j})}))$$
$$=\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}(1-e^{-\lambda_l h})(2-(1-e^{-\lambda_l h})e^{-2\lambda_l(s+h_{k,j})})$$

$$\leqslant\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}(1-e^{-\lambda_l h})$$
$$=\sigma^2(h) \tag{4.5.40}$$

and

$$E\xi_{i,j}^2(0)=\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}\frac{e^{2\lambda_l 2ih_{k,j}}-e^{2\lambda_l(2i-1)h_{k,j}}}{e^{2\lambda_l 2ih_{k,j}}}=\frac{1}{2}\sigma^2(2h_{k,j}).$$

Applying Theorem 4.4.1 and Theorem 1.2.4, we have for $2\leqslant j\leqslant e^{k/4}$

$$P\{\sup_{0\leqslant s\leqslant e^{-j}}|\xi_{i,j}(s)|\leqslant 2\sigma(de^{-j-1}/(k+1))\}$$
$$\geqslant P\{\sup_{0\leqslant s\leqslant e^{-j}}|\xi_{i,j}(s)-\xi_{i,j}(0)|\leqslant\sigma(de^{-j-1}/(k+1)),\ |\xi_{i,k}(0)|\leqslant\sigma(de^{-j-1}/(k+1))\}$$
$$\geqslant P\{\sup_{0\leqslant s\leqslant e^{-j}}|\xi_{i,j}(s)-\xi_{i,j}(0)|\leqslant\sigma(de^{-j-1}/(k+1))\}\times P\{|\xi_{i,k}(0)|\leqslant\sigma(de^{-j-1}/(k+1))\}$$
$$\geqslant\exp(-ed^{-1}d_\alpha(k+1))P\left\{|N(0,1)|\leqslant\frac{\sigma(de^{-j-1}/(k+1))}{\sigma(2h_{k,j})}\right\}$$
$$\geqslant\exp(-ed^{-1}d_\alpha(k+1))P\left\{|N(0,1)|\leqslant\frac{d}{4(k+1)}\right\}$$
$$\geqslant\frac{d}{8(k+1)}\exp(-ed^{-1}d_\alpha(k+1))$$
$$\geqslant\exp(-k/4) \tag{4.5.41}$$

for every k large enough, provided that d is chosen sufficiently large, where d_α is a constant depending only on α. Now (4.5.38) follows immediately from (4.5.41).

We next show that

$$J_2=0 \quad \text{a.s.} \tag{4.5.42}$$

For $0\leqslant s\leqslant s+h\leqslant e^{-j}$ we have

$$E(\eta_{i,j}(s+h)-\eta_{i,j}(s))^2$$
$$=\sum_{l=1}^{\infty}\frac{\gamma_l}{\lambda_l}e^{-2\lambda_l h_{k,j}}(e^{-\lambda_l(s+h)}-e^{-\lambda_l s})^2$$

$$\leqslant \sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l} e^{-2\lambda_l h_{k,j}}(1-e^{-\lambda_l h})^2$$

$$\leqslant \left(\sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l}(1-e^{-\lambda_l h}) \right)^{1/2} \left(\sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l} e^{-4\lambda_l h_{k,j}}(1-e^{-\lambda_l h})^3 \right)^{1/2}$$

$$\leqslant \sigma(h) \left(\sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l} e^{-4\lambda_l h_{k,j}}(1-e^{-\lambda_l e^{-j}})^3 \right)^{1/2} \tag{4.5.43}$$

and

$$\sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l} e^{-4\lambda_l h_{k,j}}(1-e^{-\lambda_l e^{-j}})^3$$

$$= \sum_{\lambda_l e^{-j} \leqslant k^{-7}} \frac{\gamma_l}{\lambda_l} e^{-4\lambda_l h_{k,j}}(1-e^{-\lambda_l e^{-j}})^3$$

$$+ \sum_{\lambda_l e^{-j} > k^{-7}} \frac{\gamma_l}{\lambda_l} e^{-4\lambda_l h_{k,j}}(1-e^{-\lambda_l e^{-j}})^3$$

$$\leqslant k^{-14} \sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l}(1-e^{-\lambda_l e^{-j}}) + e^{-k} \sum_{l=1}^{\infty} \frac{\gamma_l}{\lambda_l}(1-e^{-\lambda_l e^{-j}})$$

$$\leqslant 2k^{-14}\sigma^2(e^{-j}). \tag{4.5.44}$$

We obtain

$$E(\eta_{i,j}(s+h) - \eta_{i,j}(s))^2 \leqslant 2\sigma(h) k^{-7} \sigma(e^{-j})$$

and

$$E\eta_{i,j}^2(h) = E(\eta_{i,j}(h) - \eta_{i,j}(0))^2 \leqslant 2\sigma(h) k^{-7}\sigma(e^{-j}).$$

By (4.5.27),

$$\int_1^{\infty} \sigma^{1/2}(e^{-j} e^{-y^2}) dy \leqslant (\theta\sigma(e^{-j}))^{1/2} \int_1^{\infty} e^{-\alpha y^2/2} dy \leqslant \frac{2\theta^{1/2}\sigma^{1/2}(e^{-j})}{\sqrt{\alpha}}.$$

Applying Fernique's inequality (Theorem 1.1.3) yields for each $x>0$

$$P\left\{ \sup_{0\leqslant s\leqslant e^{-j}} |\eta_{i,j}(s)| \geqslant k^{-7}\sigma(e^{-j}) \left(1 + \frac{2\theta^{1/2}}{\sqrt{\alpha}} \right) x \right\}$$

$$\leqslant A \exp(-x^2/2). \tag{4.5.45}$$

Recalling that $\sigma(h) \geqslant \sigma(mh)/m$ for any integer $m \geqslant 1$, $h>0$, by Minkowski's inequality we derive from (4.5.45) that

$$P\left\{ \sup_{0\leqslant s\leqslant e^{-j}} |\eta_{i,j}(s)| \geqslant \frac{1}{k}\sigma(e^{-j} d/(k+1)) \right\}$$

$$\leqslant P\left\{ \sup_{0\leqslant s\leqslant e^{-j}} |\eta_{i,j}(s)| \geqslant \frac{d\sigma(e^{-j})}{(k+1)^2} \right\}$$

$$\leqslant P\left\{ \sup_{0\leqslant s\leqslant e^{-j}} |\eta_{i,j}(s)| \geqslant k^{-7}\sigma(e^{-j}) \left(1 + \frac{2\theta^{1/2}}{\sqrt{\alpha}} \right) k^4 \right\}$$

$$\leqslant A \exp(-k^8/2) \tag{4.5.46}$$

for every k sufficiently large.

This proves (4.5.42) by (4.5.46) and the Borel-Cantelli lemma.

Proof of Theorem 4.5.6 First we prove that for any $\varepsilon>0$ small enough,

$$\liminf_{h\to 0} \inf_{0\leqslant t\leqslant 1-h} \sup_{0\leqslant s\leqslant h} \left(\frac{8 \log h^{-1}}{\pi^2 \Gamma_1 h} \right)^{1/2} |X(t+s) - X(t)| \geqslant 1-\varepsilon \text{ a.s.} \tag{4.5.47}$$

Let

$$\rho_h = [h^{-1}(\log h^{-1})^3], \quad t_i = ih(\log h^{-1})^{-3},\ i = 0,1,\cdots,\rho_h.$$

By Lemma 4.3.5, we can choose h_0 small enough such that for $0 \leqslant h \leqslant h_0$

$$P\left\{ \min_{0\leqslant i\leqslant \rho_h} \sup_{0\leqslant s\leqslant h} \left(\frac{8 \log h^{-1}}{\pi^2 \Gamma_1 h} \right)^{1/2} |X(t_i+s) - X(t_i)| \leqslant 1-\varepsilon \right\}$$

$$\leqslant (\rho_h+1) P\left\{ \sup_{0\leqslant s\leqslant h} \left(\frac{8 \log h^{-1}}{\pi^2 \Gamma_1 h} \right)^{1/2} |X(s) - X(0)| \leqslant 1-\varepsilon \right\}$$

$$\leqslant 2h^{-1}(\log h^{-1})^3 \frac{4}{\pi} \exp\left(-\frac{1}{(1+\delta)(1-\varepsilon)^2} \log h^{-1} \right)$$

$$= O(h^{\theta}(\log h^{-1})^3),$$

where $\theta = (1+\delta)^{-1}(1-\varepsilon)^{-2} - 1 > 0$ (for δ small enough). Let $h_n = n^{-T}$ $(T\theta > 1)$, by the Borel-Cantelli lemma we have

$$\liminf_{h\to 0}\min_{1\le i\le \rho h_n}\sup_{0\le s\le h_n}\left(\frac{8\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}|X(t_i+s)-X(t_i)|$$
$$\ge 1-\varepsilon \text{ a.s.} \tag{4.5.48}$$

By the continuity modulus theorem of $X(\cdot)$ (cf. Theorem 2.2.5), it is easy to finish the proof of (4.5.47).

Now we prove that for any $\varepsilon>0$ small enough

$$\limsup_{h\to 0}\inf_{1\le t\le 1/2}\sup_{0\le s\le h}\left(\frac{8\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(t+s)-X(t)|$$
$$\le (1+\varepsilon)^2 \quad \text{a.s.} \tag{4.5.49}$$

Recall (4.5.34) and let $v>1$ near one enough. Put

$$h_k=v^{-k}k^{11/2},\quad n_k=[1/(4h_k)].$$

Then

$$\limsup_{h\to 0}\inf_{0\le t\le 1/2}\sup_{0\le s\le h}\left(\frac{8\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}|X(t+s)-X(t)|$$
$$\le \limsup_{k\to\infty}\sup_{v^{-k-1}\le h\le v^{-k}}\min_{1\le i\le n_k}\sup_{0\le s\le h}\left(\frac{8\log h^{-1}}{\pi^2\Gamma_1 h}\right)^{1/2}$$
$$\times|X(2ih_k+s)-X(2ih_k)|$$
$$\le \limsup_{k\to\infty}\min_{1\le i\le n_k}\sup_{0\le s\le v^{-k}}\left(\frac{8\log v^{k+1}}{\pi^2\Gamma_1 v^{-k}}\right)^{1/2}|X(2ih_k+s)-X(2ih_k)|$$
$$\le (1+\varepsilon)\limsup_{k\to\infty}\min_{1\le i\le n_k}\sup_{0\le s\le v^{-k}}\left(\frac{8\log v^{k+1}}{\pi^2\Gamma_1 v^{-k}}\right)^{1/2}$$
$$\times|X(2ih_k+s)-X(2ih_k)|$$
$$\le (1+\varepsilon)\left(\limsup_{k\to\infty}\min_{1\le i\le n_k}\sup_{0\le s\le v^{-k}}\left(\frac{8\log v^{k+1}}{\pi^2\Gamma_1 v^{-k}}\right)^{1/2}|\xi_{i,k}(s)|\right.$$
$$\left.+\limsup_{k\to\infty}\max_{1\le i\le n_k}\sup_{0\le s\le v^{-k}}\left(\frac{8\log v^{k+1}}{\pi^2\Gamma_1 v^{-k}}\right)^{1/2}|\eta_{i,k}(s)|\right)$$
$$=:(1+\varepsilon)(I_1+I_2), \tag{4.5.50}$$

where

$$\xi_{i,k}(s)=\sum_{j=1}^{\infty}\left(\frac{\gamma_j}{\lambda_j}\right)^{1/2}\left(\frac{W_j(e^{2\lambda_j(s+2ih_k)})-W_j(e^{2\lambda_j(2i-1)h_k})}{e^{\lambda_j(s+2ih_k)}}\right.$$
$$\left.-\frac{W_j(e^{2\lambda_j 2ih_k})-W_j(e^{2\lambda_j(2i-1)h_k})}{e^{\lambda_j 2ih_k}}\right),$$

$$\eta_{i,k}(s)=\sum_{j=1}^{\infty}\left(\frac{\gamma_j}{\lambda_j}\right)^{1/2}\frac{W_j(e^{2\lambda_j(2i-1)h_k})}{e^{\lambda_j 2ih_k}}(1-e^{-\lambda_j s}).$$

For $0\le s\le s+h\le v^{-k}$, we have

$$E(\eta_{i,k}(s+h)-\eta_{i,k}(s))^2\le\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}e^{-2\lambda_j h_k}(1-e^{-\lambda_j h})^2$$
$$\le h\sum_{j=1}^{\infty}\gamma_j e^{-2\lambda_j h_k}(1-e^{-\lambda_j v^{-k}})$$
$$\le h\Big(\sum_{j:\lambda_j v^{-k}\le k^{-6}}+\sum_{j:\lambda_j v^{-k}>k^{-6}}\Big)\gamma_j e^{-2\lambda_j h_k}(1-e^{-\lambda_j v^{-k}})$$
$$\le h\Big(k^{-6}\sum_{j=1}^{\infty}\gamma_j+e^{-2k^{3/2}}\sum_{j=1}^{\infty}\gamma_j\Big)\le hk^{-6}\Gamma_1$$

and $E\eta_{i,k}^2(s)\le\Gamma_1 k^{-6}v^{-k}$. Thus by Feinique's inequality (Theorem 1.1.3), we have

$$\sum_{k=1}^{\infty}P\left\{\max_{1\le i\le n_k}\sup_{0\le s\le v^{-k}}\frac{\log v^k}{v^{-k/2}}|\eta_{i,k}(s)|>3\Gamma_1^{1/2}\varepsilon\right\}$$
$$\le c\sum_{k=1}^{\infty}v^k\exp\left(-\frac{\varepsilon^2k^4}{2(\log v)^2}\right)<\infty.$$

It follows that

$$I_2=0\quad\text{a.s.} \tag{4.5.51}$$

Next we consider I_1. Note that $\xi_{i,k}(0)=0$, $E\xi_{i,k}(s)=0$, and for $0\le s\le s+h\le v^{-k}$

$$E(\xi_{i,k}(s+h)-\xi_{i,k}(s))^2$$
$$=\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h})(2-(1-e^{-\lambda_k h})e^{-2\lambda_k(s)})\le\Gamma_1 h.$$

For all $0\le x_1\le x_2\le x_3\le x_4$, we have

$$E(\xi_{i,k}(x_4)-\xi_{i,k}(x_3))(\xi_{i,k}(x_2)-\xi_{i,k}(x_1))$$

$$= - \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} e^{-\lambda_k x_3} (1 - e^{-\lambda_k (x_4 - x_3)}) (1 - e^{\lambda_k (x_2 - x_1)})$$
$$\times (e^{\lambda_k x_2} + e^{-\lambda_k (x_1 + 2h_k)})$$
$$\leqslant 0.$$

For any integer m, $1 \leqslant i \leqslant m$ and any $0 < x \leqslant v^{-k}$, we have

$$0 \geqslant E\left(\xi_{i,k}\left(\frac{i}{m}x\right) - \xi_{i,k}\left(\frac{i-1}{m}x\right)\right)\left(\xi_{i,k}\left(\frac{i-1}{m}x\right) - \xi_{i,k}(0)\right)$$
$$+ E\left(\xi_{i,k}(x) - \xi_{i,k}\left(\frac{i}{m}x\right)\right)\left(\xi_{i,k}\left(\frac{i}{m}x\right) - \xi_{i,k}\left(\frac{i-1}{m}x\right)\right)$$
$$= - \sum_{j=1}^{\infty} \frac{\gamma_j}{\lambda_j}\left(1 - e^{\frac{-\lambda_j x}{m}}\right)\left\{\left(1 - e^{-\frac{\lambda_j (i-1) x}{m}}\right)\left(1 + e^{-\frac{\lambda_j (i-1) x}{m} - 2\lambda_j h_k}\right)\right.$$
$$\left. + \left(1 - e^{-\lambda_j (1 - \frac{i}{m}) x}\right)\left(1 + e^{-\frac{\lambda_j (2i-1) x}{m} - 2\lambda_j h_k}\right)\right\}$$
$$\geqslant - \frac{2}{m} x H(x),$$

where $H(x) = 2\sum_{j=1}^{\infty} \gamma_j (1 - e^{-\lambda_j x/2}) \to 0 \ (x \to 0)$. Hence, by Remark 4.3.4 we can choose k_0 large enough such that for $x \geqslant 0$, $k \geqslant k_0$, $1 \leqslant i \leqslant n_k$

$$P\{\sup_{0 \leqslant s \leqslant v^{-k}} |\xi_{i,k}(s)| \leqslant x(\Gamma_1 v^{-k})^{1/2}\} \geqslant \frac{2}{\pi} \exp\left(-\frac{\pi^2}{8(1-\delta)x^2}\right),$$

which together with the independence of $\{\xi_{i,k}(s); i = 1, \cdots, n_k\}$ implies that

$$P\left\{\min_{1 \leqslant i \leqslant n_k} \sup_{0 \leqslant s \leqslant v^{-k}} \left(\frac{8 \log v^k}{\pi^2 \Gamma_1 v^{-k}}\right)^{1/2} |\xi_{i,k}(s)| > 1 + \varepsilon\right\}$$
$$\leqslant \left(1 - \frac{2}{\pi} \exp\left(-\frac{1}{(1-\delta)(1+\varepsilon)^2} \log v^k\right)\right)^{[\frac{1}{4h_k}]}$$
$$\leqslant \exp\left(-\frac{1}{2\pi} v^{k\theta/2}\right),$$

where $\theta = 1 - (1-\delta)^{-1}(1+\varepsilon)^{-2} > 0$ (for δ small enough). By the Borel-Cantelli lemma we obtain

$$I_1 \leqslant 1 + \varepsilon \quad \text{a.s.} \tag{4.5.52}$$

Combining it with (4.5.50) and (4.5.51) we have (4.5.49). The proof of Theorem 4.5.6 is over.

4.6 Liminfs for Two-parameter Gaussian Processes

4.6.1 Liminfs for a two-parameter Wiener process

Let $\{W(x,y); 0 \leqslant x, y < \infty\}$ be a two-parameter Wiener process, $0 < a_T \leqslant T$ and $b_T \geqslant T^{1/2}$ be two non-decreasing functions of T. Let

$$D_T = \{(x,y); xy \leqslant T, 0 \leqslant x, y \leqslant b_T\},$$
$$D_T^* = \{(x,y); xy = T, 0 \leqslant x, y \leqslant b_T\}.$$

For the rectangle $R = [x_1, x_2] \times [y_1, y_2]$, define

$$\lambda(R) = (x_2 - x_1)(y_2 - y_1),$$
$$W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1).$$

Let $L_T = \{R; R \subset D_T, \lambda(R) \leqslant a_T\}$ be a set of rectangles $R = [x_1, x_2] \times [y_1, y_2]$. Define

$$\lambda_T = \{2T(\log(1 + \log b_T T^{-1/2}) - \log\log\log T)\}^{-1/2},$$
$$\beta_T = T^{-1/2}\left(\frac{\log\log T}{\log b_T T^{-1/2}}\right)^{1/2}\left(\log \frac{\log\log T}{\log b_T T^{-1/2}}\right)^{-3/2}.$$

Lacey (1989) established the following law of the iterated logarithm.

Theorem 4.6.1 *Suppose that* $\lambda'_T = \{2T\log(1 + \log b_T T^{-1/2})\}^{-1/2}$ *satisfies*

(a) $\lim_{\theta \downarrow 1} \limsup_{k \to \infty} \lambda'_{\theta^k} / \lambda'_{\theta^{k+1}} = 1$

and that $b_T' = b_T T^{-1/2}$ is a non-decreasing function of T, $b_T' \to \infty$. Assume also that for any $0<\varepsilon<1$ and some $0<a<1$, we have

$$\sum_k \exp\{-(\log b'_{m_k})^\varepsilon\} < \infty. \tag{4.6.1}$$

where $m_k = \exp(k^a)$, $k \in \mathbf{N}$. Then

$$\liminf_{T\to\infty} \sup_{(x,y)\in D_T^*} \lambda_T' W(x,y) = 1 \quad \text{a.s.} \tag{4.6.2}$$

Particularly,

$$\liminf_{T\to\infty} \sup_{\substack{0\le x,y\le T\\ xy=T}} \frac{W(x,y)}{\sqrt{2T\log\log T}} = 1 \quad \text{a.s.} \tag{4.6.3}$$

Using Corollary 4.4.1, one can obtain a Chung type law of the iterated logarithm for the two-parameter Wiener process (cf. Talagrand 1994) which reads as follows.

Theorem 4.6.2 *We have*

$$0 < \liminf_{T\to\infty} \frac{(\log\log T)^{1/2}}{T(\log\log\log T)^{3/2}} \sup_{0\le x,y\le T} |W(x,y)| < \infty \quad \text{a.s.} \tag{4.6.4}$$

Lacey (1989) asked whether (4.6.1) is necessary for (4.6.2). It is of interest to investigate whether there are any results similar to (4.6.2) possible in the case one was to weaken (4.6.1). It is also of interest to sort out the nature of having (4.6.3) type liminf results versus having (4.6.4) type liminf results. Zhang (1996b) gave a positive answer to solve these problems. He found the watershed between these two different kinds of liminfs. He also showed that condition (a) in Theorem 4.6.1 is superfluous. The results read as follows.

Theorem 4.6.3 *If*

$$\Delta_T := \frac{\log b_T T^{-1/2}}{\log\log T} \to \infty \quad (T\to\infty), \tag{4.6.5}$$

then

$$\begin{aligned}\liminf_{T\to\infty} \lambda_T \sup_{R\subset D_T} |W(R)| &= \liminf_{T\to\infty} \lambda_T \sup_{R\subset D_T^*} |W(R)| \\ &= \liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T} |W(x,y)| \\ &= \liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T^*} |W(x,y)| = 1 \quad \text{a.s.}\end{aligned} \tag{4.6.6}$$

If

$$\Delta_T \to 0 \quad (T\to\infty), \tag{4.6.7}$$

then for some positive constants C_1, C_2 we have

$$C_1 \le \liminf_{T\to\infty} \beta_T \sup_{R\subset D_T} |W(R)| \le C_2 \quad \text{a.s.,} \tag{4.6.8}$$

$$C_1 \le \liminf_{T\to\infty} \beta_T \sup_{(x,y)\in D_T} |W(x,y)| \le C_2 \quad \text{a.s.} \tag{4.6.9}$$

The truncated hyperbola interpolating between these two liminf results works as a bridge. If we take $b_T = T^{1/2}$, then (4.6.9) is just (4.6.4). And, if we take $b_T = T$, then (4.6.6) is the Lacey's type LIL (4.6.3). Also, our condition (4.6.5) is much weaker than Lacey's condition (4.6.1). To verify this fact, it is enough to note that under the condition in Theorem 4.6.1, (4.6.1) is equivalent to

$$\lim_{T\to\infty} \frac{\log\log b_T'}{\log\log\log T} = \infty. \tag{4.6.10}$$

In fact, if (4.6.10) is true, then

$$\liminf_{k\to\infty} \frac{\log\log b'_{m_k}}{\log\log k} = \liminf_{k\to\infty} \frac{\log\log b'_{m_k}}{\log\log\log m_k} > \frac{2}{\varepsilon}.$$

So, for k large enough, we have

$$(\log b'_{m_k})^\varepsilon \ge (\log k)^2.$$

which implies (4.6.1).

On the other hand, if (4.6.1) is true, noting that b_T' is non-decreasing, we have

$$k \exp(-(\log b'_{m_k})^{\varepsilon}) \leqslant \sum_{k=1}^{\infty} \exp(-(\log b'_{m_k})^{\varepsilon}) < \infty.$$

So,

$$\exp((\log b'_{m_k})^{\varepsilon}) \geqslant ck,$$

which implies

$$\liminf_{k\to\infty} \frac{\log\log b'_{m_k}}{\log\log\log m_k} \geqslant \frac{1}{\varepsilon}.$$

For $m_k \leqslant T \leqslant m_{k+1}$ we have

$$\frac{\log\log b'_T}{\log\log\log T} \geqslant \frac{\log\log b'_{m_k}}{\log\log\log m_k} \frac{\log\log m_k}{\log\log\log m_{k+1}}.$$

Hence (4.6.10) holds true.

Form Theorem 4.6.1, we also conclude that, if

$$\lim_{T\to\infty} \frac{\log\log b_T T^{-1/2}}{\log\log\log T} = r \geqslant 1, \tag{4.6.11}$$

then

$$\liminf_{T\to\infty} \lambda'_T \sup_{(x,y)\in D_T} |W(x,y)| = \left(\frac{r-1}{r}\right)^{1/2} \quad \text{a. s.} \tag{4.6.12}$$

Hence, from (4.6.11) it is easy to see that (4.6.1) or (4.6.10) is very close to being necessary for (4.6.2).

Remark 4.6.1 Put $\theta(T) = (2+2\log b_T)/\log\log T$. Csáki and Shi (1998) investigate liminf behaviors of the Wiener sheet when $\theta(T)$ admits a finite limit and give the following results.

Assume that $b_T \geqslant 1$ is non-decreasing, such that $\theta(T)/T$ is non-increasing. If $\theta := \lim_{T\to\infty}\theta(T) \in [0,\infty)$. Then

$$\liminf_{T\to\infty} \sqrt{\frac{\theta(T)}{T}} \sup_{st=T, 0\leqslant s,t\leqslant \sqrt{T}b_T} W(s,t) = \begin{cases} -2 & \text{if } \theta = 0, \\ -\sqrt{\theta}\,\beta(1/\theta) & \text{if } 0<\theta<\infty, \end{cases}$$

$$\liminf_{T\to\infty} \frac{1}{\sqrt{T\theta(T)}} \sup_{st=T, 0\leqslant s,t\leqslant \sqrt{T}b_T} |W(s,t)| = \begin{cases} \pi/2 & \text{if } \theta = 0, \\ \theta^{-1/2}\gamma(1/\theta) & \text{if } 0<\theta<\infty, \end{cases}$$

where $\gamma(u) > 0$ is the largest real zero of Kummer's function $M(-u/2, 1/2, \gamma^2/2)$:

$$M(a,b,x) := 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)} \cdot \frac{x^n}{n!},$$

and $\beta(u) \in (-\infty,\infty)$ is the largest real zero of the parabolic cylinder function $D_u(\cdot)$:

$$D_u(x) := 2^{\frac{u}{2}} e^{-\frac{x^2}{4}} \left[\frac{\Gamma(1/2)}{\Gamma((1-u)/2)} M\left(-\frac{u}{2}, \frac{1}{2}, \frac{x^2}{2}\right) + \frac{x}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-u/2)} M\left(\frac{1-u}{2}, \frac{3}{2}, \frac{x^2}{2}\right)\right].$$

The proofs of Theorems 4.6.1 and 4.6.2 will not be given here. The proof of Theorem 4.6.3 is based upon the following probability inequalities.

Lemma 4.6.1 *For any $\varepsilon > 0$, there exist constants $C = C(\varepsilon) > 0$, $u_0 = u_0(\varepsilon) > 0$ and $T_0 = T_0(\varepsilon) > 0$ such that*

$$P\{\sup_{(x,y)\in D_T^*} |W(x,y)| \leqslant u T^{1/2}\} \leqslant \exp(-C(1+\log b_T T^{-1/2}) e^{-u^2/(2-\varepsilon)}) \tag{4.6.13}$$

holds for any $u \geqslant u_0$, $T \geqslant T_0$.

Proof Let $L = L(T)$ be the largest integer for which we have

$$T^{1/2} M^{L+1} < b_T \quad (M > 1).$$

Define the rectangles

$$S_i = S_i(T) = [x_1(i), x_2(i)] \times [y_1(i), y_2(i)] = [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}],\ i = 0, 1, \cdots, L.$$

Then $S_i \subset D_T, \lambda(S_i) = T(1-i/M), i=0,1,\cdots,L$, and $L \geqslant (\log b_T T^{-1/2})/\log M$. Let

$$\widetilde{S}_i = [0, T^{1/2}M^i] \times [0, T^{1/2}M^{-i-1}], \quad i = 0,1,\cdots,L.$$

Then

$$\begin{aligned}
&P\{\sup_{(x,y)\subset D_T^*} |W(x,y)| \leqslant uT^{1/2}\} \\
&\leqslant P\{\max_{0\leqslant i\leqslant L} |W(x_2(i), y_2(i))| \leqslant uT^{1/2}\} \\
&= P\{\max_{0\leqslant i\leqslant L} |W(\widetilde{S}_i) + W(S_i)| \leqslant uT^{1/2}\}. \qquad (4.6.14)
\end{aligned}$$

We employ a conditioning argument. Let $\sigma_i = \sigma\{W(x,y); 0\leqslant y\leqslant b_T, 0\leqslant x\leqslant T^{1/2}M^{i+1}\}$, then $W(S_i)\in\sigma_i$, $W(\widetilde{S}_i)\in\sigma_{i-1}$, and $W(S_i)$ is independent of σ_{i-1}. So for M large enough, we have

$$\begin{aligned}
&P\{\max_{0\leqslant i\leqslant L} |W(\widetilde{S}_i) + W(S_i)| \leqslant uT^{1/2}\} \\
&= E\{I(\max_{0\leqslant i\leqslant L-1} |W(\widetilde{S}_i) + W(S_i)| \leqslant uT^{1/2}) \\
&\quad \times P(|W(\widetilde{S}_L) + W(S_L)| \leqslant uT^{1/2} | \sigma_{L-1})\} \\
&\leqslant E[I(\max_{0\leqslant i\leqslant L-1} |W(\widetilde{S}_i) + W(S_i)| \leqslant uT^{1/2}) P(|W(S_L)| \leqslant uT^{1/2})] \\
&= P(\max_{0\leqslant i\leqslant L-1} |W(\widetilde{S}_i) + W(S_i)| \leqslant uT^{1/2}) P(|W(S_L)| \leqslant uT^{1/2}) \\
&\leqslant \cdots \leqslant \prod_{i=0}^{L} P(|W(S_i)| \leqslant uT^{1/2}) \\
&\leqslant \left\{1 - 2\Phi\left(-\left(\frac{M}{M-1}\right)^{1/2} u\right)\right\}^{L+1} \\
&\leqslant \left\{1 - \exp\left(-\frac{u^2}{2-\epsilon}\right)\right\}^{L+1} \\
&\leqslant \exp\left\{-c\frac{1}{\log M}(\log b_T T^{-1/2}) e^{-u^2/(2-\epsilon)}\right\}. \qquad (4.6.15)
\end{aligned}$$

Hence by (4.6.14) and (4.6.15),

$$\begin{aligned}
&P\{\sup_{(x,y)\in D_T^*} |W(x,y)| \leqslant uT^{1/2}\} \\
&\leqslant \exp\left\{-c\frac{1}{\log M}(\log b_T T^{-1/2}) e^{-u^2/(2-\epsilon)}\right\},
\end{aligned}$$

which implies (4.6.13).

The following lemma comes from Corollary 4.4.1 immediately.

Lemma 4.6.2 *There exists a constant $C>0$ such that for any $0<u\leqslant 1/2$ and $T_1, T_2>0$ we have*

$$P\left\{\sup_{(x,y)\in[0,T_1]\times[0,T_2]} |W(x,y)| \leqslant (T_1T_2)^{1/2}\left(u(\log\frac{1}{u})^3\right)^{1/2}\right\} \overset{\text{KS}}{\geqslant} \exp\left(-\frac{C}{u}\right).$$

Lemma 4.6.3 *There exists a constant $C_2>0$ such that for any $0<u\leqslant 1/2$ and $T>0$,*

$$\begin{aligned}
&\exp\left(-C_2\frac{1}{u}\log b_T T^{-1/2}\right) \\
&\leqslant P\left\{\sup_{R\subset D_T} |W(R)| \leqslant T^{1/2}\left(u\left(\log\frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\leqslant \exp\left(-\frac{1}{C_2 u}\log b_T T^{-1/2}\right), \qquad (4.6.16) \\
&\exp\left(-C_2\frac{1}{u}\log b_T T^{-1/2}\right) \\
&\leqslant P\left\{\sup_{(x,y)\in D_T} |W(x,y)| \leqslant T^{1/2}\left(u\left(\log\frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\leqslant \exp\left(-\frac{1}{C_2 u}\log b_T T^{-1/2}\right). \qquad (4.6.17)
\end{aligned}$$

Proof Noting that

$$\sup_{(x,y)\in D_T} |W(x,y)| \leqslant \sup_{R\subset D_T} |W(R)| \leqslant 4\sup_{(x,y)\in D_T} |W(x,y)|, \quad (4.6.18)$$

we need only to prove (4.6.16). First we establish the lower bound. Without loss of generality, we can assume $T=1$. Write D for $D_T=D_1$, and so on. Take $R_j=[0,2^j]\times[0,2^{-j+1}]$, for $j=-1-[\log_2 b], \cdots, 1+[\log_2 b]$. Notice that each R_j has area 2, and

$$R^* := \bigcup_{j=-1-[\log_2 b]}^{1+[\log_2 b]} R_j \supset D = \{(x,y); xy \leqslant 1, 0 \leqslant x,y \leqslant b\}.$$

Then, by Lemma 4.6.2 and Theorem 1.2.4, we have

$$\begin{aligned}
&P\left\{\sup_{(x,y)\in D} |W(x,y)| \leqslant \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\geqslant P\left\{\sup_{(x,y)\in R^*} |W(x,y)| \leqslant \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\overset{\text{KS}}{\geqslant} \prod_{j=-1-[\log_2 b]}^{1+[\log_2 b]} P\left\{\sup_{(x,y)\in R_j} |W(x,y)| \leqslant \left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\overset{\text{KS}}{\geqslant} \exp\left(-\frac{C}{u}\log b\right).
\end{aligned}$$

Hence we obtain the lower bound. For the upper bound, we define

$$S_i = [T^{1/2}M^i, T^{1/2}M^{i+1}] \times [0, T^{1/2}M^{-i-1}], \quad i = 0,1,\cdots,L,$$

where L is the largest integer for which $T^{1/2}M^{L+1} < b_T\,(M>1)$. Then, by (4.4.25) of Corollary 4.4.1, we have

$$\begin{aligned}
&P\left\{\sup_{R\subset D_T} |W(R)| \leqslant T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\leqslant P\left\{\sup_i \sup_{R\subset S_i} |W(R)| \leqslant T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&= \prod_{i=0}^{L} P\left\{\sup_{R\subset S_i} |W(R)| \leqslant T^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&= \prod_{i=0}^{L} P\left\{\sup_{R\subset[0,1]\times[0,1]} |W(R)| \leqslant \left(\frac{M}{M-1}\right)^{1/2}\left(u\left(\log \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\leqslant \prod_{i=0}^{L} P\left\{\sup_{0\leqslant x,y\leqslant 1} |W(x,y)| \leqslant \left(\frac{M}{M-1}\right)^{1/2}\left(u\left(\lg \frac{1}{u}\right)^3\right)^{1/2}\right\} \\
&\leqslant \exp\left(-\frac{C}{u}\frac{M-1}{M}L\right) \\
&\leqslant \exp\left(-\frac{C}{u}\log b_T T^{1/2}\right).
\end{aligned}$$

Lemma 4.6.3 is proved.

Proof of Theorem 4.6.3

(1) Proof of (4.6.6)

The upper bound of (4.6.6) comes from Step 1 of the proof of Theorem 2.3.2 with $a_T = T$ (cf. 2.3.23). We now verify the lower bound. It is sufficient to show that if $\Delta_T \to \infty$, then

$$\liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T^*} |W(x,y)| \geqslant 1 \quad \text{a.s.} \tag{4.6.19}$$

We can write $\lambda_T = \{2T(\log\log b_T T^{-1/2} - \log\log\log T)\}^{-1/2}$. Let

$$\begin{aligned}
&A_k = \{T; e^k \leqslant b_T/T^{1/2} \leqslant e^{k+1}\},\ k \geqslant 0, \\
&A_{k,j} = \{T; e^{\sqrt{j}} \leqslant T \leqslant e^{\sqrt{j+1}}, T \in A_k\},\ k \geqslant 0,\ j \geqslant 0, \\
&b(T_{k,j}) = \inf\{b_T; T \in A_{k,j}\}, \\
&b(T_{k,j}^*) = \sup\{b_T; T \in A_{k,j}\}, \\
&D_{k,j}^* = \{(x,y); xy = e^{\sqrt{j}}, 0 \leqslant x,y \leqslant b(T_{k,j})\}, \qquad (4.6.20) \\
&L_{k,j} = \{R \subset \{(x,y); xy \leqslant e^{\sqrt{j+1}}, 0 \leqslant x,y \leqslant b(T_{k,j}^*)\}; \\
&\qquad \lambda(R) \leqslant e^{\sqrt{j+1}} - e^{\sqrt{j}}\}.
\end{aligned}$$

Note that $\Delta_T \to \infty$ and $\dfrac{k+1}{(1/2)\log j} \geqslant \Delta_T$ for $T \in A_{k,j}$. Moreover for any $M > 4$, if j is large enough, then

$$A_{k,j} = \varnothing \text{ if } k \leqslant M\log j.$$

Therefore

$$\begin{aligned}
&\liminf_{T\to\infty} \lambda_T \sup_{(x,y)\in D_T^*} |W(x,y)| \\
&\geqslant \liminf_{j\to\infty} \inf_k \inf_{T\in A_{k,j}} \lambda_T \sup_{(x,y)\in D_T^*} |W(x,y)| \\
&\geqslant \liminf_{j\to\infty} \inf_k \inf_{T\in A_{k,j}} \lambda_T \inf_{(x,y)\in D_{k,j}^*} |W(x,y)| \\
&\quad -4 \limsup_{j\to\infty} \sup_k \sup_{T\in A_{k,j}} \lambda_T \sup_{R\subset L_{k,j}} |W(R)| \\
&\geqslant \liminf_{j\to\infty} \inf_{k\geqslant M\log j} \{2e^{\sqrt{j+1}}(\log(k+1) - \log\log j^{1/2}\}^{-1/2} \\
&\quad \times \sup_{(x,y)\in D_{k,j}^*} |W(x,y)|
\end{aligned}$$

$$-4\limsup_{j\to\infty}\sup_{k\geq M\log j}\{2e^{\sqrt{j}}(\log k-\log\log(j+1)^{1/2})\}^{-1/2}$$
$$\times\sup_{R\subset L_{k,j}}|W(R)|$$
$$=:I_1-4I_2. \tag{4.6.21}$$

By Lemma 4.6.1, we have for j large enough

$$P\Big\{\{2e^{\sqrt{j+1}}(\log(k+1)-\log\log j^{1/2})\}^{-1/2}$$
$$\times\sup_{(x,y)\in D^*_{k,j}}|W(x,y)|\leqslant 1-2\varepsilon\Big\}$$
$$\leqslant\exp\Big\{-C(\log b(T_{k,j})e^{-\sqrt{j}/2})$$
$$\times\exp\Big(-\frac{2(1-2\varepsilon)^2e^{\sqrt{j+1}}(\log(k+1)-\log\log j^{1/2})}{(2-2\varepsilon)e^{\sqrt{j}}}\Big)\Big\}$$
$$\leqslant\exp\Big\{-C(\log e^k)\exp\Big(-\frac{2(1-\varepsilon)^2(\log(k+1)-\log\log j^{1/2})}{2-2\varepsilon}\Big)\Big\}$$
$$\leqslant\exp\Big\{-C\Big(\frac{k}{\log j}\Big)^{\varepsilon}\log j\Big\}. \tag{4.6.22}$$

Note that for M and j_0 large enough we have

$$\sum_{j=j_0}^{\infty}\sum_{k\geq M\log j}\exp\Big\{-C\Big(\frac{k}{\log j}\Big)^{\varepsilon}\log j\Big\}$$
$$\leqslant\sum_{j=j_0}^{\infty}\sum_{k\geq M\log j}\exp\Big\{-C\Big(\frac{k}{\log j}\Big)^{\varepsilon/2}\log j-\Big(\frac{k}{\log j}\Big)^{\varepsilon/2}\Big\}$$
$$\leqslant\sum_{j=j_0}^{\infty}\exp\{-CM^{\varepsilon/2}\log j\}\cdot\sum_{k\geq M\log j}\exp\Big\{-\Big(\frac{k}{\log j}\Big)^{\varepsilon/2}\Big\}$$
$$\leqslant\sum_{j=j_0}^{\infty}(\log j)\exp\{-CM^{\varepsilon/2}\log j\}<\infty, \tag{4.6.23}$$

which together with (4.6.22) implies

$$I_1\geqslant 1-2\varepsilon\quad\text{a.s.} \tag{4.6.24}$$

Consider I_2. By (4.6.12) or Theorem 1.12.6 of Csörgő and Révész (1981), we have

$$P\{\sup_{R\subset L_{k,j}}|W(R)|\geqslant u(e^{\sqrt{j+1}}-e^{\sqrt{j}})^{1/2}\}$$
$$\leqslant C\frac{e^{\sqrt{j+1}}}{e^{\sqrt{j+1}}-e^{\sqrt{j}}}\Big(1+\log\frac{e^{\sqrt{j+1}}}{e^{\sqrt{j+1}}-e^{\sqrt{j}}}\Big)$$
$$\times(1+\log b(T^*_{k,j}))(e^{\sqrt{j+1}}-e^{\sqrt{j}})^{-1/2})e^{-u^2/(2+\varepsilon)}$$
$$\leqslant C\sqrt{j}(1+\log j)(1+k+\log j)e^{-u^2/(2+\varepsilon)}$$
$$\leqslant Cjke^{-u^2/(2+\varepsilon)}, \tag{4.6.25}$$

so that we obtain

$$\sum_{j=j_0}^{\infty}P\{\sup_{k\geq M\log j}\{2e^{\sqrt{j}}(\log k-\log\log(j+1)^{1/2})\}^{-1/2}$$
$$\times\sup_{R\subset L_{k,j}}|W(R)|>\frac{\varepsilon}{4}\Big\}$$
$$\leqslant C\sum_{j=j_0}^{\infty}\sum_{k\geq M\log j}jk\exp\Big(-\varepsilon'\sqrt{j}\log\frac{k}{\log j}\Big)$$
$$\leqslant C\sum_{j=j_0}^{\infty}j(\log j)^{\varepsilon'\sqrt{j}}\sum_{k\geq M\log j}k^{-\varepsilon'\sqrt{j}+1}$$
$$\leqslant C\sum_{j=j_0}^{\infty}j(\log j)^2M^{-\varepsilon'\sqrt{j}}<\infty, \tag{4.6.26}$$

where $\varepsilon'>0$ depends only on ε, which implies

$$I_2\leqslant\varepsilon/4\quad\text{a.s.} \tag{4.6.27}$$

Hence (4.6.19) is proved by (4.6.24) and (4.6.27). The proof of (4.6.6) is now completed.

(2) Proof of (4.6.8) and (4.6.9)

Let C_2 be as in Lemma 4.6.3. First we prove

$$\liminf_{T\to\infty}\beta_T\sup_{R\subset D_T}|W(x,y)|\leqslant C_2^{1/2}\quad\text{a.s.} \tag{4.6.28}$$

Let $T_n=e^{n^p}(p>1)$, $D'_{T_{n+1}}=D_{T_{n+1}}\cap D^c_{T_n}$, $D''_{T_{n+1}}=\{(x,y);0\leqslant x,y\leqslant b_{T_{n+1}},xy\leqslant 2T_n\}$. By (2.3.27), there exists a sequence of independent random variables $\{Y_n\}$ such that

$$\sup_{R\subset D_{T_{n+1}}}|W(R)|$$

$$\leqslant Y_{n+1}+\left\{4\log\left(\frac{T_{n+1}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D^*_{T_{n+1}}}|W(x,y)|,$$

$$Y_{n+1}\leqslant\sup_{R\subset D_{T_{n+1}}}|W(R)|$$
$$+\left\{4\log\left(\frac{T_{n+1}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D^*_{T_{n+1}}}|W(x,y)|. \quad (4.6.29)$$

Now, by (4.6.12) or Theorem 1.12.6 of Csörgő and Révész (1981), we have that for n large enough

$$J''_{n+1}:=P\left\{\left(4\log\left(\frac{T_{n+1}}{T_n}+1\right)+12\right)\sup_{(x,y)\in D^*_{T_{n+1}}}\beta_{T_{n+1}}|W(x,y)|>\varepsilon\right\}$$
$$\leqslant P\left\{\sup_{(x,y)\in D^*_{T_{n+1}}}|W(x,y)|>\varepsilon' n^{-p}\beta^{-1}_{T_{n+1}}\right\}$$
$$\leqslant P\left\{\sup_{(x,y)\in D^*_{T_{n+1}}}|W(x,y)|>\varepsilon' n^{-p}T^{1/2}_{n+1}/(\log\log T_{n+1})^{1/2}\right\}$$
$$\leqslant C(1+\log(b_{T_{n+1}}(2T_n)^{-1/2}))\exp\left\{\frac{-\varepsilon' T_{n+1}}{n^{2p}(\log\log T_{n+1})T_n}\right\}$$
$$\leqslant C\log\log T_{n+1}\exp\left\{-\varepsilon'\frac{e^{(n+1)^p}}{e^{n^p}n^{3p}}\right\}$$
$$\leqslant C(\log n)e^{-2n}\leqslant Ce^{-n}, \quad (4.6.30)$$

where ε' is a positive constant depending only on ε and p, whose value can differ from line to line. By the Borel-Cantelli lemma we have

$$\limsup_{n\to\infty}\left\{4\log\left(\frac{T_{n+1}}{T_n}+1\right)+12\right\}\sup_{(x,y)\in D^*_{T_{n+1}}}\beta_{T_{n+1}}|W(x,y)|$$
$$\leqslant\varepsilon \quad \text{a.s.} \quad (4.6.31)$$

Let

$$u=\frac{1}{C_2(1+\varepsilon)^2}\,\frac{\log(b_{T_{n+1}}T^{-1/2}_{n+1})}{\log\log T_{n+1}}.$$

By Lemma 4.6.3, we have that for n large enough

$$J_{n+1}:=P\left\{\sup_{R\subset D_{T_{n+1}}}\beta_{T_{n+1}}|W(R)|\leqslant C_2^{1/2}(1+\varepsilon)\right\}$$
$$\geqslant\exp\left\{-\frac{C_2(1+\varepsilon)}{u}\log\left(b_{T_{n+1}}T^{-1/2}_{n+1}\right)\right\}$$
$$=\exp\left\{-\frac{1}{1+\varepsilon}\log\log T_{n+1}\right\}$$
$$=(n+1)^{-p/(1+\varepsilon)}. \quad (4.6.32)$$

Choose p such that $1<p<1+\varepsilon/2$. Then

$$\sum_{n=1}^{\infty}J_{n+1}=\infty. \quad (4.6.33)$$

By (4.6.29), (4.6.30) and (4.6.32), we have

$$J'_{n+1}:=P\{Y_{n+1}\leqslant C_2^{1/2}(1+\varepsilon)+\varepsilon\}$$
$$\geqslant J_{n+1}-J''_{n+1}\geqslant(n+1)^{-p/(1+\varepsilon)}-Ce^{-n} \quad (4.6.34)$$

for n large enough. Hence $\sum_{n=1}^{\infty}J'_{n+1}=\infty$. Then, by the Borel-Cantelli lemma and the independence of $\{Y_n\}_{n=1}^{\infty}$, we obtain

$$\liminf_{n\to\infty}Y_{n+1}\leqslant C_2^{1/2}(1+\varepsilon)+\varepsilon \quad \text{a.s.} \quad (4.6.35)$$

By (4.6.29), (4.6.31) and (4.6.35) we obtain

$$\liminf_{T\to\infty}\sup_{R\subset D_T}\beta_T|W(R)|$$
$$\leqslant\liminf_{n\to\infty}\sup_{R\subset D_{T_{n+1}}}\beta_{T_{n+1}}|W(R)|\leqslant C_2^{1/2}(1+\varepsilon)+2\varepsilon \quad \text{a.s.}$$
$$(4.6.36)$$

which implies (4.6.28).

Next, we prove

$$\liminf_{T\to\infty}\sup_{(x,y)\in D_T}\beta_T|W(x,y)|\geqslant\frac{1}{(2C_2)^{1/2}} \quad \text{a.s.} \quad (4.6.37)$$

Let

$$A_k=\{T;e^k\leqslant b_TT^{-1/2}\leqslant e^{k+1}\},\ k\geqslant 0,$$
$$A_{k,j}=\{T;e^{j^p}\leqslant T\leqslant e^{(j+1)^p},T\in A_k\},\ k\geqslant 0,\ j\geqslant 0,$$
$$b(T_{k,j})=\inf\{b_T;T\in A_{k,j}\}, \quad (4.6.38)$$

$$D_{k,j}=\{(x,y);xy\leqslant e^{j^p},0\leqslant x,y\leqslant b(T_{k,j})\},$$

where $0<p<1$ will be defined later. Note that $\Delta_T\to 0$ and

$$\frac{k}{p\log(j+1)}\leqslant\Delta_T\quad\text{for } T\in A_{k,j}.$$

Also, for any $\varepsilon>0$, there exists j_0 such that

$$A_{k,j}=\varnothing \text{ for } k\geqslant\varepsilon\log j,\ j\geqslant j_0.$$

Hence

$$\begin{aligned}&\liminf_{T\to\infty}\beta_T\sup_{(x,y)\in D_T}|W(x,y)|\\ &\geqslant\liminf_{j\to\infty}\inf_{k\leqslant\varepsilon\log j}\inf_{T\in A_{k,j}}\beta_T\sup_{(x,y)\in D_{k,j}}|W(x,y)|\\ &\geqslant\liminf_{j\to\infty}\inf_{k\leqslant\varepsilon\log j}e^{-j^p/2}\left(\frac{\log j^p}{k+1}\right)^{1/2}\left(\log\frac{\log j^p}{k+1}\right)^{-3/2}\\ &\quad\times\sup_{(x,y)\in D_{k,j}}|W(x,y)|.\end{aligned}\tag{4.6.39}$$

By Lemma 4.6.3 we have

$$\begin{aligned}J_j:&=P\left\{\inf_{k\leqslant\varepsilon\log j}e^{-j^p/2}\left(\frac{\log j^p}{k+1}\right)^{1/2}\left(\log\frac{\log j^p}{k+1}\right)^{-3/2}\right.\\ &\qquad\left.\times\sup_{(x,y)\in D_{k,j}}|W(x,y)|\leqslant\frac{1}{(2C_2)^{1/2}(1+\varepsilon)}\right\}\\ &\leqslant\sum_{k\leqslant\varepsilon\log j}\exp\left(-2(1+\varepsilon)\frac{\log j^p}{k+1}\log b(T_{k,j})e^{-j^p/2}\right)\\ &\leqslant\sum_{k\leqslant\varepsilon\log j}\exp\left(-\frac{k\vee 1}{k+1}2(1+\varepsilon)\log j^p\right)\\ &\leqslant\varepsilon(\log j)j^{-p(1+\varepsilon)}.\end{aligned}\tag{4.6.40}$$

Choose $0<p<1$ such that $p(1+\varepsilon)>1$. Then

$$\sum_{j=j_0}^{\infty}J_j<\infty,\tag{4.6.41}$$

which implies

$$\liminf_{j\to\infty}\inf_{k\leqslant\varepsilon\log j}e^{-j^p/2}\left(\frac{\log j^p}{k+1}\right)^{1/2}\left(\log\frac{\log j^p}{k+1}\right)^{-3/2}\sup_{(x,y)\in D_{k,j}}|W(x,y)|$$
$$\geqslant\frac{1}{(2C_2)^{1/2}(1+\varepsilon)}\quad\text{a.s.}$$

Thus (4.6.37) is proved. By (4.6.28), (4.6.37) and (4.6.18) we get (4.6.8) and (4.6.9). The proof of Theorem 4.6.3 is completed.

4.6.2 The moduli of non-differentiability for a two-parameter Ornstein-Uhlenbeck process

Let $\{X(t,v);t\geqslant 0,v\geqslant 0\}$ be a two-parameter Ornstein-Ulenbeck process (OUP_2)

$$X(t,v)=e^{-\alpha t-\beta v}\left\{X_0+\int_0^t\int_0^v e^{\alpha x+\beta y}dW(x,y)\right\},\ t\geqslant 0,\ v\geqslant 0.\tag{4.6.42}$$

Lin (1995d) gave the moduli of non-differentiability for $\{X(t,v)\}$. Assume that there exists $\delta>0$ such that $E|X_0|^\delta<\infty$. Let

$$\beta(v)=(1-e^{-2\beta v})/2\beta.$$

Theorem 4.6.4 *We have*

$$\lim_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t+s,v)-X(t,v)|=1\quad\text{a.s.},\tag{4.6.43}$$

$$\lim_{h\to 0}\inf_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t+s,u)-X(t,u)|=1\quad\text{a.s.}\tag{4.6.44}$$

for any $v>0$.

Remark 4.6.2 By symmetry of $X(t,v)$ on t and v, we also have

$$\lim_{h\to 0}\inf_{0\leqslant v\leqslant 1}\sup_{0\leqslant u\leqslant h}\left(\frac{8\log h^{-1}}{\pi^2\beta(t)h}\right)^{1/2}|X(t,v+u)-X(t,v)|=1\quad\text{a.s.},$$

$$\lim_{h\to 0}\inf_{0\leqslant v\leqslant 1}\sup_{0\leqslant u\leqslant h}\sup_{t\leqslant s\leqslant t+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(t)h}\right)^{1/2}|X(s,v+u)-X(s,v)|=1\quad\text{a.s.}$$

for any $t>0$.

Theorem 4.6.4 and Remark 4.6.1 imply that almost all sample functions of $X(t,v)$ in t (v) are nowhere differentiable for any v (t). From the proof below, it is easy to see that the range $[0,1]$ of t (or v) can be replaced by any finite intervals.

The proof of Theorem 4.6.4 needs the following lemma. First, we write

$$X(t+s,v)-X(t,v)=\xi_1(t,s,v)+\xi_2(t,s,v)+\xi_3(t,s,v) \tag{4.6.45}$$

where

$$\xi_1(t,s,v)=e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})X_0,$$

$$\xi_2(t,s,v)=e^{-\alpha(t+s)-\beta v}(1-e^{\alpha s})\int_0^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y),$$

$$\xi_3(t,s,v)=e^{-\alpha t-\beta v}\int_t^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y).$$

Lemma 4.6.4 *For any $\varepsilon<0$, there exist $h=h(\varepsilon)>0$ and $C=C(\varepsilon)>0$ such that for $i=1,2$*

$$P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}|\xi_i(t,s,u)|\geqslant\varepsilon\right\}\leqslant C(h\log h^{-1})^{\delta/2}.$$

Proof For $i=1$, we have

$$|\xi_1(t,s,u)|\leqslant(e^{\alpha s}-1)|X_0|$$

for any $t\geqslant 0$, $s\geqslant 0$ and $u\geqslant 0$. Hence

$$\begin{aligned}&P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}|\xi_1(t,s,u)|\geqslant\varepsilon\right\}\\&\leqslant P\left\{|X_0|\geqslant\frac{\varepsilon}{2\alpha}(h\log h^{-1})^{-1/2}\right\}\\&\leqslant cE|X_0|^\delta(h\log h^{-1})^{\delta/2}.\end{aligned} \tag{4.6.46}$$

For $i=2$, let

$$Y_{t,s}(v)=e^{-\alpha(t+s)}(e^{\alpha h}-1)\int_0^{t+s}\int_0^v e^{\alpha x+\beta y}dW(x,y),$$

which is a Gaussian process with independent increments for any fixed t and s, and

$$\begin{aligned}EY_{t,s}^2(v)&=\frac{1}{4\alpha\beta}(e^{\alpha h}-1)^2(1-e^{-2\alpha(t+s)})(e^{2\beta v}-1)\\&\leqslant\frac{1}{4\alpha\beta}(e^{\alpha h}-1)^2e^{2\beta v}.\end{aligned}$$

We have

$$\begin{aligned}&P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}|\xi_2(t,s,u)|\geqslant\varepsilon\right\}\\&\leqslant P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}e^{-\beta v}|Y_{t,s}(u)|\geqslant\varepsilon\right\}.\end{aligned} \tag{4.6.47}$$

Let $t_j=[t2^j/h]h/2^j$, $j=0,1,\cdots$, for any $t\geqslant 0$. Write

$$\begin{aligned}|Y_{t,s}(u)|&\leqslant|Y_{t_0,s_0}(u)|\\&\quad+\sum_{j=0}^{\infty}e^{-\alpha(t_{j+1}+s_{j+1})}(e^{\alpha h}-1)\left|\int_{t_j+s_j}^{t_{j+1}+s_{j+1}}\int_0^u e^{\alpha x+\beta y}dW(x,y)\right|.\end{aligned} \tag{4.6.48}$$

We have

$$\begin{aligned}&P\left\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant h}\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}e^{-\beta v}|Y_{t_0,s_0}(u)|\geqslant\frac{\varepsilon}{2}\right\}\\&\leqslant\frac{1}{h}\sup_{0\leqslant t\leqslant 1}P\left\{\sup_{v\leqslant u\leqslant v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}|Y_{t_0,s_0}(u)|\geqslant\frac{\varepsilon}{2}e^{\beta v}\right\}\\&\leqslant\frac{2}{h}\sup_{0\leqslant t\leqslant 1}P\left\{\left(\frac{\log h^{-1}}{h}\right)^{1/2}|Y_{t_0,s_0}(v+h)|\geqslant\frac{\varepsilon}{2}e^{\beta v}\right\}\\&\leqslant\frac{c}{h}\exp\left\{-\frac{\varepsilon^2h}{8\log h^{-1}}e^{2\beta v}/EY_{t_0,s_0}^2(v+h)\right\}\\&\leqslant\frac{c}{h}\exp\left\{-\frac{\varepsilon^2\beta}{4\alpha h\log h^{-1}}e^{-2\beta h}\right\}\\&\leqslant c\exp(-h^{-1/2}).\end{aligned} \tag{4.6.49}$$

Similarly,

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}\sum_{j=0}^{\infty}e^{-\alpha(t_{j+1}+s_{j+1})}(e^{\alpha h}-1)\right.$$
$$\left.\times\left|\int_{t_j+s_j}^{t_{j+1}+s_{j+1}}\int_0^u e^{\alpha x+\beta y}dW(x,y)\right|\ge\frac{\varepsilon}{2}(1-2^{-1/4})e^{\beta v}\sum_{j=0}^{\infty}2^{-j/4}\right\}$$
$$\le\frac{1}{h}\sum_{j=0}^{\infty}2^{2(j+1)}\sup_{0\le t\le 1}\sup_{0\le s\le h}P\left\{\sup_{v\le u\le v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}e^{-\alpha(t_{j+1}+s_{j+1})}\right.$$
$$\times(e^{\alpha h}-1)\left|\int_{t_j+s_j}^{t_{j+1}+s_{j+1}}\int_0^u e^{\alpha x+\beta y}dW(x,y)\right|\ge\frac{\varepsilon}{2}$$
$$\left.\times(1-2^{-1/4})e^{\beta v}2^{-j/4}\right\}$$
$$\le\frac{c}{h}\sum_{j=0}^{\infty}2^{2(j+1)}\exp\left\{-\frac{\varepsilon^2}{8}(1-2^{-1/4})^2 2^{-j/2}e^{2\beta v}\frac{h}{\log h^{-1}}\right.$$
$$\left.\Big/\left(\frac{1}{4\alpha\beta}(e^{\alpha h-1})^2(1-e^{4\alpha h/2^{j+1}})(e^{2\beta(v+h)}-1)\right)\right\}$$
$$\le\frac{c}{h}\sum_{j=0}^{\infty}2^{2(j+1)}\exp\left\{-\frac{\varepsilon^2\beta}{8\alpha^2}(1-2^{-1/4})^2\frac{1}{h^2\log h^{-1}}\right.$$
$$\left.\times 2^{j/2}e^{-2\beta h}(1-e^{-2\beta v})\right\}$$
$$\le\frac{c}{h}\sum_{j=0}^{\infty}2^{2(j+1)}\exp\{-h^{-1}(1+2^{j/2})\}$$
$$\le\frac{c}{h}\exp(-h^{-1}). \tag{4.6.50}$$

Combining (4.6.47)—(4.6.50) yields the required inequality.

Proof of Theorem 4.6.4

At first, we prove

$$\lim_{h\to 0}\sup_{0\le t\le 1}\inf_{0\le s\le h}\sup_{v\le u\le v+h}\sup\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|X(t+s,u)-X(t,u)|$$
$$\le 1 \text{ a. s.} \tag{4.6.51}$$

Let $h_n=\theta^n$, $0<\theta<1$. Using Lemma 4.6.4 we have

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h_n}\sup_{v\le u\le v+h_n}\left(\frac{\log h_n^{-1}}{h_n}\right)^{1/2}|\xi_1(t,s,u)+\xi_2(t,s,u)|\ge\varepsilon\right\}$$

$$\le\theta^{n\delta/4},$$

which, in combination with the Borel-Cantelli lemma, implies

$$\lim_{n\to\infty}\sup_{0\le t\le 1}\sup_{0\le s\le h_n}\sup_{v\le u\le v+h_n}\sup\left(\frac{\log h_n^{-1}}{h_n}\right)^{1/2}|\xi_1(t,s,u)+\xi_2(t,s,u)|$$
$$\le\varepsilon \text{ a. s.}$$

Furthermore, we obtain

$$\lim_{h\to 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\xi_1(t,s,u)+\xi_2(t,s,u)|$$
$$=0 \text{ a. s.} \tag{4.6.52}$$

Hence, from (4.6.45), it is enough for the proof of (4.6.51) to show that

$$\lim_{h\to 0}\inf_{0\le t\le 1}\sup_{0\le s\le h}\sup_{v\le u\le v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\xi_3(t,s,u)|\le 1 \text{ a. s.} \tag{4.6.53}$$

Let

$$\eta_1(t,s,u)=e^{-\beta u}\int_t^{t+s}\int_0^u e^{\beta y}dW(x,y),$$
$$\eta_2(t,s,u)=e^{-\alpha t-\beta u}\int_t^{t+s}\int_0^u(e^{\alpha x}-e^{\alpha t})e^{\beta y}dW(x,y).$$

Then

$$\xi_3(t,s,u)=\eta_1(t,s,u)+\eta_2(t,s,u).$$

Consider $\eta_2(t,s,u)$ at first. We have

$$E\eta_2^2(t,s,u)=e^{-2\alpha t-2\beta u}\int_t^{t+s}\int_0^u(e^{\alpha x}-e^{\alpha t})^2e^{2\beta y}dxdy$$
$$=\frac{1}{2\beta}\left\{\frac{1}{2\alpha}(e^{2\alpha s}-1)-\frac{2}{\alpha}(e^{\alpha s}-1)+s\right\}(1-e^{-2\beta u})$$
$$=\frac{\alpha^2}{6\beta}(1-e^{-2\beta u})s^3(1+o(1))\quad\text{as } s\to 0.$$

Along the lines of the investigation for $\xi_2(t,s,u)$, we can obtain

$$\lim_{h\to 0}\sup_{0\leq t\leq 1}\sup_{0\leq s\leq h}\sup_{v\leq u\leq v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_2(t,s,u)|=0 \text{ a.s.}$$

(4.6.54)

Now we need only to show

$$\limsup_{h\to 0}\inf_{0\leq t\leq 1}\sup_{0\leq s\leq h}\sup_{v\leq u\leq v+h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t,s,u)|\leq 1 \text{ a.s.}$$

(4.6.55)

Note that

$$E\eta_1^2(t,s,u)=\frac{1}{2\beta}s(1-e^{-2\beta u})=\beta(u)s.$$

Consequently, $\eta_1(t,s,u)/\beta(u)^{1/2}$ is a Wiener process on s for any fixed $t\geq 0$ and $v>0$. Let $t^i=ih$, $i=0,1,\cdots,[h^{-1}]$. Then using Lemma 1.6.1 of Csörgő and Révész (1981), we have

$$\begin{aligned}
&P\left\{\min_{0\leq i\leq h^{-1}}\sup_{0\leq s\leq h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t^i,s,v)|\geq 1+\varepsilon\right\}\\
&=\prod_{i=0}^{[h^{-1}]}P\left\{\sup_{0\leq s\leq h}\left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t^i,s,v)|\geq 1+\varepsilon\right\}\\
&\leq\left\{1-\frac{2}{\pi}\exp\left(-\frac{1}{(1+\varepsilon)^2}\log h^{-1}\right)\right\}^{1/h}\\
&\leq\exp\left\{-\frac{2}{\pi}h^{(1+\varepsilon)^{-2}-1}\right\}.
\end{aligned}\tag{4.6.56}$$

Let $h_n=1/n$. Then (4.6.56) implies

$$\limsup_{n\to\infty}\min_{0\leq i\leq h_n^{-1}}\sup_{0\leq s\leq h_n}\left(\frac{8\log h_n^{-1}}{\pi^2\beta(v)h}\right)^{1/2}|\eta_1(t^i,s,v)|\leq 1+\varepsilon\quad\text{a.s.}$$

(4.6.57)

For $v\leq u\leq v+h$, let

$$\zeta_1(t^i,s,u)=e^{-\beta v}\int_{t^i}^{t^i+s}\int_v^u e^{\beta y}dW(x,y),$$

$$\zeta_2(t^i,s,u)=(e^{-\beta(u-v)}-1)e^{-\beta v}\int_{t^i}^{t^i+s}\int_0^u e^{\beta y}dW(x,y).$$

Then

$$\eta_1(t^i,s,u)=\eta_1(t^i,s,v)+\zeta_1(t^i,s,u)+\zeta_2(t^i,s,u).$$

(4.6.58)

Similar to (4.6.48), write

$$|\zeta_1(t^i,s,u)|\leq\sum_{k=0}^{\infty}|\zeta_1(t^i+s_k,s_{k+1}-s_k,u)|.$$

We have

$$\begin{aligned}
E\zeta_1^2(t^i+s_k,s_{k+1}-s_k,u)&\leq\frac{1}{2\beta}h2^{-(k+1)}(e^{2\beta h}-1)\\
&\leq h^2 2^{-k}.
\end{aligned}$$

Then

$$\begin{aligned}
&P\left\{\max_{0\leq i\leq h^{-1}}\sup_{0\leq s\leq h}\sup_{v\leq u\leq v+h}\left(\frac{\log h^{-1}}{h}\right)^{\frac{1}{2}}|\zeta_1(t^i,s,u)|\geq\varepsilon\sum_{k=0}^{\infty}2^{-\frac{(k+1)}{4}}\right\}\\
&\leq\sum_{i=0}^{[h^{-1}]}\sum_{k=0}^{\infty}P\left\{\sup_{0\leq s\leq h}\sup_{v\leq u\leq v+h}\left(\frac{\log h^{-1}}{h}\right)^{1/2}\right.\\
&\qquad\left.\times|\zeta_1(t^i+s_k,s_{k+1}-s_k,u)|\geq\varepsilon 2^{-(k+1)/4}\right\}\\
&\leq ch^{-1}\sum_{k=0}^{\infty}2^{k+1}\exp\left\{-\frac{\varepsilon^2}{2}2^{-(k+1)/2}h(\log h^{-1})^{-1}h^{-2}2^k\right\}\\
&\leq ch^{-1}\sum_{k=0}^{\infty}2^{k+1}\exp\left\{-\frac{\varepsilon^2}{4\sqrt{2}}(h\log h^{-1})^{-1}(1+2^{k/2})\right\}\\
&\leq ch^{-1}\exp(-h^{-1/2})\sum_{k=0}^{\infty}2^{k+1}\exp(-2^{k/2})\\
&\leq c\exp(-h^{-1/4}),
\end{aligned}$$

which implies

$$\begin{aligned}
&\limsup_{n\to\infty}\max_{0\leq i\leq h_n^{-1}}\sup_{0\leq s\leq h_n}\sup_{v\leq u\leq v+h_n}\left(\frac{\log h_n^{-1}}{h_n}\right)^{1/2}|\zeta_1(t^i,s,u)|\\
&\leq\varepsilon\sum_{k=0}^{\infty}2^{-(k+1)/4}\quad\text{a.s.}
\end{aligned}\tag{4.6.59}$$

For $\zeta_2(t^i,s,u)$ we have a similar inequality. Therefore we obtain from (4.6.57) and (4.6.58) that

$$\limsup_{n\to\infty} \min_{0\leqslant i\leqslant h_n^{-1}} \sup_{0\leqslant s\leqslant h_n} \sup_{v\leqslant u\leqslant v+h_n} \left(\frac{8\log h_n^{-1}}{\pi^2\beta(v)h_n}\right)^{1/2} |\eta_1(t^i,s,u)|$$
$$\leqslant 1 \quad \text{a.s.} \tag{4.6.60}$$

And further, (4.6.60) implies that (4.6.55), hence (4.6.53) holds true. (4.6.51) is proved.

Next, we prove that for any $v>0$

$$\liminf_{h\to 0} \inf_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2} |X(t+s,v)-X(t,v)|$$
$$\geqslant 1 \quad \text{a.s.} \tag{4.6.61}$$

From (4.6.52) and (4.6.54), it follows that (4.6.61) is equivalent to

$$\liminf_{h\to 0} \inf_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \left(\frac{8\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2} |\eta_1(t,s,v)| \geqslant 1 \text{ a.s.} \tag{4.6.62}$$

Put

$$\eta_v(t) = e^{-\beta v}\int_0^t\int_0^v e^{\beta y} dW(x,y).$$

Noting that $\eta_v(t)/\beta(v)^{1/2}$ is a Wiener process in t and $\eta_1(t,s,v)$ are increments of $\eta_v(t)$, we conclude that (4.6.62) is true from the moduli of non-differetiability of Wiener process, and hence (4.6.61) is proved. Combining (4.6.51) with (4.6.61) completes the proof of Theorem 4.6.4.

By the same discussion, Lu and Yu (1997) obtain a Chung's law of the iterated logarithm for a two-parameter Ornstein-Uhlenbeck process as follows:

Theorem 4.6.5 *Under the assume of Theorem* 4.6.4, *we have*

$$\liminf_{h\to 0} \sup_{0\leqslant t\leqslant h} \left(\frac{8\log\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2} |X(t,v) - X(0,v)| = 1 \text{ a.s.},$$

$$\liminf_{h\to 0} \sup_{0\leqslant t\leqslant h} \sup_{v\leqslant u\leqslant v+h} \left(\frac{8\log\log h^{-1}}{\pi^2\beta(v)h}\right)^{1/2} |X(t,u) - X(0,u)| = 1 \text{ a.s.}$$

Chapter 5

Other Path Properties of Gaussian Processes

In this chapter, we introduce the p-variation of Gaussian processes and the fractal nature of Gaussian processes.

5.1 The p-variation of Gaussian Processes

Let $\{X(t);\ t\geqslant 0\}$ be a mean zero Gaussian process with stationary increments. Put

$$\sigma^2(h) = E(X(t+h) - X(t))^2. \qquad (5.1.1)$$

Let $\pi = \{0 = x_0 < x_1 < \cdots < x_{k_n} = a\}$ denote a partition of $[0,a]$ and $m(\pi) = \max_{1\leqslant i\leqslant k_n}(x_i - x_{i-1})$ denote the length of the largest interval in π.

5.1.1 Introdution and Results

The interest in the p-variation of stochastic processes was initiated by Lévy's elegant result on the 2-variation of Wiener process $\{W(t);t\geqslant 0\}$, i.e.

$$\lim_{n\to\infty}\sum_{i=0}^{2^n-1}\left(W\left(\frac{i}{2^n}\right) - W\left(\frac{i+1}{2^n}\right)\right)^2 = 1 \quad \text{a.s.}$$

There are many generalizations of this result. Dudley (1973) showed that for any partitions sequence $\{\pi(n)\}$ of $[0,a]$ with $m(\pi(n)) = o(1/\log n)$,

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}(W(x_i) - W(x_{i-1}))^2 = a \quad \text{a.s.} \qquad (5.1.2)$$

However, de la Vega (1974) showed that this is no longer true if the condition on $m(\pi(n))$ is relaxed to $m(\pi(n)) = O(1/\log n)$. Taylor (1972) showed that

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\overline{\psi}(|W(x_i) - W(x_{i-1})|) = 1 \quad \text{a.s.},$$

where $\overline{\psi}(x) = |x/\sqrt{2\log\log(1/x)}|^2$ and $Q_a(\delta) = \{$partitions π of $[0,a]$; $m(\pi)\leqslant\delta\}$. Similar results for the stochastic processes, with 2 replaced by p, are refered to as results about the p-variation of these processes.

Results about various aspects of the p-variation of Gaussian processes appeared in Kôno (1969), Kawata and Kôno (1973), Giné and Klein (1975), Jain and Monrad (1983), and Adler and Pyke (1993). Recently, Marcus and Rosen (1992b) considered the p-variation of Gaussian process $X(\cdot)$ and proved the following theorem.

Theorem 5.1.1 *If $\sigma^2(h)$ is concave on $[0,a]$ and satisfies $\lim_{h\to 0}\sigma(h)/h^{1/p}=b$ for some $p\geqslant 2$ and $0<b<\infty$, then for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$ with $m(\pi)=o((1/\log n)^{p/2})$,*

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^p=EN(0,1)^p b^p a \quad \text{a.s.} \tag{5.1.3}$$

Also

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^p$$

$$=E|N(0,1)^p(2b)|^p\int_0^a|X(x)|^p\mathrm{d}x \quad \text{a.s.} \tag{5.1.4}$$

Shao (1996 b) weakened the condition on $\sigma(h)$ and gave more general results.

Theorem 5.1.2 *Let $p>1$. Assume that $\sigma^2(h)$ is non-decreasing. Then for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$.*

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}\frac{(x_i-x_{i-1})|X(x_i)-X(x_{i-1})|^p}{\sigma^p(x_i-x_{i-1})}=aE|N(0,1)|^p \quad \text{a.s.} \tag{5.1.5}$$

If one of the following two conditions is satisfied:

(A1) *$\sigma^2(h)$ is concave on $[0,a]$, and*

$$m(\pi(n))=o((\log n)^{-(1\vee p/2)});$$

(A2) *$\sigma^2(h)$ is convex on $[0,a+\varepsilon_0]$ for some $\varepsilon_0>0$, and*

$$\max_{x_i\in\pi(n)}(x_i-x_{i-1})^{\frac{1}{2}+\frac{1}{2}\wedge\frac{1}{p}}/\sigma(x_i-x_{i-1})=o(\log^{-1/2}n).$$

Theorem 5.1.3 *Assume that $\sigma^2(h)$ is continuous on $[0,a]$ satisfying*

$$\int_1^\infty\sigma(\mathrm{e}^{-z^2})\mathrm{d}z<\infty.$$

Then, under the condition of Theorem 5.1.2, we have

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}\frac{(x_i-x_{i-1})|X^2(x_i)-X^2(x_{i-1})|^p}{\sigma^p(x_i-x_{i-1})}$$

$$=2^pE|N(0,1)|^p\int_0^a|X(x)|^p\mathrm{d}x \quad \text{a.s.} \tag{5.1.6}$$

The following results are immediate consequences of Theorems 5.1.2 and 5.1.3.

Corollary 5.1.1 *Assume that $\sigma^2(h)$ is non-decreasing and concave on $[0,a]$ and satisfies $\lim_{h\to 0}\sigma(h)/h^{\alpha}=c$ for $0<\alpha<\infty$ and $0<c<\infty$. Then for $p>1$ and for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$ with $m(\pi(n))=o((\log n)^{-(1\vee p/2)})$,*

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$

$$=ac^pE|N(0,1)|^p \quad \text{a.s.}, \tag{5.1.7}$$

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$

$$=a(2c)^p\int_0^a|X(x)|^p\mathrm{d}x \quad \text{a.s.} \tag{5.1.8}$$

Corollary 5.1.2 *Asume that $\sigma^2(h)$ is non-decreasing and convex on $[0,a+\varepsilon_0]$ for some $\varepsilon_0>0$ and satisfies $\lim_{h\to 0}\sigma(h)/h^{\alpha}=c$ for $1/2\leqslant\alpha<1$ and $0<c<\infty$. Then for $1<p<2/(2\alpha-1)$ and for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$ with $m(\pi(n))=o((\log n)^{-1/(1-2\alpha+1\wedge 2/p)})$,*

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$

$$=ac^pE|N(0,1)|^p \quad \text{a.s.} \tag{5.1.9}$$

and

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$

$$=a(2c)^p\int_0^a|X(x)|^p\mathrm{d}x \quad \text{a.s.} \tag{5.1.10}$$

In particular, for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$

with $m(\pi(n))=o((\log n)^{-1/(2(1-\alpha))})$,

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^{1/\alpha}=ac^{1/\alpha}E|N(0,1)|^{1/\alpha}\quad\text{a. s.}\tag{5.1.11}$$

and

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^{1/\alpha}$$
$$=a(2c)^{1/\alpha}\int_0^a|X(x)|^{1/\alpha}\mathrm{d}x\quad\text{a. s.}\tag{5.1.12}$$

Applying Corollaries 5. 1. 1 and 5. 1. 2 to the fractional Wiener process, we have

Corollary 5. 1. 3 *Let* $\{Z(t);t\geqslant0\}$ *be the fractional Wiener process of order* α, $0<\alpha<1$, *i. e.* $\sigma(h)=h^\alpha$. *Let* $1<p<2/((2\alpha-1)\vee 0)$ *and* $r_{p,\alpha}=(1\vee p/2)I(0<\alpha\leqslant1/2)+1/(1-2\alpha+1\wedge 2/p)\times I(1/2<\alpha<1)$. *Then for any sequence of partitions* $\{\pi(n)\}$ *of* $[0,a]$ *with* $m(\pi(n))=o((\log n)^{-r_{p,\alpha}})$,

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$
$$=aE|N(0,1)|^p\quad\text{a. s.},$$
$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^p(x_i-x_{i-1})^{1-p\alpha}$$
$$=a2^p\int_0^a|X(x)|^p\mathrm{d}x\quad\text{a. s.}$$

In particular, for any sequence of partitions $\{\pi(n)\}$ *of* $[0,a]$ *with* $m(\pi(n))=o((\log n)^{-1/2(\alpha\wedge(1-\alpha))})$,

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X(x_i)-X(x_{i-1})|^{1/\alpha}=aE|N(0,1)|^{1/\alpha}\quad\text{a. s.},$$

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}|X^2(x_i)-X^2(x_{i-1})|^{1/\alpha}=a2^{1/\alpha}\int_0^a|X(x)|^{1/\alpha}\mathrm{d}x\quad\text{a. s.}$$

Corollary 5. 1. 1 with $p=1/\alpha$ coincides Theorem 5. 1. 1

(Marcus and Rosen 1992 b). The results in (5. 1. 7) and (5. 1. 9) for $p=2$ overlap Theorem 2 of Giné and Klein (1975), but the assumption here looks simpler.

5. 1. 2 Proofs of Theorems

The proofs of the Theorems need the following lemmas, some of which are of independent interests.

Lemma 5. 1. 1 *Let* (X,Y) *be a normal random vector with* $EX=EY=0$. *Then*

$$E|XY|^p\leqslant E|X|^pE|Y|^p+C_p|EXY|E|X|^{p-1}E|Y|^{p-1}$$

for $p\geqslant1$, *where*

$$C_p=2^p\left\{\frac{E|N|^{2p}}{(E|N|^{p-1})^2}+\frac{E|N|^{p+1}}{E|N|^{p-1}}\right\}$$

and N *is the standard normal random variable.*

Proof Put $\rho=E(XY)/EY^2$. Using the elementary inequality

$$(1+x)^p\leqslant1+2^p(x+x^p)\text{ for every }x>0$$

and noting that $X-\rho Y$ and Y are independent, we have

$$\begin{aligned}E|XY|^p&=E|X-\rho Y+\rho Y|^p|Y|^p\\
&\leqslant E\{|X-\rho Y|^p+2^p(|X-\rho Y|^{p-1}|\rho Y|+|\rho Y|^p)\}|Y|^p\\
&=E|X-\rho Y|^pE|Y|^p+2^p\{|\rho|E|X-\rho Y|^{p-1}E|Y|^{p+1}+|\rho|^pE|Y|^{2p}\}\\
&\leqslant(EX^2-\rho^2EY^2)^{p/2}E|N|^pE|Y|^p+2^p\{|\rho|^p(EY^2)^pE|N|^{2p}\\
&\quad+|\rho|(EX^2-\rho^2EY^2)^{(p-1)/2}(EY^2)^{(p+1)/2}E|N|^{p-1}E|N|^{p+1}\}\\
&\leqslant(EX^2)^{p/2}E|N|^pE|Y|^p+2^p\{|EXY|^pE|N|^{2p}\\
&\quad+|\rho|(EX^2)^{(p-1)/2}(EY^2)^{(p+1)/2}E|N|^{p-1}E|N|^{p+1}\}\\
&\leqslant E|X|^pE|Y|^p+2^p\{|EXY|(EX^2EY^2)^{(p-1)/2}E|N|^{2p}\\
&\quad+|EXY|(EX^2)^{(p-1)/2}(EY^2)^{(p-1)/2}E|N|^{p-1}E|N|^{p+1}\}\end{aligned}$$

$$= E|X|^p E|Y|^p + 2^p |EXY| E|X|^{p-1} E|Y|^{p-1} \times \left\{ \frac{E|N|^{2p}}{(E|N|^{p-1})^2} + \frac{E|N|^{p+1}}{E|N|^{p-1}} \right\},$$

as desired.

Lemma 5.1.2 *Let $B=(b_{ij})$ be an $n\times n$ real symmetric matrix. Then for any $x=(x_1,\cdots,x_n)$*

$$xBx' \leqslant \sum_{i=1}^n x_i^2 b_i^*, \tag{5.1.13}$$

where $b_i^* = \sum_{j=1}^n |b_{ij}|, i = 1,2,\cdots,n.$

Proof Let B^* be a diagonal matrix with diagonal elements $\{b_i^*\}$. It is well known that B^*-B is a symmetric positive semi-definitive matrix. Hence (5.1.13) follows.

Lemma 5.1.3 *Let $p\geqslant 1$, and $\{\xi_i; 1\leqslant i\leqslant n\}$ be a Gaussian sequence with mean zero. Then for any $x>0$ and any sequence of positive numbers $\{a_i;1\leqslant i\leqslant n\}$,*

$$P\left\{ \left| \left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} - E\left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} \right| \geqslant x \right\} \leqslant \begin{cases} 2\exp\left(- \dfrac{x^2}{2\left(\sum_{i=1}^n a_i^{2/(2-p)} \rho_i^{p/(2-p)} \right)^{(2-p)/p}} \right) & \text{if } 1\leqslant p<2, \\ 2\exp\left(- \dfrac{x^2}{2\max_{1\leqslant i\leqslant n}(a_i^{2/p})\rho_i} \right) & \text{if } p\geqslant 2, \end{cases} \tag{5.1.14}$$

where $\rho_i = \sum_{j=1}^n |E\xi_i\xi_j|, i=1,2,\cdots,n.$

Proof Put $q = p/(p-1)$ and define $\| b \|_q = \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}$ for $b=(b_1,\cdots,b_n)$. It is well-known that

$$\left(\sum_{i=1}^n a_i |\xi_i|^p \right)^{1/p} = \sup_{\|b\|_q\leqslant 1} \sum_{i=1}^n a_i^{1/p} b_i \xi_i.$$

By Borell's inequality (Theorem 1.1.1), we have

$$\begin{aligned} &P\left\{ \left| \left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} - E\left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} \right| \geqslant x \right\} \\ &= P\left\{ \left| \sup_{\|b\|_q\leqslant 1} \sum_{i=1}^n a_i^{1/p} b_i\xi_i - E \sup_{\|b\|_q\leqslant 1} \sum_{i=1}^n a_i^{1/p} b_i\xi_i \right| \geqslant x \right\} \\ &\leqslant 2\exp\left(- \frac{x^2}{2\sup_{\|b\|_q\leqslant 1} E\left(\sum_{i=1}^n a_i^{1/p} b_i\xi_i \right)^2} \right). \end{aligned} \tag{5.1.15}$$

Let $B=(\rho_{ij})$ be the covariance matrix of $(\xi_1,\cdots,\xi_n)$ and $y=(a_1^{1/p}b_1,\cdots,a_n^{1/p}b_n)$. It follows from Lemma 5.1.2 that

$$\begin{aligned} \sup_{\|b\|_q\leqslant 1} E\left(\sum_{i=1}^n a_i^{1/p} b_i\xi_i \right)^2 &= \sup_{\|b\|_q\leqslant 1} yBy' \leqslant \sup_{\|b\|_q\leqslant 1} \sum_{i=1}^n a_i^{2/p} b_i^2 \rho_i \\ &\leqslant \begin{cases} \sup_{\|b\|_q\leqslant 1} \|b\|_q^2 \left(\sum_{i=1}^n (a_i^{2/p}\rho_i)^{1/(1-2/q)} \right)^{1-2/q} & \text{if } 1\leqslant p<2, \\ \sup_{\|b\|_q\leqslant 1} \|b\|_2^2 \max_{1\leqslant i\leqslant n} (a_i^{2/p}\rho_i) & \text{if } p\geqslant 2 \end{cases} \\ &\leqslant \begin{cases} \left(\sum_{i=1}^n (a_i^{2/p}\rho_i)^{1/(1-2/q)} \right)^{1-2/q} & \text{if } 1\leqslant p<2, \\ \max_{1\leqslant i\leqslant n}(a_i^{2/p}\rho_i) & \text{if } p\geqslant 2. \end{cases} \end{aligned} \tag{5.1.16}$$

This proves (5.1.14), by (5.1.15) and (5.1.16).

From Hölder's inequality it follows immediately that

$$E\left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} \leqslant \left(\sum_{i=1}^n a_i E|\xi_i|^p \right)^{1/p}.$$

The next lemma gives a lower bound of $E\left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p}$.

Lemma 5.1.4 *Let $p>1$, $\{\xi_i;1\leqslant i\leqslant n\}$ be a Gaussian sequence with mean zero. Then for any sequence of positive numbers $\{a_i;1\leqslant i\leqslant n\}$*

$$E\left(\sum_{i=1}^n a_i|\xi_i|^p \right)^{1/p} \geqslant \left(\sum_{i=1}^n a_i E|\xi_i|^p \right)^{1/p} \times \left(1 + C_p \sum_{i=1}^n a_i^2 E|\xi_i|^{2p-2}\rho_i \Big/ \left(\sum_{i=1}^n a_i E|\xi_i|^p \right)^2 \right)^{(1-p)/p}, \tag{5.1.17}$$

where $\rho_i = \sum_{j=1}^{n} |E\xi_i\xi_j|$, *and* C_p *is defined as in Lemma* 5.1.1.

Proof By Lyapunov's inequality, for any random variable X

$$(E|X|^p)^{1/p} \leqslant (E|X|)^{1/(2p-1)}(E|X|^{2p})^{(p-1)/(2p-1)}.$$

Hence,

$$E\Big(\sum_{i=1}^{n} a_i|\xi_i|^p\Big)^{1/p} \geqslant \frac{(E\sum_{i=1}^{n} a_i|\xi_i|^p)^{(2p-1)/p}}{\Big(E\Big(\sum_{i=1}^{n} a_i|\xi_i|^p\Big)^2\Big)^{(p-1)/p}}$$

$$= \frac{\Big(\sum_{i=1}^{n} a_i E|\xi_i|^p\Big)^{(2p-1)/p}}{\Big(E\Big(\sum_{i=1}^{n} a_i|\xi_i|^p\Big)^2\Big)^{(p-1)/p}}. \tag{5.1.18}$$

By Lemmas 5.1.1 and 5.1.2, we have

$$\begin{aligned}
&E\Big(\sum_{i=1}^{n} a_i|\xi_i|^p\Big)^2 \\
&= \Big(\sum_{i=1}^{n} a_i E|\xi_i|^p\Big)^2 + E\Big(\sum_{i=1}^{n} a_i(|\xi_i|^p - E|\xi_i|^p)\Big)^2 \\
&\leqslant \Big(\sum_{i=1}^{n} a_i E|\xi_i|^p\Big)^2 + C_p \sum_{1\leqslant i,j\leqslant n} a_i a_j E|\xi_i|^{p-1}E|\xi_j|^{p-1}|E\xi_i\xi_j| \\
&\leqslant \Big(\sum_{i=1}^{n} a_i E|\xi_i|^p\Big)^2 + C_p \sum_{1\leqslant i\leqslant n} a_i^2(E|\xi_i|^{p-1})^2\rho_i \\
&\leqslant \Big(\sum_{i=1}^{n} a_i E|\xi_i|^p\Big)^2 + C_p \sum_{1\leqslant i\leqslant n} a_i^2 E|\xi_i|^{2p-2}\rho_i.
\end{aligned} \tag{5.1.19}$$

(5.1.17) follows from (5.1.18) and (5.1.19).

Proof of Theorem 5.1.2

We first collect two facts:

(A) If $\sigma^2(h)$ is concave on $[0,a]$, then

$$E(X(x_2) - X(x_1))(X(x_4) - X(x_3)) \leqslant 0$$

for $0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant a$;

(B) If $\sigma^2(h)$ is convex on $[0,a+\varepsilon_0]$ for some $\varepsilon_0>0$, then

$$E(X(x_2) - X(x_1))(X(x_4) - X(x_3)) \geqslant 0$$

for $0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant a$, and there is $K > 0$ such that

$$\sigma^2(x+y) - \sigma^2(x) \leqslant Ky \quad \text{for all} \quad 0 \leqslant x \leqslant x+y \leqslant a.$$

Let

$$E_{\pi(n)} = E\Big(\sum_{x_i\in\pi(n)} \frac{(x_i - x_{i-1})|X(x_i) - X(x_{i-1})|^p}{\sigma^p(x_i - x_{i-1})}\Big)^{1/p},$$

$$\rho_i = \sum_{x_j\in\pi(n)} |E(X(x_i) - X(x_{i-1}))(X(x_j) - X(x_{j-1}))|,$$

$$\delta_n = \begin{cases} \Big(\sum_{x_i\in\pi(n)}\Big(\dfrac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\Big)^{\frac{2}{(2-p)}} \rho_i^{\frac{p}{(2-p)}}\Big)^{\frac{(2-p)}{p}} & \text{for } 1 \leqslant p \leqslant 2, \\ \max\limits_{x_i\in\pi(n)} \rho_i\Big(\dfrac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\Big)^{\frac{2}{p}} & \text{for } p > 2. \end{cases}$$

We next prove that

$$\delta_n = o(1/\log n) \quad \text{as} \quad n\to\infty. \tag{5.1.20}$$

I. $\sigma^2(\cdot)$ is concave on $[0,a]$. From (A) we get

$$\begin{aligned}
\rho_i &= \sigma^2(x_i - x_{i-1}) \\
&\quad - \sum_{j\neq i, x_j\in\pi(n)} E(X(x_i) - X(x_{i-1}))(X(x_j) - X(x_{j-1})) \\
&= \sigma^2(x_i - x_{i-1}) - E(X(x_i) - X(x_{i-1}))(X(a) - X(x_i)) \\
&\quad - E(X(x_i) - X(x_{i-1}))(X(x_{i-1}) - X(0)) \\
&\leqslant 2\sigma^2(x_i - x_{i-1}).
\end{aligned} \tag{5.1.21}$$

Thus, by condition (A 1)

$$\delta_n \leqslant \begin{cases} \Big(\sum_{x_i\in\pi(n)}\Big(\dfrac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\Big)^{2/(2-p)} \\ \quad \times (2\sigma^2(x_i - x_{i-1}))^{p/(2-p)}\Big)^{(2-p)/p} & \text{for} \quad 1 \leqslant p \leqslant 2, \\ \max\limits_{x_i\in\pi(n)} 2\sigma^2(x_i - x_{i-1})\Big(\dfrac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\Big)^{2/p} & \text{for} \quad p > 2 \end{cases}$$

$$\leqslant \begin{cases} 2\left(\sum_{x_i \in \pi(n)} (x_i - x_{i-1})^{2/(2-p)}\right)^{(2-p)/p} & \text{for } 1 \leqslant p \leqslant 2, \\ 2 \max_{x_i \in \pi(n)} (x_i - x_{i-1})^{2/p} & \text{for } p > 2 \end{cases}$$

$$\leqslant \begin{cases} 2m(\pi(n))\left(\sum_{x_i \in \pi(n)} (x_i - x_{i-1})\right)^{(2-p)/p} & \text{for } 1 \leqslant p \leqslant 2, \\ 2m(\pi(n))^{2/p} & \text{for } p > 2 \end{cases}$$

$$= o(1/\log n). \tag{5.1.22}$$

This proves (5.1.20).

Ⅱ. $\sigma^2(\cdot)$ is convex on $[0, a+\varepsilon_0]$. Using (B), we have

$$\rho_i = E(X(x_i) - X(x_{i-1})) \sum_{x_i \in \pi(n)} (X(x_i) - X(x_{i-1}))$$

$$= E(X(x_i) - X(x_{i-1}))(X(a) - X(0))$$

$$= \frac{1}{2}(\sigma^2(a - x_{i-1}) - \sigma^2(a - x_i) + \sigma^2(x_i) - \sigma^2(x_{i-1}))$$

$$\leqslant c(x_i - x_{i-1}). \tag{5.1.23}$$

Similar to (5.1.21), by condition (AⅡ)

$$\delta_n \leqslant$$

$$\begin{cases} \left(\sum_{x_i \in \pi(n)} \left(\frac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\right)^{\frac{2}{2-p}} (c(x_i - x_{i-1}))^{\frac{p}{2-p}}\right)^{\frac{2-p}{p}} & \text{for } 1 \leqslant p \leqslant 2, \\ c \max_{x_i \in \pi(n)} (x_i - x_{i-1}) \left(\frac{x_i - x_{i-1}}{\sigma^p(x_i - x_{i-1})}\right)^{2/p} & \text{for } p > 2 \end{cases}$$

$$\leqslant \begin{cases} c \max_{x_j \in \pi(n)} \frac{(x_j - x_{j-1})}{\sigma^2(x_j - x_{j-1})} \left(\sum_{x_j \in \pi(n)} (x_i - x_{i-1})\right)^{\frac{2}{(2-p)}} & \text{for } 1 \leqslant p \leqslant 2, \\ c \max_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})^{1+2/p}}{\sigma^2(x_i - x_{i-1})} & \text{for } p > 2 \end{cases}$$

$$= o(1/\log n), \tag{5.1.24}$$

from which (5.1.20) follows.

From Lemma 5.1.3 and (5.1.20) we obtain for any $\varepsilon > 0$

$$P\left\{\left|\left(\sum_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})|X(x_i) - X(x_{i-1})|^p}{\sigma^p(x_i - x_{i-1})}\right)^{1/p} - E_{\pi(n)}\right| > \varepsilon\right\}$$

$$\leqslant 2\exp(-\varepsilon^2/(2\delta_n)) = 2\exp(-\varepsilon^2 \log n / o(1)).$$

Therefore, by the Borel-Cantelli lemma

$$\lim_{n\to\infty} \left\{\left(\sum_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})|X(x_i) - X(x_{i-1})|^p}{\sigma^p(x_i - x_{i-1})}\right)^{1/p} - E_{\pi(n)}\right\}$$

$$= 0 \quad \text{a.s.} \tag{5.1.25}$$

Since $\sum_{x_i \in \pi(n)} (x_i - x_{i-1}) E|X(x_i) - X(x_{i-1})|^p / \sigma^p(x_i - x_{i-1}) = aE|N|^p$, to finish the proof, by (5.1.25) and (5.1.16), it suffices to show that

$$\liminf_{n\to\infty} E_{\pi(n)} \geqslant (aE|N|^p)^{1/p}. \tag{5.1.26}$$

In terms of (5.1.21) and (5.1.23), we have

$$\sum_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})^2 |X(x_i) - X(x_{i-1})|^{2p-2}}{\sigma^{2p}(x_i - x_{i-1})} \rho_i$$

$$= E|N|^{2p-2} \sum_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})^2 \rho_i}{\sigma^2(x_i - x_{i-1})}$$

$$\leqslant \begin{cases} 2E|N|^{2p-2} \sum_{x_i \in \pi(n)} (x_i - x_{i-1})^2 & \text{if } \sigma^2 \text{ is concave,} \\ 2cE|N|^{2p-2} \sum_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})^2}{\sigma^2(x_i - x_{i-1})} & \text{if } \sigma^2 \text{ is convex} \end{cases}$$

$$\leqslant \begin{cases} 2aE|N|^{2p-2} m(\pi(n)) & \text{if } \sigma^2 \text{ is concave,} \\ 2acE|N|^{2p-2} \max_{x_i \in \pi(n)} \frac{(x_i - x_{i-1})^2}{\sigma^2(x_i - x_{i-1})} & \text{if } \sigma^2 \text{ is convex} \end{cases}$$

$$= o(1) \quad \text{as} \quad n \to \infty. \tag{5.1.27}$$

Hence, (5.1.26) holds by lemma 5.1.4 and (5.1.27).

The proof is now completed.

Proof of Theorem 5.1.3

The proof follows from that of Theorem 5.1.1 (cf. Theorem 1.2 of Marcus and Rosen 1992b).

5.2 The Fractal Nature of Image and Graph of Gaussian Fields

The study of the random fractals began in nineteen forties, though there was not the notion of the random fractal at that time. In nineteen forties, Lévy P studied the sample properties of a Wiener process. Later, Besicovith A S and Taylor S J also studied similar problems. By combining their results, it is known that the Hausdorff dimension of the set of the zero points of the one-dimensional Wiener process $W(\cdot)$ is $1/2$, i.e.,

$$\dim\{t \in [0,1]; W(t) = 0\} = \frac{1}{2} \quad \text{a.s.},$$

which is possibly the first one of the beautiful results on the random fractals. Since then, there has been considerable interest in the fractal nature of the image, graph, level sets and multiple point of Gaussian fields, including fractional Wiener processes. Another early result is that the Hausdorff dimension of the image of the one-dimensional Wiener process is 1, and while, the Hausdorff dimensions of the images of high-dimensional Wiener processes are all 2, i.e.,

$$\dim\{W(t); t \in [0,1]\} = \begin{cases} 1 & \text{if } d = 1, \\ 2 & \text{if } d \geqslant 2, \end{cases}$$

where $W(t) = (W_1(t), \cdots, W_d(t))$ is the d-dimensional Wiener process. Xiao (1995) studied the Hausdorff dimension and packing dimension of the image and the graph of index-α Gaussian fields. We state his results here. For more and other studies one can refer to Adler(1981), Cuzick(1978,1981), Goldman(1981), Kahane(1985), Taylor and Tricot(1985), Talagrand(1995) and others.

Let $X(t) = (X_1(t), \cdots, X_d(t))$ be a $\mathbf{R}^d$-valued mean zero Gaussian vector fields on $\mathbf{R}^N$. We assume that the coordinate fields $X_1, \cdots, X_d$ have stationary increments. Denote

$$\sigma_j^2(t) = E(X_j(t) - X_j(0))^2.$$

If for each $j = 1, 2, \cdots, d$, there exists $0 < \alpha_j \leqslant 1$ such that

$$\begin{aligned} \alpha_j &= \sup\{\alpha > 0; \lim_{|t| \to 0} |t|^{-\alpha}\sigma_j(t) = 0\} \\ &= \inf\{\alpha > 0; \lim_{|t| \to 0} |t|^{-\alpha}\sigma_j(t) = \infty\}, \end{aligned}$$

where $|\cdot|$ is the Euclidean norm, we call $X(t)$ an index-$\alpha(N, d)$ Gaussian field for $\alpha = (\alpha_1, \cdots, \alpha_d)$. For simplicity, we shall assume that all $\sigma_j(t)$ are bounded away from zero on $[-1,1]^N$ for t bounded away from the origin. To avoid degeneration, we make the following restriction on the type of dependence allowed between the coordinate fields $X_1, \cdots, X_d$: there exists a constant $\varepsilon > 0$ such that

$$\det \operatorname{Cov}(X(t) - X(s)) \geqslant \varepsilon \prod_{j=1}^{d} \sigma_j^2(t - s), \tag{5.2.1}$$

where $\operatorname{Cov}(Y)$ denotes the covariance matrix of the random vector Y. This condition will be satisfied if the coordinate fields are independent. A specific example of index-α Gaussian fields is the fractional Wiener process.

Recall briefly the definition of Hausdorff dimension and packing dimension. For any continuous increasing function φ:

$[0,1]\to[0,\infty)$ with $\varphi(0)=0$, the φ-Hausdorff measure of $E\subset \mathbf{R}^N$ is defined by

$$\varphi\text{-mes}E = \lim_{\delta\to 0}\inf\left\{\sum_i \varphi(2r_i);E\subset \bigcup_i B(x_i,r_i),r_i\leqslant\delta\right\}, \tag{5.2.2}$$

where $B(x_i,r_i)$ denotes the open ball of radius r_i centered at x_i. Here, the $B(x_i,r_i)$ constitute an δ-cover of E (i.e., a collection of balls with radius not exceeding δ, whose union includes E), and infimum in (5.2.2) is taken over all δ-covers of E. φ-mes is metric outer measure and all Borel sets are measurable. The Hausdorff dimension of a subset E is defined by

$$\begin{aligned}\dim E &= \inf\{\alpha>0;s^\alpha-\text{mes}E=0\}\\ &=\sup\{\alpha>0;s^\alpha-\text{mes}E=\infty\}.\end{aligned}$$

Taylor and Tricot (1985) defined another set function φ-pack E in which economical coverings are replaced by disjoint packings,

$$\begin{aligned}&\varphi\text{-pack } E\\ &=\lim_{\delta\to 0}\sup\left\{\sum_i \varphi(2r_i);B(x_i,r_i)\text{ are disjoint},x_i\in E,r_i\leqslant\delta\right\}.\end{aligned}$$

φ-pack is not an outer measure because it fails to be countably subadditive. However, φ-pack is a premeasure, so we can obtain a metric outer measure on $\mathbf{R}^N$ by

$$\varphi\text{-pack } E = \inf\left\{\sum_i \varphi\text{-pack } E_i;E\subset\bigcup_i E_i\right\}.$$

φ-pack E is called the φ-packing measure of E. The packing dimension of E is defined by

$$\begin{aligned}\text{Dim } E &= \inf\{\alpha>0;s^\alpha\text{-pack } E=0\}\\ &=\sup\{\alpha>0;s^\alpha\text{-pack } E=\infty\}.\end{aligned}$$

It is known (cf. Taylor and Tricot 1985) that φ-mes $E\leqslant\varphi$-

pack E, so

$$0\leqslant\dim E\leqslant\text{Dim } E\leqslant N. \tag{5.2.3}$$

For each $\varepsilon>0$ and bounded set $E\subset\mathbf{R}^N$, let $M(\varepsilon,E)$ be the smallest number of balls of radius ε needed to cover E, and

$$\delta(E)=\lim_{\varepsilon\to 0}\inf\frac{\log M(\varepsilon,E)}{-\log\varepsilon},$$

$$\Delta(E)=\lim_{\varepsilon\to 0}\sup\frac{\log M(\varepsilon,E)}{-\log\varepsilon}.$$

δ and Δ are called the upper and lower entropy indices of Kolmogorov. Tricot (1982) proved that

$$\text{Dim } E=\hat{\Delta}(E):=\inf\{\sup\Delta(E_i);E\subset\bigcup_i E_i\}. \tag{5.2.4}$$

The following is the result on the Hausdorff dimension of the image $X(E)=\{X(t);t\in E\}$ and Gr $X(E)=\{(t,X(t));t\in E\}$ of the (N,d) index-α Gaussian fields $X(t)$, where $E\subset\mathbf{R}^N$ is an arbitrary compact set.

Theorem 5.2.1 *Let $X(t)$ be an (N,d) Gaussian field of index α with coordinates so arranged that the α satisfy*

$$0=\alpha_0<\alpha_1\leqslant\cdots\leqslant\alpha_d\leqslant 1$$

and let $E\subset\mathbf{R}^N$ be a compact set. If for any $(s,t)\in E\times E$ (5.2.1) holds, then with probability 1,

$$\begin{aligned}\dim X(E)&=\min\left\{d;\frac{\dim E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leqslant j\leqslant d\right\}\\ &=\begin{cases}(\dim E+\sum\limits_{i=1}^{k}(\alpha_k-\alpha_i))/\alpha_k & if\ \sum\limits_{i=0}^{k-1}\alpha_i\leqslant\dim E\leqslant\sum\limits_{i=1}^{k}\alpha_i,\\ d & if\ \dim E>\sum\limits_{i=1}^{d}\alpha_i,\end{cases}\end{aligned} \tag{5.2.5}$$

$$\dim \mathrm{Gr}X(E)$$
$$= \min\left\{\frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \leqslant j \leqslant d; \dim E + \sum_{i=1}^{d}(1-\alpha_i)\right\}$$
$$= \begin{cases} \dim X(E) & \text{if } \dim X(E) < d, \\ \dim E + \sum_{i=1}^{d}(1-\alpha_i) & \text{if } \dim X(E) = d. \end{cases} \tag{5.2.6}$$

To prove Theorem 5.2.1, we need some lemmas on the Hausdorff dimension. The first one is the Frostman theorem, whose proof can be found in Falconer (1990).

Lemma 5.2.1 *Let K be a subset of $\mathbf{R}^d$. Assume that there exists a positive measure μ on K with $\mu(K)=1$ such that*

$$\int_K\int_K \frac{1}{|x-y|^\gamma}\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty.$$

Then s^γ-mes $K=\infty$ (so, $\dim K \geqslant \gamma$).

If s^γ-mes $K>0$, then there a compact set $K' \subset K$ and a positive measure μ on K' with $\mu(K')=1$ such that

$$\int_{K'}\int_{K'} \frac{1}{|x-y|^\beta}\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty \quad \text{for any } \beta < \gamma.$$

The second one will be used to deal with the upper bounds of $\dim X(E)$ and $\dim \mathrm{Gr}\, X(E)$ in (5.2.5) and (5.2.6).

Lemma 5.2.2 *Let $E \subset \mathbf{R}^N$ be a compact set, $f=(f_1,\cdots,f_d): E \to \mathbf{R}^d$ satisfy a uniform Hölder's condition of order $\alpha=(\alpha_1,\cdots,\alpha_d)$, that is,*

$$|f_j(x)-f_j(y)| \leqslant c|x-y|^{\alpha_j} \quad \text{for any } x,y \in E, \quad j=1,2,\cdots,d, \tag{5.2.7}$$

where $c>0$, $0<\alpha_j\leqslant 1$ $(j=1,\cdots,d)$ are constants. If

$$\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_d, \tag{5.2.8}$$

then

$$\dim f(E) \leqslant \min\left\{d; \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \leqslant j \leqslant d\right\}, \tag{5.2.9}$$

$$\dim \mathrm{Gr}f(E) \leqslant \min\left\{\frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\right.$$
$$\left. 1 \leqslant j \leqslant d; \dim E + \sum_{i=1}^{d}(1-\alpha_i)\right\}. \tag{5.2.10}$$

Proof Clearly $\dim f(E) \leqslant d$, $\dim f(E) \leqslant \dim \mathrm{Gr} f(E)$, we only need to prove (5.2.10). Take any $\gamma > \dim E$. Then by (5.2.2), for each $\delta>0$, there exists a sequence of balls $\{B_l\}$ with $\mathrm{diam} B_l \leqslant \delta$, such that

$$E \subset \bigcup_l B_l, \quad \sum_l (\mathrm{diam} B_l)^\gamma < \infty. \tag{5.2.11}$$

By (5.2.7), each $f(B_l)$ can be covered by a rectangle C_l of sides $c(\mathrm{diam}\ B_l)^{\alpha_i}$ $(i=1,\cdots,d)$. For each fixed $1\leqslant j\leqslant d$, C_l can be covered by $O((\mathrm{diam}\ B_l)^{\sum_{i=1}^{j}(\alpha_i-\alpha_j)})$ cubes C_{lk} of edge $(\mathrm{diam}\ B_l)^{\alpha_j}$. Since

$$\mathrm{Gr}f(E) \subset \bigcup_l \bigcup_k (B_l \times C_{lk}), \quad \mathrm{diam}(B_l \times C_{lk}) \leqslant c(\mathrm{diam}\ B_l)^{\alpha_j}$$

by (5.2.11), we have

$$\sum_{l,k} (\mathrm{diam}(B_l \times C_{lk}))^{(\gamma + \sum_{i=1}^{j}(\alpha_j-\alpha_i))/\alpha_j}$$
$$\leqslant c\sum_l (\mathrm{diam}\ B_l)^{\sum_{i=1}^{j}(\alpha_j-\alpha_i)}(\mathrm{diam}\ B_l)^{\gamma+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}$$
$$= c\sum_l (\mathrm{diam}\ B_l)^\gamma < \infty.$$

This proves that

$$\dim \mathrm{Gr}\ f(E)$$
$$\leqslant \min\left\{\frac{\dim E + \sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j}, j=1,\cdots,d\right\}. \tag{5.2.12}$$

On the other hand, each rectangle C_l can be covered by $O((\operatorname{diam} B_l)^{\sum_{i=1}^{d}(\alpha_i-1)})$ cubes C'_{lk} of edge diam B_l, and Gr $f(E)\subset \bigcup_l \bigcup_k (B_l\times C'_{lk})$,

$$\sum_{l,k}(\operatorname{diam}(B_l\times C'_{lk}))^{\gamma+\sum_{i=1}^{j}(1-\alpha_i)}$$
$$\leqslant c\sum_l(\operatorname{diam} B_l)^{\gamma}<\infty.$$

Hence,

$$\dim \operatorname{Gr} f(E)\leqslant \dim E+\sum_{i=1}^{d}(1-\alpha_i). \qquad (5.2.13)$$

From (5.2.12) and (5.2.13) we prove (5.2.9) and (5.2.10).

For the packing dimension, we have a similar lemma:

Lemma 5.2.3 *Under the conditions of Lemma 5.2.2, we have*

$$\operatorname{Dim} f(E)\leqslant \min\left\{d;\frac{\operatorname{Dim} E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\leqslant j\leqslant d\right\}, \qquad (5.2.14)$$

$$\operatorname{Dim} \operatorname{Gr} f(E)\leqslant \min\left\{\frac{\operatorname{Dim} E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},\ 1\leqslant j\leqslant d;\operatorname{Dim} E+\sum_{i=1}^{d}(1-\alpha_i)\right\}. \qquad (5.2.15)$$

Proof We need only to show (5.2.15). Let $N(\varepsilon,E)$ be the smallest number of cubes with edge ε that cover E. It is easily seen that in the definitions of $\delta(\cdot)$ and $\Delta(\cdot)$, we can replace $M(\varepsilon,E)$ by $N(\varepsilon,E)$. Take any $\gamma>\operatorname{Dim} E$. By (5.2.4), there exists $\{E_l\}$ with $E\subset\bigcup E_l$ and $\sup\Delta(E_l)<\gamma$. For fixed l, by the definition of $\Delta(\cdot)$, there exists $\varepsilon_l>0$ such that for any $0<\varepsilon<\varepsilon_l$, $N(\varepsilon,E_l)<\varepsilon^{-\gamma}$. It follows that there exist cubes $B_{lm}, m=1,\cdots,N(\varepsilon,E_l)$, of edge ε which cover E_l. By (5.2.7), each $f(B_{lm})$ can be covered by a rectangle C_{lm} of sides $c\varepsilon^{\alpha_i}(i=1,\cdots,d)$, where c is

the constant such that (5.2.7) holds. For each fixed $1\leqslant j\leqslant d$, C_{lm} can be covered by $O\left(\varepsilon^{\sum_{i=1}^{j}(\alpha_j-\alpha_i)}\right)$ cubes C_{lmk} of edge ε^{α_j}. Let

$$\overline{E}_l=\bigcup_m\bigcup_k(B_{lm}\times C_{lmk}).$$

Then

$$\operatorname{Gr} f(E)\subset\bigcup_l\overline{E}_l$$

and

$$N(c\varepsilon^{\alpha_j},\overline{E}_l)<\varepsilon^{-\gamma}\cdot O\left(\varepsilon^{\sum_{i=1}^{j}(\alpha_j-\alpha_i)}\right),$$

for $\varepsilon<\varepsilon_l$. It follows that

$$\Delta(\overline{E}_l)=\limsup_{\varepsilon\to0}\frac{\log N(c\varepsilon^{\alpha_j},\overline{E}_l)}{-\log c\varepsilon^{\alpha_j}}\leqslant\frac{\gamma+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j}.$$

By (5.2.4) again, we have

$$\dim \operatorname{Gr} f(E)\leqslant \min\left\{\frac{\dim E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},j=1,\cdots,d\right\}. \qquad (5.2.16)$$

On the other hand, each rectangle C_{lm} can be covered by $O\left(\varepsilon^{\sum_{i=1}^{d}(\alpha_i-1)}\right)$ cubes C'_{lm} of edge ε, and

$$\operatorname{Gr} f(E)\subset\bigcup_l\overline{E}'_l,$$

where

$$\overline{E}'_l=\bigcup_m\bigcup_k(B_{lm}\times C'_{lmk}).$$

It follows that

$$N(\varepsilon,\overline{E}'_l)<\varepsilon^{-\gamma}\cdot O\left(\varepsilon^{\sum_{i=1}^{d}(\alpha_i-1)}\right),$$

which implies

$$\Delta(\overline{E}'_l)\leqslant\gamma+\sum_{i=1}^{d}(1-\alpha_i)$$

by (5.2.4). This proves

$$\dim \operatorname{Gr} f(E)\leqslant \operatorname{Dim} E+\sum_{i=1}^{d}(1-\alpha_i). \qquad (5.2.17)$$

From (5.2.16) and (5.2.17) we obtain (5.2.15).

Remark 5.2.1 From the proofs of Lemmas 5.2.1 and 5.2.2, it is easy to show that if f satisfies a uniform Hölder's condition of every order $\beta<\alpha$, that is, for every $\beta=(\beta_1,\cdots,\beta_d)$ with $0<\beta_j<\alpha_j$, $j=1,\cdots,d$ (5.2.7) holds, then (5.2.9), (5.2.10), (5.2.14) and (5.2.15) are still valid.

By a simple calculation, we can prove the following lemma.

Lemma 5.2.4 *If* $0=\alpha_0<\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_d$ *and* $\sum_{i=0}^{k-1}\alpha_i<\dim E\leqslant\sum_{i=1}^{k}\alpha_i$, *then*

$$\lambda:=\min\left\{d;\frac{\dim E+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},j=1,\cdots,d\right\}$$
$$=\frac{\dim E+\sum_{i=1}^{k}(\alpha_k-\alpha_i)}{\alpha_k}$$

and $k-1<\lambda\leqslant k$. *If* $\dim E\geqslant\sum_{i=1}^{d}\alpha_i$, *then* $\lambda=d$.

Now we turn to the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1 Fix j. For any $0<\gamma<\alpha_j$, by the definition we have $\lim_{|t|\to0}|t|^{-\gamma}\sigma_j(t)=0$, so there exists c_0 such that $\sigma_j(t)\leqslant c_0|t|^\gamma$ for $|t|\leqslant1$, which implies $\sup_{d(s,t)\leqslant h}\sigma_j(t-s)\leqslant c_0h^\gamma$, where $d(s,t)=\max_{1\leqslant i\leqslant N}|t_i-s_i|$. It follows from (2.1.5) that

$$E\sup_{d(s,t)\leqslant h}|X_j(t)-X_j(s)|\leqslant Kh^\gamma(\log(1/h))^{1/2}.$$

Hence

$$E\sup_{|s-t|\leqslant h}|X_j(t)-X_j(s)|\leqslant Kh^\gamma(\log(1/h))^{1/2}.$$

By Theorem 1.1.1 (Borell's inequality), we have

$$P\{\sup_{|t-s|\leqslant h}|X_j(t)-X_j(s)|\geqslant Kh^\gamma(\log(1/h))^{1/2}+\lambda\}$$
$$\leqslant2\exp\left(-c\frac{\lambda^2}{h^{2\gamma}}\right).$$

With this inequality, along the lines of the proof of Theorem 2.2.1, we can get that

$$\lim_{h\to0}\frac{\sup_{|t-s|\leqslant h}|X_j(t)-X_j(s)|}{h^\gamma\log(1/h)}=0\quad\text{a.s.}$$

By the arbitrariness of $0<\gamma<\alpha_i$, it follows that for any $0<\beta_j<\alpha_j$, with probability one there exists a positive constant c such that

$$|X_j(t)-X_j(s)|\leqslant c|t-s|^{\beta_j},\text{ for any }t,s\in E,j=1,\cdots,d.$$

Then by Lemma 5.2.2 and Remark 5.2.1, we conclude that the right-hand sides of (5.2.5) and (5.2.6) serve as a.s. upper bounds to dim $X(E)$ and dim Gr $X(E)$, respectively.

Now we show that the right-hand sides of (5.2.5) and (5.2.6) also serve as lower bounds almost surely. Consider $\dim X(E)$. If $\dim E=0$, there is nothing to prove. We assume $\sum_{i=0}^{k-1}\alpha_i<\dim E\leqslant\sum_{i=1}^{k}\alpha_i$. Then by Lemma 5.2.4, $k-1<\lambda\leqslant k$. It is sufficient to show that for any $k-1<\gamma<\lambda$, there exists a positive measure σ on E, such that

$$\int_E\int_E E(|X(t)-X(s)|^{-\gamma})\sigma(dt)\sigma(ds)<\infty.\quad(5.2.18)$$

In fact if (5.2.18) holds, we can assume that $\sigma(E)=1$. By the Fubini Theorem, with probability 1,

$$\int_E\int_E\frac{1}{|X(t)-X(s)|^\gamma}\sigma(dt)\sigma(ds)<\infty.$$

Let $\mu=\sigma X(\cdot,\omega)^{-1}$, i.e. $\mu(A)=\sigma(X(t,\omega)\in A)$ for $A\subset\mathbf{R}^d$. Then $\mu(X(E,\omega))=\sigma(E)=1$ and

$$\int_{X(E,\omega)}\int_{X(E,\omega)}\frac{1}{|x-y|^\gamma}\mu(dx)\mu(dy)<\infty,$$

which together with Lemma 5.2.1 yields that with probability 1,

$$\dim X(E,\omega) \geqslant \gamma.$$

Now let

$$Y_j(t,s) = \frac{X_j(t) - X_j(s)}{\sigma_j(t-s)}, \quad j = 1,\cdots,d.$$

By (5.2.1), we have $\det\,(\mathrm{Cov}(Y(t,s))) \geqslant \varepsilon$. Also,

$$E|X(t) - X(s)|^{-\gamma}$$

$$= \int_{\mathbf{R}^d} \Big(\sum_{i=1}^{d} (x_i\sigma_i(t-s))^2 \Big)^{-\gamma/2}$$

$$\times \frac{1}{(2\pi)^{d/2}\sqrt{\det \mathrm{Cov}(Y)}} \exp\Big\{ -\frac{1}{2} X\mathrm{Cov}(Y)^{-1}X' \Big\} \mathrm{d}x_1\cdots\mathrm{d}x_d, \tag{5.2.19}$$

where X' is the transpose of $X=(x_1,\cdots,x_d)$. Take $\beta_j (j=1,\cdots,d)$ with $\beta_j > \alpha_j$ for $\alpha_j < 1, \beta_j = 1$ when $\alpha_j = 1$ and

$$\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_d,$$

$$\gamma\beta_k < \dim E + \sum_{i=1}^{d} (\beta_k - \beta_i). \tag{5.2.20}$$

Then there exist $\delta > 0$, $c_1 > 0$ such that for $|t| < \delta$,

$$\sigma_j(t) \geqslant c_1 |t|^{\beta_j} \quad (j = 1,\cdots,d). \tag{5.2.21}$$

It is known that for any $d \times d$ real symmetric matrix B, there is a constant $c_2 > 0$, such that for every $X \in \mathbf{R}^d$

$$|XBX'| \leqslant c_2 XX'. \tag{5.2.22}$$

Since $\mathrm{Cov}(Y)$ is positive definite with rank d, there is a $d\times d$ invertible matrix A such that $\mathrm{Cov}(Y)^{-1} = AA'$. Then by (5.2.22) with $B = A^{-1}(A')^{-1}$, we have

$$X\mathrm{Cov}(Y)^{-1}X' \geqslant \frac{1}{c_2} XX'. \tag{5.2.23}$$

By (5.2.19)—(5.2.23), we have for $|t-s| < \delta$,

$$E(|X(t) - X(s)|^{-\gamma})$$

$$\leqslant c\int_{\mathbf{R}^d} \Big(\sum_{j=1}^{d} (x_j|t-s|^{\beta_j})^2 \Big)^{-\gamma/2}$$

$$\times \exp\{-(x_1^2 + \cdots + x_d^2)/2c_2\} \mathrm{d}x_1\cdots\mathrm{d}x_d$$

$$= c|t-s|^{-\gamma\beta_1} \int_{\mathbf{R}^d} (x_1^2 + (x_2|t-s|^{\beta_2-\beta_1})^2 + \cdots$$

$$+ (x_d|t-s|^{\beta_d-\beta_1})^2)^{-\gamma/2}$$

$$\times \exp\{-(x_1^2 + \cdots + x_d^2)/2c_2\} \mathrm{d}x_1\cdots\mathrm{d}x_d. \tag{5.2.24}$$

The integral in (5.2.24) is convergent since $\gamma < d$. Using the fact that

$$\int_0^\infty (y^2 + a^2)^{-\gamma/2} \mathrm{d}y = c_1(\gamma) a^{-\gamma+1} \quad \text{for } \gamma > 1, \tag{5.2.25}$$

$$\int_0^\infty (y^2 + a^2)^{-\gamma/2} \exp\{-y^p\} \mathrm{d}y$$

$$\sim c_2(\gamma) a^{-\gamma+1} + c_3(\gamma), \quad \text{for } 0 < \gamma < 1, p > 0, \tag{5.2.26}$$

where $c_1(\gamma), c_2(\gamma)$ and $c_3(\gamma)$ are positive constants depending only on γ. We first integrate out x_1 to obtain that (5.2.24) is less than a constant times

$$|t-s|^{-\gamma\beta_1} \int_{\mathbf{R}^{d-1}} \big((x_2|t-s|^{\beta_2-\beta_1})^2 + \cdots + (x_d|t-s|^{\beta_d-\beta_1})^2 \big)^{-(\gamma-1)/2}$$

$$\times \exp\Big(-\frac{x_2^2 + \cdots + x_d^2}{2c_2} \Big) \mathrm{d}x_2\cdots\mathrm{d}x_d, \tag{5.2.27}$$

hence repeating this argument for $\mathrm{d}x_2,\cdots,\mathrm{d}x_{k-1}$, we find that (5.2.27) is less than a constant times

$$|t-s|^{-\gamma\beta_1 - (\gamma-1)(\beta_2-\beta_1) - \cdots - (\gamma-k+1)(\beta_k-\beta_{k-1})}$$

$$\times \int_{\mathbf{R}^{d-k+1}} \Big(x_k^2 + \sum_{i=k+1}^{d} (x_i|t-s|^{\beta_i-\beta_k})^2 \Big)^{-(\gamma-k+1)/2}$$

$$\times \exp\Big\{ \frac{-\sum_{i=k}^{d} x_i^2}{2c_2} \Big\} \mathrm{d}x_k\cdots\mathrm{d}x_d$$

$$\leqslant c|t-s|^{-\gamma\beta_1 - (\gamma-1)(\beta_2-\beta_1) - \cdots - (\gamma-k+1)(\beta_k-\beta_{k-1})}$$

$$\times \int_{\mathbf{R}^{d-k}} \Big(\Big(\sum_{i=k+1}^{d} (x_i |t-s|^{\beta_i-\beta_k})^2 \Big)^{-(\gamma-k)/2} + c(\gamma) \Big)$$

$$\times \exp\left\{ \frac{-\sum_{i=k+1}^{d} x_i^2}{2c_2} \right\} \mathrm{d}x_{k+1} \cdots \mathrm{d}x_d$$

$$\leqslant c|t-s|^{-\gamma\beta_k + \sum_{i=1}^{k}(\beta_k-\beta_i)}.$$

Since $\gamma\beta_k - \sum_{i=1}^{k}(\beta_k - \beta_i) < \dim E$, by the second part of Lemma 5.2.1 there is a positive measure σ on E with

$$\int_E\int_E \frac{\sigma(\mathrm{d}t)\sigma(\mathrm{d}s)}{|t-s|^{\gamma\beta_k - \sum_{i=1}^{k}(\beta_k-\beta_i)}} < \infty.$$

Thus we have (5.2.18). If $\dim E > \sum_{i=1}^{d}\alpha_i$, the same computation shows that (5.2.18) holds for any $\gamma < d$. Therefore, (5.2.5) holds.

To find $\dim \mathrm{Gr}\, X(E)$, we first consider the case $\dim X(E) < d$. Since $\dim \mathrm{Gr} X(E) \geqslant \dim X(E)$, it follows from Lemma 5.2.2 and (5.2.5) that $\dim \mathrm{Gr}\, X(E) = \dim X(E)$ almost surely. In the case of $\dim X(E) = d$, which implies that $\dim E \geqslant \sum_{i=1}^{d}\alpha_i$, the dimension of the graph can be larger. Similar manipulation as above shows that for any $d < \gamma < \dim E + \sum_{i=1}^{d}(1-\alpha_i)$, there exists a positive measure μ on E such that

$$\int_E\int_E E\big((|t-s|^{\alpha} + |t-s|^2 + |X(t)-X(s)|^2)^{-\gamma/2} \big) \mu(\mathrm{d}t)\mu(\mathrm{d}s) < \infty.$$

Therefore,

$$\dim \mathrm{Gr}\, X(E) \geqslant \dim E + \sum_{i=1}^{d}(1-\alpha_i) \quad \text{a. s.}$$

This completes the proof of (5.2.6).

If $E = [0,1]^N$, then by Lemma 5.2.3 and (5.2.3) we have

the following theorem.

Theorem 5.2.2 *Let $X(t)$ be an (N,d) Gaussian field as in Theorem 5.2.1. If (5.2.1) holds for all $s,t \in [0,1]^N$, then with probability 1,*

$$\dim X([0,1]^N) = \mathrm{Dim} X([0,1]^N) = \min\left\{ d; \frac{N + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \leqslant j \leqslant d \right\},$$

$$\dim \mathrm{Gr} X([0,1]^N) = \mathrm{Dim}\ \mathrm{Gr}\ X([0,1]^N) = \min\left\{ N + \sum_{i=1}^{d}(1-\alpha_i); \frac{N + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \leqslant j \leqslant d \right\}.$$

Remark 5.2.2 Obviously, Theorem 5.2.2 remains true when $[0,1]^N$ is replaced by any compact set E with non-empty interior.

If $X(t)$ is a Gaussian process from $\mathbf{R}^N$ to $\mathbf{R}^d$ such that

$$E(|X(t) - X(s)|^2) = d|t-s|^{2\alpha},$$

where $0 < \alpha < 1$, then it is an (N,d) Gaussian field of index $(\alpha_1, \cdots, \alpha_d)$. We call this process the (N,d,α) Gaussian process. The $\mathbf{R}^d$-valued Wiener process is the $(1,d,1/2)$ Gaussian process. The $\mathbf{R}^d$-valued fractional Wiener process is the $(1,d,\alpha)$ Gaussian process. It should be clear that the components of the (N,d,α) Gaussian process $X(t)$, which are processes from $\mathbf{R}^N$ to $\mathbf{R}$, are independent copies of the $(N,1,\alpha)$ Gaussian process. It follows that $X(t)$ satisfies (5.2.1). So, we have the following corollary.

Corollary 5.2.1 *Let $X(T)$ be an (N,d,α) Gaussian process, $E \subset \mathbf{R}^N$ a compact set. Then with probability 1,*

$$\dim X(E) = \min\left\{ d, \frac{\dim E}{\alpha} \right\},$$

$$\dim GrX(E) = \min\left\{ \frac{\dim E}{\alpha}, \dim E + d(1-\alpha) \right\},$$

$$\dim X([0,1]^N) = \mathrm{Dim} X([0,1]^N) = \min\{d, N/\alpha\},$$
$$\dim \mathrm{Gr}X([0,1]^N) = \mathrm{Dim}\,\mathrm{Gr}X([0,1]^N)$$
$$= \min\left\{\frac{N}{\alpha}, N + d(1-\alpha)\right\}.$$

Remark 5. 2. 3 Talagrand (1995) showed that if $X(t)$ is an (N,d,α) Gaussian process with $N<\alpha d$, then with probability 1,
$$0 < \varphi - \mathrm{mes}\{X(t); |t| \leqslant a\} < \infty,$$
where $a>0$ and $\varphi(h)=h^{N/\alpha}\log\log h^{-1}$.

5. 3 The Fractal Nature of Increments of l^p-valued Gaussian Processes

Let $\{W(t);t\geqslant 0\}$ be a standard Wiener process. By the law of the iterated logarithm,
$$\limsup_{h\to 0} \frac{|W(t+h)-W(t)|}{(2h\log\log h^{-1})^{1/2}} = 1 \quad \text{a. s.} \ \forall\, t\in[0,1]. \tag{5. 3. 1}$$
On the other hand, by Lévy's moduli of continuity (Theorem 0. 1), we have
$$\lim_{h\to 0}\sup_{t\in[a,b]} \frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}} = 1 \quad \text{a. s.} \ \forall\, 0\leqslant a\leqslant b\leqslant 1.$$
It can also be shown that
$$\sup_{t\in[a,b]}\limsup_{h\to 0} \frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}} = 1 \quad \text{a. s.} \ \forall\, 0\leqslant a\leqslant b\leqslant 1. \tag{5. 3. 2}$$
In fact, we can assume that $a=0$, $b=1$ without loss of generali-

ty. Let $h_n=2^{-n}[(\log n)^3]$ and
$$Y_{t,h} = \frac{W(t+h)-W(t)}{(2h\log h^{-1})^{1/2}}.$$
Then for any $0<\varepsilon<1$,
$$P\{\max_{0\leqslant k\leqslant 2^n-[(\log n)^3]} Y_{k2^{-n},h_n} \leqslant 1-\varepsilon\}$$
$$\leqslant P\{\max_{0\leqslant m\leqslant 2^n/[(\log n)^3]} Y_{mh_n,h_n} \leqslant 1-\varepsilon\}$$
$$\leqslant \prod_{m=0}^{2^n/[(\log n)^3]} P\{N(0,1)\leqslant (1-\varepsilon)(2\log h_n^{-1})^{1/2}\}$$
$$\leqslant (1-ce^{-(1-\varepsilon)\log h_n^{-1}})^{h_n}$$
$$\leqslant \exp(-ch_n^{-\varepsilon}) \to 0, \quad n\to\infty.$$
Hence
$$\limsup_{n\to\infty} \max_{0\leqslant k\leqslant 2^n-[(\log n)^3]} Y_{k2^{-n},h_n} \geqslant 1-\varepsilon \quad \text{a. s.}$$
By letting
$$J_{k,n}(\omega) = \begin{cases}[k2^{-n},(k+1)2^{-n}) & \text{if } Y_{k2^{-n},h_n}(\omega)\geqslant 1-\varepsilon,\\ \varnothing & \text{otherwise},\end{cases}$$
$$A_n(\omega) = \bigcup_{k=0}^{2^n-[(\log n)^3]} J_{k,n}(\omega),$$
it follows that $P\{A_n(\omega)\neq\varnothing, \text{i,o.}\}=1$, i. e. ,
$$P\{\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n(\omega)\neq\varnothing\} = 1.$$
Hence there exists $t\in[0,1]$ with probability one such that
$$t\in \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n(\omega) = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\bigcup_{k=0}^{2^n-[(\log n)^3]} J_{k,n}(\omega),$$
which means that for infinite many n's, there exists k such that
$$t\in[k2^{-n},(k+1)2^{-n}) \quad\text{and}\quad Y_{k2^{-n},h_n}(\omega)\geqslant 1-\varepsilon.$$
Thus
$$Y_{[t2^n]2^{-n},h_n}(\omega) \geqslant 1-\varepsilon.$$
It follows that

$$\sup_{t\in[0,1]}\limsup_{n\to\infty} Y_{[t2^n]2^{-n},h_n}\geqslant 1 \quad \text{a. s.}$$

Also, by the moduli of continuity, we have

$$\max_{t\in[0,1]}\limsup_{n\to\infty}|Y_{[t2^n]2^{-n},h_n}-Y_{t,h_n}|$$

$$\leqslant 2\limsup_{n\to\infty}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant 2^{-n}}\frac{|W(t+s)-W(t)|}{(2h_n\log h_n^{-1})^{1/2}}=0 \quad \text{a. s.}$$

It follows that

$$\sup_{t\in[0,1]}\limsup_{n\to\infty} Y_{t,h_n}\geqslant 1 \quad \text{a. s.}$$

So,

$$\sup_{t\in[0,1]}\limsup_{h\to 0} Y_{t,h}\geqslant 1 \quad \text{a. s.}$$

On the other hand, it is clear that

$$\sup_{t\in[0,1]}\limsup_{h\to 0}|Y_{t,h}|\leqslant\limsup_{h\to 0}\sup_{t\in[0,1]}|Y_{t,h}|\leqslant 1 \quad \text{a. s.}$$

Hence (5. 3. 2) is proved.

Now by (5. 3. 2), it follows that for any $0<\alpha<1$ there exists $t\in[a,b]$ with probability one such that

$$\limsup_{h\to 0}\frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}}\geqslant\alpha.$$

By the arbitrariness of a and b , it follows that there are many (at least a countable number of) such t's in $[0,1]$. On the other hand, by (5. 3. 1) and Fubini's theorem, the Lebesgue measure of the set of such t's is zero. So, it is nature to ask how many such t's are and to study the property of the set of such t's. Orey and Taylor (1974) studied the fractal nature of the set

$$B(\alpha):=\left\{t\in[0,1];\limsup_{h\to 0}\frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}}\geqslant\alpha\right\}$$

$(0\leqslant\alpha\leqslant 1)$. They showed that for each $0\leqslant\alpha\leqslant 1$, $B(\alpha)$ is a random fractal and proved:

Theorem 5. 3. 1. *For any $\alpha\in[0,1]$ we have almost surely*

$$\dim B(\alpha)=1-\alpha^2. \tag{5. 3. 3}$$

This theorem tells us that for any $0\leqslant\alpha\leqslant 1$, the t's in $B(\alpha)$ are nearly as many as those in $[0,1]$.

As we have motioned, Theorem 5. 3. 1 is corresponding to the law of the iterated logarithm and the Lévy moduli of continuity of $W(t)$. In Chapters 2 and 3, it has been shown that a lot of Gaussian processes have moduli of continuity similar to that of the Wiener process $W(t)$. For example, in Chapter 3 we studied the increments of l^p Gaussian processes, and, Theorems 3. 3. 3, 3. 3. 4 provided the moduli of continuity for l^p-valued Gaussian processes. What about the fractal nature of the type (5. 3. 3)? This section will give the answer. Also, Deheuvels and Mason (1994, 1995) studied the fractal nature for empirical increments and processes with independent increments. But their methods are not effective for studying Gaussian processes with dependent increments.

Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent Gaussian processes with $EX_k(t)=0$ and stationary increments $\sigma_k^2(h)=E(X_k(t+h)-X(t))^2$, where, and throughout this section, $\sigma_k(h)$ is assumed to be a non-decreasing continuous function for each $k\geqslant 1$, as usual. Recall

$$\sigma(p,h)=\Big(\sum_{k=1}^{\infty}\sigma_k^p(h)\Big)^{1/p},\quad \sigma^*(h)=\max_{k\geqslant 1}\sigma_k(h),$$

$$\tilde{\sigma}(p,h)=\begin{cases}\sigma\left(\dfrac{2p}{2-p},h\right) & \text{if } 1\leqslant p<2,\\ \sigma^*(h) & \text{if } p\geqslant 2,\end{cases}$$

$$\delta_p^p=E|N(0,1)|^p,\quad p\geqslant 1.$$

If the conditions in Theorem 3. 3. 4 are satisfied, then we

have

$$\begin{aligned}
&\limsup_{h\to 0}\inf_{0\leqslant t\leqslant 1}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\\
&\geqslant\limsup_{h\to 0}\min_{0\leqslant n\leqslant h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}\\
&\quad-2\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 2}\sup_{0\leqslant s\leqslant h^2}\frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h^2)}\frac{\sigma(p,h^2)}{\sigma(p,h)}\\
&=\limsup_{h\to 0}\min_{0\leqslant n\leqslant h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}.
\end{aligned}$$

Similar to the proof of (3. 3. 33), for any $1<\theta<2$ we have that for h sufficiently small,

$$\begin{aligned}
&P\left\{\min_{0\leqslant n\leqslant h^{-2}}\frac{\|Y(nh^2+h)-Y(nh^2)\|_{l^p}}{\delta_p\sigma(p,h)}\leqslant 2-\theta\right\}\\
&\leqslant 2(h^{-2}+1)\exp\left(-\frac{(\theta-1)^2\delta_p^2\sigma^2(p,h)}{8\tilde{\sigma}^2(p,h)}\right)\\
&\leqslant 4h^{-2}\exp\left(-4\log\frac{1}{h}\right)=4h^2\to 0\quad\text{as } h\to 0.
\end{aligned}$$

It follows that

$$\limsup_{h\to 0}\inf_{0\leqslant t\leqslant 1}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\geqslant 1 \text{ a. s.}$$

Hence, if we define a random set similar to $B(\alpha)$, by

$$E(\alpha)=\left\{t\in[0,1];\ \limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\delta_p\sigma(p,h)}\geqslant\alpha\right\},\ 0\leqslant\alpha\leqslant 1,$$

then $E(\alpha)=[0,1]$ a. s. for any $0\leqslant\alpha\leqslant 1$. So, in this case there is nothing for us to consider on the fractal nature of $E(\alpha)$.

Now, we suppose the conditions in Theorem 3. 3. 3 are satisfied and define a random set similar to $B(\alpha)$, by

$$E(\alpha)=\left\{t\in[0,1];\ \limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}}\geqslant\alpha\right\},\ 0\leqslant\alpha\leqslant 1.$$

The following result on the Hausdorff dimension of $E(\alpha)$ is due to Zhang (1997d).

Theorem 5. 3. 2 *Assume that $\tilde{\sigma}(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that*

$$\sigma(p,h)=o(\tilde{\sigma}(p,h)(\log h^{-1})^{1/2})\quad\text{as } h\to 0,\tag{5. 3. 4}$$

and that

$$\limsup_{h\to 0}\max_{h^{-\varepsilon}\leqslant j\leqslant h^{-1}}\max_{k\geqslant 1}\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{(\log h^{-1})^{-2}\sigma_k^2(h)}\leqslant 0,\tag{5. 3. 5}$$

for each $\varepsilon>0$. Then for any $\alpha\in[0,1]$, we have almost surely

$$\dim E(\alpha)=1-\alpha^2.\tag{5. 3. 6}$$

Also, for the random set

$$E^*(\alpha)=\left\{t\in[0,1];\limsup_{h\to 0}\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}}=\alpha\right\},\quad 0\leqslant\alpha\leqslant 1\tag{5. 3. 7}$$

we have the following result.

Theorem 5. 3. 3 *Assume that $\tilde{\sigma}(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that* (5. 3. 4) *holds and*

$$\limsup_{h\to 0}\max_{(\log h^{-1})^{p}\leqslant j\leqslant h^{-1}}\max_{k\geqslant 1}\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{(\log h^{-1})^{-2}\sigma_k^2(h)}\leqslant 0,\tag{5. 3. 8}$$

for some $p>0$. Then for any $\alpha\in[0,1]$, we have almost surely

$$\dim E^*(\alpha)=1-\alpha^2.\tag{5. 3. 9}$$

To prove the results, we first state three lemmas required in the proof. The first one is due to Orey and Taylor (1974).

Lemma 5. 3. 1 *Suppose that φ: $[0,1]\to[0,\infty)$ is a continuous function with $\varphi(0)=0$. Let $K\subset[0,1]$ be such that $K=\bigcap_{m=1}^{\infty}E_m$, where $E_1\supset\cdots\supset E_m\cdots$ and $E_m=\bigcup_{k=1}^{M_m}I_{m,k}$ with $\{I_{m,k};\ 1\leqslant k\leqslant M_m\}$ being, for each $m\geqslant 1$, a collection of disjoint closed subintervals of*

[0,1]. *Then, if there exist two constants $\Delta>0$ and $d>0$ such that, for every interval $I\subset[0,1]$ with $|I|\leqslant\Delta$ there is a constant $m(I)$ such that for all $m\geqslant m(I)$,*

$$M_m(I):=\mathrm{Card}\{I_{m,k}\subset I;1\leqslant k\leqslant M_m\}\leqslant d\varphi(|I|)M_m, \quad (5.3.10)$$

we have φ-mes$(K)>0$. Here $|I|$ denotes the Lebesgue measure of I.

Proof Since K is closed and φ is continuous, it is suffcient to consider only finite covers of K by disjoint open intervals of length at most Δ. Let J_i, $i=1,\cdots,n$ be any such cover. Then for $m\geqslant\max m(J_i)$ we have

$$M_m(J_i)\leqslant d\varphi(|J_i|)M_m \quad (i=1,2,\cdots,n). \quad (5.3.11)$$

Further, since $J=\bigcup J_i$ is open and K is compact, there exists $m_0(J)\geqslant\max m(J)$ such that $E_m\subset J$ for $m\geqslant m_0(J)$, so that

$$\sum_{i=1}^n M_m(J_i)=M_m \quad (m\geqslant m_0(J)).$$

By summation, (5.3.11) now yields $\sum_i\varphi(|J_i|)\geqslant d^{-1}$, which implies that φ-mes$K\geqslant d^{-1}>0$ by noting that

$$\varphi\text{-mes}B=\lim_{h\to0}\Big(\inf\Big\{\sum_i\varphi(|I_i|);B\subset\bigcup_i I_i,|I_i|\leqslant h\Big\}\Big).$$

Lemma 5.3.2 *Assume that $\tilde\sigma(p,h)/h^\Lambda$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$ and (5.3.4) holds. Then for any $0<\alpha_2<\alpha_1$, there exists $\delta=\delta(\alpha_1,\alpha_2)$ such that*

$$P\{\sup_{a\leqslant s<t\leqslant a+h}\|Y(t)-Y(s)\|_{l^p}>\alpha_1\tilde\sigma(p,h)(2\log h^{-1})^{1/2}\}<h^{\alpha_2}$$

for all $a\geqslant0$ and $0<h<\delta$.

Proof For any fixed $\varepsilon>0$, put

$$\sigma_*(p,h)=\varepsilon\sup_{0\leqslant s\leqslant h}\tilde\sigma(p,h)(\log s^{-1})^{1/2},\ 0<h\leqslant1.$$

By Lemma 3.1.3, Lemma 3.2.1, Remark 3.3.1 and (3.3.6), there exist $C=C(\varepsilon,\Lambda)$, $h_0=h_0(\varepsilon,\Lambda)$ and a constant c_0 independent of ε such that

$$\begin{aligned}&P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|Y(t+s)-Y(t)\|_{l^p}\geqslant\\&\qquad x\tilde\sigma(p,h)+(1+\varepsilon)\varepsilon c_0\tilde\sigma(p,h)(\log h^{-1})^{1/2}\}\\&\leqslant P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|Y(t+s)-Y(t)\|_{l^p}\geqslant\\&\qquad x\tilde\sigma(p,h)+(1+\varepsilon)\sigma_*(p,h)\}\\&\leqslant C\Big(\frac{T}{h}+1\Big)\exp\Big(-\frac{x^2}{2(1+\varepsilon)}\Big)\end{aligned}$$

for every $x\geqslant1$, $T\geqslant0$ and $0<h\leqslant h_0$. And then the proof is completed immediately.

Lemma 5.3.3 *Suppose that $\{\xi_i;i=1,\cdots,n\}$ are mean zero Gaussian variables with $E\xi_i^2=1$ and $E\xi_i\xi_j\leqslant\beta^2(j\neq i)$, where $0\leqslant\beta<1$. Then for any $\varepsilon>0$, $\lambda>0$, $0<\theta\leqslant1/2$ and all t, we have*

$$\begin{aligned}&P\Big\{\sum_{i=1}^n I\{\xi_i>t\}-np_0\geqslant\lambda np_0\Big\}\\&\leqslant\exp\{-\theta n((\lambda+1)p_0-(1+\theta)p_1)\}+2nP\Big\{N(0,1)>\frac{\varepsilon}{\beta}\Big\},\end{aligned} \quad (5.3.12)$$

$$\begin{aligned}&P\Big\{np_0-\sum_{i=1}^n I\{\xi_i>t\}\geqslant\lambda np_0\Big\}\\&\leqslant\exp\{-\theta n((1-\theta)p_2-(1-\lambda)p_0)\}+2nP\Big\{N(0,1)>\frac{\varepsilon}{\beta}\Big\},\end{aligned} \quad (5.3.13)$$

where $p_0=P\{N(0,1)>t\}$, $p_1=P\{N(0,1)>(t-\varepsilon)/(1-\beta^2)^{1/2}\}$ and $p_2=P\{N(0,1)>(t+\varepsilon)/(1-\beta^2)^{1/2}\}$.

Proof Let $\{\tau,\eta_i;i=1,\cdots,n\}$ are independent mean zero Gaussian variables with $E\tau^2=\beta^2$ and $E\eta_i^2=1-\beta^2$. Then $E(\tau+\eta_i)^2=E\xi_i^2=1$ and $E\xi_i\xi_j\leqslant E(\tau+\eta_i)(\tau+\eta_j)=\beta^2(i\neq j)$. Define

$$f(x)=\begin{cases}e^x, & \text{for } 0\leqslant x<m,\\ e^m(x-m+1), & \text{for } x\geqslant m,\end{cases}$$

where $m>0$ will be specified later on. It is easy to see that $f(x)\leqslant e^x$ for $x\geqslant 0$, $f(x)\leqslant ne^m$ for $0\leqslant x\leqslant n/2$ and $f(x)(x\geqslant 0)$ is an increasing convex function. It follows that

$$g(x_1,\cdots,x_n):=f\Big(\theta\sum_{i=1}^n I(x_i>t)\Big)\leqslant e^{\theta\sum_{i=1}^n I(x_i>t)},$$

$$g(x_1,\cdots,x_n)\leqslant ne^m$$

and g is a function on $\mathbf{R}^n$ such that its second derivatives in the sense of distribution satisfy

$$D_{ij}g=\theta^2\frac{d^2f}{dx^2}\cdot\frac{d}{dx_j}I(x_i>t)\cdot\frac{d}{dx_j}I(x_j>t)\geqslant 0\quad(i\neq j).$$

By Theorem 1.2.1 (the comparison property), we have

$$Eg(\xi_1,\cdots,\xi_n)\leqslant Eg(\tau+\eta_1,\cdots,\tau+\eta_n).$$

Choose $m=\theta(\lambda np_0+np_0)$, we conclude that

$$\begin{aligned}
&P\Big\{\sum_{i=1}^n I(\xi_i>t)-np_0\geqslant\lambda np_0\Big\}\\
&=P\Big\{f\Big(\theta\sum_{i=1}I(\xi_i>t)\Big)\geqslant f(m)\Big\}\\
&=P\{g(\xi_1,\cdots,\xi_m)\geqslant e^m\}\\
&=P\{g(\xi_1,\cdots,\xi_m)\geqslant e^{\theta(\lambda+1)np_0}\}\\
&\leqslant e^{-\theta(\lambda+1)np_0}Eg(\xi_1,\cdots,\xi_m)\\
&\leqslant e^{-\theta(\lambda+1)np_0}Eg(\tau+\eta_1,\cdots,\tau+\eta_m)\\
&\leqslant e^{-\theta(\lambda+1)np_0}\{Ee^{\theta\sum_{i=1}^n I(\tau+\eta_i>t)}I(|\tau|\leqslant\varepsilon)+ne^{\theta(\lambda+1)np_0}P\{|\tau|>\varepsilon\}\}\\
&\leqslant e^{-\theta(\lambda+1)np_0}Ee^{\theta\sum_{i=1}^n I(\eta_i>t-\varepsilon)}+2nP\{|\tau|>\varepsilon\}.
\end{aligned}$$

By the fact that $\{\eta_i;i=1,\cdots,n\}$ are independent, it is easy to see that

$$\begin{aligned}
&Ee^{\theta\sum_{i=1}^n I(\eta_i>t-\varepsilon)}\\
&=e^{\theta np_1}(Ee^{\theta(I(\eta_1>t-\varepsilon)-p_1)})^n\\
&\leqslant e^{\theta np_1}(1+p_1(1-p_1)\theta^2)^n\leqslant e^{\theta np_1+\theta^2np_1(1-p_1)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&P\Big\{\sum_{i=1}^n I(\xi_i>t)-np_0\geqslant\lambda np_0\Big\}\\
&\leqslant e^{-\theta(\lambda+1)np_0}e^{\theta^2np_1(1-p_1)+\theta np_1}+2nP\{\tau>\varepsilon\}\\
&\leqslant e^{-\theta n((\lambda+1)p_0-(1+\theta)p_1)}+2nP\{\tau>\varepsilon\},
\end{aligned}$$

which implies (5.3.12) immediately. Note that

$$\begin{aligned}
&P\Big\{np_0-\sum_{i=1}^n I(\xi_i>t)\geqslant\lambda np_0\Big\}\\
&=P\Big\{\sum_{i=1}^n I(\xi_i\leqslant t)-n(1-p_0)\geqslant\lambda np_0\Big\}.
\end{aligned}$$

If we choose $m=\theta(\lambda np_0+n(1-p_0))$ and define g by

$$g(x_1,\cdots,x_n)=f\Big(\theta\sum_{i=1}^n I(x_i\leqslant t)\Big),$$

we can obtain (5.3.13) similarly.

Proof of Theorem 5.3.2 We first show that dim $E(\alpha_0)\leqslant 1-\alpha_0^2$ a.s. We may assume that $\alpha_0>0$ as otherwise there is nothing to prove. For $I=[a,b]$, let

$$R(I)=\sup_{a\leqslant s<t\leqslant b}\|Y(t)-Y(s)\|_{l^p}.$$

Take $0<\alpha_2<\alpha_1<\alpha_0$ and consider the collection $\mathscr{S}=\mathscr{S}(\alpha_1)$ of intervals $I\subset[0,1]$ such that

$$R(I)>\alpha_1\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}\quad(|I|=h).$$

It follows from Lemma 5.3.2 that there is a $\delta=\delta(\alpha_2)>0$ such that

$$P\{I\in\mathscr{S}\}<|I|^{\alpha_2}\quad(0<|I|<\delta).\qquad(5.3.14)$$

Now let $\mathscr{C}_n$ consist of all closed intervals of lenght $h_n=\exp(-n/\log n)$, and left-hand endpoints $ih_n/\log n$ $(i=0,1,\cdots,[h_n^{-1}\log n])$. For any $t\in E(\alpha_0)$ there is a sequence of intervals $I_n=[t,t+u_n]$ with $u_n\to 0$ and

$$R(I_n)>(\alpha_1+\alpha_0)\tilde{\sigma}(p,u_n)(2\log u_n^{-1})^{1/2}/2.$$

When u_n is small enough, each such I_n will be contained in one of the intervals of $\mathscr{C}_m \cap \mathscr{S}$ for suitable m. Hence every point of $E(\alpha_0)$ is covered infinitely often by intervals from the collection $\bigcup_{m=1}^{\infty} \mathscr{C}_m \cap \mathscr{S}$. Now let T_m denote the number of intervals in $\mathscr{C}_m \cap \mathscr{S}$. The estimate (5.3.14) yields

$$ET_m \leqslant h_m^{\alpha_2^2-1} \log m.$$

So, for $\varepsilon > 0$

$$\sum_{m=1}^{\infty} h_m^{1-\alpha_2^2+\varepsilon} ET_m < \infty.$$

Hence with probability 1,

$$\sum_{m=1}^{\infty} T_m h_m^{1-\alpha_2^2+\varepsilon} < \infty,$$

so that $\dim E(\alpha_0) \leqslant 1-\alpha_2^2+\varepsilon$. If we now let $\varepsilon \to 0$ and $\alpha_2 \to \alpha_0$ through a countable set, we obtain, with probability 1

$$\dim E(\alpha_0) \leqslant 1 - \alpha_0^2.$$

Now, we turn to the proof of the opposite inequality. It is sufficient to show that for any $0<\alpha_0<1$, we have almost surely

$$\dim E(\alpha_0) \geqslant 1 - \alpha_0^2. \tag{5.3.15}$$

For each fixed $1>\alpha>\alpha_0$ and $\varepsilon>0$, we will apply Lemma 5.3.1 with K chosen as a suitable subset of $E(\alpha_0)$ and $\varphi(s) = s^{\beta-2\varepsilon}$, where $\beta = 1-\alpha^2$, $0<\varepsilon<\beta/2$. This will enable us to establish (5.3.15). The remainder of the proof is devoted to the construction of K and is inspired by, and accurately is a generalized version of, the arguments in Section 4 of Orey and Taylor (1974).

Let $\mathscr{T}$ denote the collection of intervals $[u,v] \subset [0,1]$ such that

$$\| Y(v) - Y(u) \|_{l^p} \geqslant \alpha_0 \tilde{\sigma}(p, v-u)(2\log(v-u)^{-1})^{1/2}. \tag{5.3.16}$$

Theorem 3.3.3 tells us that

$$\| Y(t) - Y(s) \|_{l^p} \leqslant 2\tilde{\sigma}(p, |t-s|)(2\log|t-s|^{-1})^{1/2}$$

for all s, $t \in [0,1]$ with $|t-s|$ sufficiently small. Hence there exists $b>0$ depending only on α_0 and α such that, for every sufficiently small $I=[u,v] \subset [0,1]$,

$$\| Y(v) - Y(u) \|_{l^p} \geqslant \alpha \tilde{\sigma}(p, v-u)(2\log(v-u)^{-1})^{1/2} \tag{5.3.17}$$

implies that $[t,v] \in \mathscr{T}$ for all $t \in I(b) = [u, u+b(v-u)]$. For convenience we assume that b is the reciprocal of an integer.

Suppose that ρ_m is the reciprocal of an integer, $\rho_{m+1} < b\rho_m$ and $b\rho_m/\rho_{m+1}$ is an integer for $m=1,2,\cdots$. Let δ be a positive number such that $\delta < \varepsilon/16$. For each $m \geqslant 1$, define $A_m = [\rho_m^{-\delta}]$, $l_m = [(\rho_m^{-1}-1)/A_m]+1$ and

$$t_m(i) = iA_m\rho_m, \quad i=0,1,\cdots,l_m-1, \tag{5.3.18}$$

$$\mathscr{F}_m = \{[t_m(i), t_m(i)+\rho_m]; \quad i=0,1,\cdots,l_m-1\}. \tag{5.3.19}$$

We proceed with the proof by considering the cases of $1\leqslant p<2$ and $2\leqslant p<\infty$ separately.

Case I $\quad 1\leqslant p<2$.

In this case, for each $m\geqslant 1$ and any $I=[t_m(i), t_m(i)+\rho_m] \in \mathscr{F}_m$ we have

$$\begin{aligned}
\frac{\| Y(I) \|_{l^p}}{\tilde{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}} &= \frac{\| Y(t_m(i)+\rho_m) - Y(t_m(i)) \|_{l^p}}{\tilde{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}} \\
&= \frac{\sup_{\|a\|_{l^q}\leqslant 1} \sum_{j=1}^{\infty} a_j (X_j(t_m(i)+\rho_m) - X_j(t_m(i)))}{\tilde{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}} \\
&\geqslant \frac{\sum_{j=1}^{\infty} \sigma_j(\rho_m)^{\frac{2(p-1)}{2-p}} (X_j(t_m(i)+\rho_m) - X_j(t_m(i)))}{\tilde{\sigma}(p,\rho_m)\left(\sum_{j=1}^{\infty} \sigma_j(\rho_m)^{\frac{2p}{2-p}}\right)^{\frac{p-1}{p}} (2\log\rho_m^{-1})^{1/2}} \\
&= \frac{\sum_{j=1}^{\infty} \sigma_j(\rho_m)^{\frac{2(p-1)}{2-p}} (X_j(t_m(i)+\rho_m) - X_j(t_m(i)))}{\bar{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}}
\end{aligned}$$

$$=:Y_{m,I}/(2\log\rho_m^{-1})^{1/2}, \tag{5.3.20}$$

where $\bar{\sigma}(p,\rho_m) = (\sum_{j=1}^{\infty}\sigma_j(\rho_m)^{\frac{2p}{2-p}})^{1/q}$, $q = p/(p-1)$ and

$$Y_{m,I} = \bar{\sigma}(p,\rho_m)^{-1}\sum_{j=1}^{\infty}\sigma_j(\rho_m)^{\frac{2(p-1)}{2-p}}(X_j(t_m(i)+\rho_m) - X_j(t_m(i))). \tag{5.3.21}$$

We define

$$\mathscr{F}_m^+ = \{I\in\mathscr{F}_m; Y_{m,I}>\alpha(2\log\rho_m^{-1})^{1/2}\},$$

$$\mathscr{F}_m^+(b) = \{I(b)=[u,u+b(v-u)]; I=[u,v]\in\mathscr{F}_m^+\},$$

$$N_m(J)=\mathrm{Card}\{I\in\mathscr{F}_m^+; I\subset J\},\quad N_m=N_m([0,1]), \tag{5.3.22}$$

$$l_m(J)=\mathrm{Card}\{I\in\mathscr{F}_m; I\subset J\},\qquad l_m=l_m([0,1]),$$

$$\rho_m^{1-\beta(m)}=P\{N(0,1)>\alpha(2\log\rho_m^{-1})^{1/2}\},$$

where $0<\beta(m)\to\beta=1-\alpha^2$ as $m\to\infty$.

From (5.3.20), we deduce that for m large enough, $I=[u,v]\in\mathscr{F}_m^+$ implies (5.3.17), and then $[t,v]\in\mathscr{F}$ for any $t\in I(b)\in\mathscr{F}_m^+(b)$.

The following lemma tells us that $N_m(J)$ has similar probability estimates to a binomial distribution with parameters $p=\rho_m^{1-\beta(m)}$ and $n=l_m(J)$.

Lemma 5.3.4 *For any $0<\theta\leqslant 1/2$, there exists an integer m_θ such that*

$$P\{|N_m(J) - EN_m(J)| > \lambda EN_m(J)\}$$
$$\leqslant 2\exp\{-\theta(\lambda-2\theta)EN_m(J)\} + \rho_m^4 \tag{5.3.23}$$

for all $J\subset[0,1]$, $m\geqslant m_\theta$ and $\lambda>0$.

Proof By condition (5.3.5), we can assume that

$$\max_{A_m\leqslant j\leqslant\rho_m^{-1}}\max_{k\geqslant 1}\frac{E(X_k(\rho_m)-X_k(0))(X_k((j+1)\rho_m)-X_k(j\rho_m))}{\sigma_k^2(h)}\leqslant b_m^2$$

for all $m\geqslant 1$, where $b_m>0$ satisfies $b_m\log\rho_m\to 0$ as $m\to\infty$. It follows that

$$\max_{I,J\in\mathscr{F}_m, I\neq J} EY_{m,I}Y_{m,J}\leqslant b_m^2.$$

For m large enough, let

$$p_0^{(m)}=P\{N(0,1)>\alpha(2\log\rho_m^{-1})^{1/2}\},$$

$$p_1^{(m)}=P\left\{N(0,1)>\frac{(\alpha-3b_m)(2\log\rho_m^{-1})^{1/2}}{(1-b_m^2)^{1/2}}\right\},$$

$$p_2^{(m)}=P\left\{N(0,1)>\frac{(\alpha+3b_m)(2\log\rho_m^{-1})^{1/2}}{(1-b_m^2)^{1/2}}\right\}.$$

Applying Lemma 5.3.3 to $\{Y_{m,I}; I\subset J\}$ with $\beta=b_m$, $t=\alpha(2\log\rho_m^{-1})^{1/2}$ and $\varepsilon=3b_m(2\log\rho_m^{-1})^{1/2}$, we conclude that for any $0<\theta\leqslant 1/2$ and $\lambda>0$,

$$\begin{aligned}&P\{|N_m(J)-EN_m(J)|>\lambda EN_m(J)\}\\&\leqslant\exp\{-\theta l_m(J)((\lambda+1)p_0^{(m)}-(1+\theta)p_1^{(m)})\}\\&\quad+\exp\{-\theta l_m(J)((1-\theta)p_2^{(m)}-(1-\lambda)p_0^{(m)})\}\\&\quad+2l_m(J)P\{N(0,1)>3(2\log\rho_m^{-1})^{1/2}\}.\end{aligned} \tag{5.3.24}$$

Note that

$$\log\frac{p_1^{(m)}}{p_0^{(m)}}\sim 2\left(\alpha^2-\frac{(\alpha-3b_m)^2}{1-b_m^2}\right)\log\rho_m^{-1}$$
$$\sim 6\alpha b_m\log\rho_m^{-1}=o(1)\text{ as } m\to\infty,$$

which implies $p_1^{(m)}\sim p_0^{(m)}$ as $m\to\infty$. Similarly, we have $p_2^{(m)}\sim p_0^{(m)}$ as $m\to\infty$. It follows that there exists an integer m_θ such that for $m\geqslant m_\theta$,

$$(1+\theta)\frac{p_1^{(m)}}{p_0^{(m)}}\leqslant 1+2\theta,\qquad (1-\theta)\frac{p_2^{(m)}}{p_0^{(m)}}\geqslant 1-2\theta.$$

Hence we deduce from (5.3.24) that

$$\begin{aligned}&P\{|N_m(J)-EN_m(J)|>\lambda EN_m(J)\}\\&\leqslant 2\exp\{-\theta(\lambda-2\theta)EN_m(J)\}+\rho_m^{-1}\rho_m^5\\&=2\exp\{-\theta(\lambda-2\theta)EN_m(J)\}+\rho_m^4\end{aligned}$$

for $m \geqslant m_0$. We have proved Lemma 5. 3. 4.

Lemma 5. 3. 5 *Given $\varepsilon_1 > 0$ and $\tau > 0$, with probability 1 there exists an integer $m_0 = m_0(\varepsilon_1, \tau)$ such that*

$$|N_m(J) - EN_m(J)| < \varepsilon_1 EN_m(J) \tag{5. 3. 25}$$

for all $J \subset [0,1]$ such that $|J| \geqslant \tau$, and all $m \geqslant m_0$.

Proof Let k be an integer larger than $4\varepsilon_1^{-1}$. Then for every J, (5. 3. 25) will follow if we can show that

$$|N_m(J) - EN_m(J)| < \varepsilon_1 EN_m(J)/2 \tag{5. 3. 26}$$

for each of the finite set of $[h^{-1}]$ intervals of the form $[ih, (i+1)h]$ $(h = \tau/4k)$.

In fact, let $J_i = [ih, (i+1)h], i = 0, \cdots, [h^{-1}] - 1$. Noting $l_m(I) \sim |I| l_m$, we have

$$EN_m(J_i) = EN_m(J_0) = l_m(J_0)\rho_m^{1-\beta(m)} \sim \frac{\tau}{4k} l_m \rho_m^{1-\beta(m)}.$$

It follows that for $m \geqslant m_0(\varepsilon_1, \tau)$,

$$EN_m(J_i) < EN_m(J)/2k$$

holds uniformly in $|J| \geqslant \tau$ and $i = 1, 2, \cdots, [h^{-1}] - 1$. Now for $|J| \geqslant \tau$, there exist i_1 and i_2 such that

$$\bigcup_{i=i_1+1}^{i_2-1} J_i \subset J \subset \bigcup_{i=i_1}^{i_2} J_i.$$

From (5. 3. 26) it follows easily that

$$|N_m(J) - EN_m(J)| < \varepsilon_1 EN_m(J)/2 + 4EN_m(J_0)$$
$$< \varepsilon_1 EN_m(J)/2 + 2EN_m(J)/k = \varepsilon_1 EN_m(J).$$

Now assume that J is the intervals of the form $[ih, (i+1)h]$. Noting that $EN_m(J) = l_m(J)\rho_m^{1-\beta(m)}$ and $l_m(J) \sim |J| l_m \sim h\rho_m^{\delta-1}$, we have $EN_m(J) \sim h\rho_m^{\delta-\beta(m)}$. By Lemma 5. 3. 4 ($\theta = \varepsilon_1/8$), we have for m large enough

$$P\{|N_m(J) - EN_m(J)| \geqslant \varepsilon EN_m(J)/2\}$$
$$\leqslant 2\exp\{-\varepsilon_1^2 EN_m(J)/32\} + \rho_m^4$$
$$\leqslant 2\exp\{-O(1)\rho_m^{\delta-\beta(m)}\} + \rho_m^4 \leqslant 2\rho_m^4$$

since $\delta - \beta(m) \to \delta - \beta > 0$, and then the Borel-Cantelli lemma implies that (5. 3. 26) holds for J of the form $[ih, (i+1)h]$, as required.

Lemma 5. 3. 6 *Given $\beta' < \beta = 1 - \alpha^2$, if $\delta < (\beta - \beta')/2$, then there exists an absolute constant c such that with probability 1 there exists $m_1 = m_1(\beta')$ such that*

$$N_m(J) \leqslant c|J|^{\beta'} N_m([0,1]) \tag{5. 3. 27}$$

for all $J \subset [0,1]$ and $m \geqslant m_1$.

Proof By Lemma 5. 3. 5, it is sufficient to show that

$$N_m(J) \leqslant c|J|^{\beta'} EN_m([0,1]) = c|J|^{\beta'} l_m \rho_m^{1-\beta(m)}$$

for $m \geqslant m_1$. Noting that $|J| < \rho_m$ implies $N_m(J) = 0$ and that $\rho_m \leqslant |J| < A_m\rho_m$ implies $N_m(J) \leqslant 1$ and $|J|^{\beta'} l_m \rho_m^{1-\beta(m)} \geqslant c\rho_m^{\delta+\beta'-\beta(m)} \to \infty$, we need only to consider the case of $|J| \geqslant A_m\rho_m$. It is clearly sufficient to consider only the class $\mathscr{D}_m$ of intervals $[iA_m\rho_m, jA_m\rho_m]$, where i, j are integers and $0 \leqslant i < j \leqslant (A_m\rho_m)^{-1}$. We deduce from (5. 3. 23) that for m large enough and all $r \geqslant 4$

$$P\{N_m(J) \geqslant rEN_m(J)\} \leqslant \exp\left(-\frac{r}{8}EN_m(J)\right) + \rho_m^4.$$

Noting that $l_m \sim \rho_m^{-1}A_m^{-1} \sim \rho_m^{\delta-1}$ and $l_m(J) \approx |J| l_m$, we have

$$P\{N_m(J) \geqslant c|J|^{\beta'} l_m \rho_m^{1-\beta(m)}; J \in \mathscr{D}_m\}$$
$$\leqslant \rho_m^{-2}\exp\left(-\frac{c}{8}\rho_m^{\beta'} l_m \rho_m^{1-\beta(m)}\right) + \rho_m^2$$
$$\leqslant \rho_m^{-2}\exp(-c_1\rho_m^{\delta+\beta'-\beta(m)}) + \rho_m^2.$$

Since $\delta + \beta' - \beta(m) \to \delta + \beta' - \beta < 0$, it follows that

$$\sum_{m=1}^{\infty} P\{N_m(J) \geqslant c|J|^{\beta'} l_m \rho_m^{1-\beta(m)}; J \in \mathscr{D}_m\} < \infty,$$

which implies almost surely there exists $m_1=m_1(\beta')$ such that

$$N_m(J)\leqslant c|J|^{\beta'}l_m\rho_m^{1-\beta(m)}$$

for all $J\in\mathscr{D}_m(m\geqslant m_1)$. This completes the proof of the lemma.

We shall now show that there exists a sequence of sets $E_1\supset E_2\supset\cdots$ fulfilling the assumptions of Lemma 5.3.1 and such that $K:=\bigcap_{m=1}^{\infty}E_m\subset E(\alpha_0)$. Since only a countable number steps of the construction are needed and each step can be carried out with probability 1, we can assume that all the steps are carried out in the same probability 1 set. Choose $\beta'=\beta-\varepsilon/4$ and define $m_1=m_1(\beta')$ such that (5.3.27) is valid for $m\geqslant m_1$. Suppose that $\{\varepsilon_k\}$ is a sequence of positive numbers with $\sum\varepsilon_k<\infty$. In the first step, we apply Lemma 5.3.5 to find an integer $Q(1)\geqslant m_1$ such that

$$|N_m-EN_m|<\varepsilon_1 EN_m\quad(m\geqslant Q(1)).\qquad(5.3.28)$$

And then we will define an increasing sequence $Q(1),Q(2),\cdots$, inductively and define for $k\geqslant 1$

$$\{I_{k,i};1\leqslant i\leqslant M_k\}=\{I(b)\in\mathscr{F}^+_{Q(k)}(b);I(b)\subset E_{k-1}\};$$

$$E_0=[0,1];\quad E_k=\bigcup_{i=1}^{M_k}I_{k,i};\qquad(5.3.29)$$

$$M_k(J)=\mathrm{Card}\{i;I_{k,i}\subset J\}\text{ for }J\subset[0,1];\quad M_k=M_k([0,1]);$$

$$\gamma(k)=\beta(Q(k));\quad\delta(k)=1-\gamma(k);\quad h_k=|I_{k,i}|=b\rho_{Q(k)}.$$

For $k\geqslant 2$, suppose that $Q(k-1)$ has been defined, we define $Q(k)$ large enough to ensure

$$Q(k)\geqslant m_0(\varepsilon,h_{k-1}^{2\delta(k-1)/\varepsilon}),\quad Q(k)\geqslant m_0(\varepsilon_k,h_{k-1}),$$

$$Q(k)\geqslant 2Q(k-1),\quad\rho_{Q(k)}\leqslant\rho^2_{Q(k-1)},\qquad(5.3.30)$$

where $m_0(\varepsilon,\tau)$ is the integer determined in Lemma 5.3.5 to invalidate (5.3.25), and

$$h_k^{1/2\varepsilon}\leqslant b^{2\beta}\prod_{i=1}^{k-1}h_i^{\delta(i)}b^{\gamma(i)}.\qquad(5.3.31)$$

Then

$$|N_m(J)-EN_m(J)|<\varepsilon_k EN_m(J)\qquad(5.3.32)$$

for all $J\subset[0,1]$ satisfying $|J|\geqslant h_{k-1}$ and all $m\geqslant Q(k)$.

The proof will be completed if the following is true.

$$M_{k+j}(J)\leqslant c\Big(\prod_{i=1}^{k}A_{Q(i)}\Big)|J|^{\beta-\varepsilon}M_{k+j}$$

$$\leqslant c\Big(\prod_{i=1}^{k}A_{Q(i)}\Big)h_k^{\varepsilon}|J|^{\beta-2\varepsilon}M_{k+j}\qquad(5.3.33)$$

for all $h_{k+1}<|J|\leqslant h_k,\quad k\geqslant 1,j\geqslant 1$.

In fact, if (5.3.33) is true, by noting that

$$\rho^2_{Q(k)}\leqslant\rho_{Q(k)}^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}}\leqslant\rho_{Q(k)}\rho_{Q(k-1)}\cdots\rho_{Q(1)}$$

and

$$\prod_{i=1}^{k}A_{Q(i)}\leqslant\Big(\prod_{i=1}^{k}\rho_{Q(i)}\Big)^{-\delta},$$

we conclude that

$$M_{k+j}(J)\leqslant cb^{\varepsilon}\rho_{Q(k)}^{\varepsilon-2\delta}|J|^{\beta-2\varepsilon}M_{k+j}$$

for all $h_{k+1}<|J|\leqslant h_k,k\geqslant 1,j\geqslant 1$, which, together with Lemma 5.3.1 and the fact that $\rho_{Q(k)}^{\varepsilon-2\delta}\to 0(k\to\infty)$, implies that

$$s^{\beta-2\varepsilon}\text{-mes}(K)>0.$$

Hence we have proved (5.3.15) in the case I.

Now we show (5.3.33). It is sufficient to consider the class $\mathscr{D}_k$ of intervals $[ih_{k+1},jh_{k+1}]$ with $0\leqslant i<j\leqslant h_{k+1}^{-1}$. By the fact that $1/b$ and $b\rho_m/\rho_{m+1}$ are integers, we can verify that for any $k\geqslant 1,j\geqslant 1,I(b)\in\mathscr{F}^+_{Q(k+j+1)}(b),I(b)\subset I_{k+j,i}$ imply

$$I\in\mathscr{F}^+_{Q(k+j+1)},\ I\subset I_{k+j,i};\qquad(5.3.34)$$

and for any $J\in\mathscr{D}_k$,

$$\text{interior }(I_{k+j,i}\cap J)\neq\varnothing\text{ implies }I_{k+j,i}\subset J.\ (5.3.35)$$

From (5.3.34) and (5.3.35), it follows that for $J\in\mathscr{D}_k$,

$$
\begin{aligned}
&M_{k+j+1}(J)\\
&= \operatorname{Card}\{I(b) \in \mathscr{F}^{+}_{Q(k+j+1)}(b); I(b) \subset \bigcup_{i=1}^{M_{k+j}} I_{k+j,i}, I(b) \subset J\}\\
&= \sum_{i=1}^{M_{k+j}} \operatorname{Card}\{I(b) \in \mathscr{F}^{+}_{Q(k+j+1)}(b); I(b) \subset I_{k+j,i}, I(b) \subset J\}\\
&= \sum_{i=1}^{M_{k+j}} \operatorname{Card}\{I \in \mathscr{F}^{+}_{Q(k+j+1)}; I \subset I_{k+j,i}, I_{k+j,i} \subset J\}\\
&= \sum_{i:I_{k+j,i}\subset J} \operatorname{Card}\{I \in \mathscr{F}^{+}_{Q(k+j+1)}; I \subset I_{k+j,i}\}\\
&= \sum_{i:I_{k+j,i}\subset J} N_{Q(k+j+1)}(I_{k+j,i}).
\end{aligned}
\tag{5.3.36}
$$

Noting that $EN_{Q(k+j+1)}(I_{k+j,i}) = l_{Q(k+j+1)}(I_{k+j,i})\rho_{Q(k+j+1)}^{1-\gamma(k+j+1)}$ and

$$
l_{Q(k+j+1)}(I_{k+j,i}) = \frac{b\rho_{Q(k+j)}}{A_{Q(k+j+1)}\rho_{Q(k+j+1)}}(1+\delta_{k+j+1,i}), \tag{5.3.37}
$$

where

$$
\begin{aligned}
|\delta_{k+j+1,i}| &\leqslant 8A_{Q(k+j+1)}\rho_{Q(k+j+1)}/(b\rho_{Q(k+j)})\\
&\leqslant 8\rho_{Q(k+j+1)}^{1-\delta}/(b\rho_{Q(k+j)})\\
&\leqslant 8b^{-1}\rho_{Q(k+j+1)}^{1/4}\\
&\leqslant b^{Q(k+j+1)/4}.
\end{aligned}
$$

From (5.3.32) and (5.3.36) we have

$$
\begin{aligned}
M_{k+j+1}(J) = M_{k+j}(J)&\frac{1}{A_{Q(k+j+1)}}h_{k+j}h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\\
&\times(1+\delta(J))(1+\eta(J)),
\end{aligned}
\tag{5.3.38}
$$

where $|\delta(J)| \leqslant b^{Q(k+j+1)/4}$, $|\eta(J)| \leqslant \varepsilon_{k+j+1}$. Noting that

$$
\prod\nolimits_{i=1}^{\infty}(1+b^{Q(i)/4})(1+\varepsilon_i) \leqslant \exp\left(\sum\nolimits_{i=1}^{\infty}(\varepsilon_i + b^{Q(i)/4})\right) < \infty
$$

and that

$$
M_1 = N_{Q(1)} \approx EN_{Q(1)} = l_{Q(1)}\rho_{Q(1)}^{1-\beta(Q(1))} = \frac{1}{A_{Q(1)}}h_1^{-\gamma(1)}b^{\gamma(1)}.
$$

By induction and (5.3.38) we have that for all $j \geqslant 1, k \geqslant 1, m \geqslant 1$

$$
M_{k+j+1} \approx \left(\prod_{i=1}^{k+j+1}\frac{1}{A_{Q(i)}}\right)h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\prod_{i=1}^{k+j}h_i^{\delta(i)}b^{\gamma(i)}. \tag{5.3.39}
$$

Also, since $Q(k) \geqslant Q(1) \geqslant m_1(\beta')$, by (5.3.27) we have

$$
\begin{aligned}
M_{k+1}(J) \leqslant N_{Q(k+1)}(J) &\leqslant c|J|^{\beta'}N_{Q(k+1)}([0,1])\\
&\leqslant c|J|^{\beta'}l_{Q(k+1)}\rho_{Q(k+1)}^{1-\beta(Q(k+1))}\\
&\leqslant c|J|^{\beta'}\frac{1}{A_{Q(k+1)}}h_{k+1}^{-\gamma(k+1)}b^{\gamma(k+1)},
\end{aligned}
$$

and so, by induction and (5.3.38) it follows that

$$
M_{k+j+1}(J) \leqslant c|J|^{\beta'}\left(\prod_{i=k+1}^{k+j+1}\frac{1}{A_{Q(i)}}\right)h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\prod_{i=k+1}^{k+j}h_i^{\delta(i)}b^{\gamma(i)}. \tag{5.3.40}
$$

A similar type of inductive argument, using (5.3.25) and $Q(k) \geqslant m_0(\varepsilon, h_{k-1}^{2\delta(k-1)/\varepsilon})$ establishes

$$
M_{k+j+1}(J) \leqslant c|J|\left(\prod_{i=k+1}^{k+j+1}\frac{1}{A_{Q(i)}}\right)h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\prod_{i=k+1}^{k+j}h_i^{\delta(i)}b^{\gamma(i)}, \tag{5.3.41}
$$

for $J \in \mathscr{D}_k$ with $h_k^{2\delta(k)/\varepsilon} \leqslant |J| \leqslant h_k$. Finally, note that

$$
|J| \leqslant c|J|^{\beta-\varepsilon}\prod_{i=1}^{k}h_i^{\delta(i)}b^{\gamma(i)} \quad (h_k^{2\delta(k)/\varepsilon} < |J| \leqslant h_k), \tag{5.3.42}
$$

$$
|J|^{\beta'} \leqslant c|J|^{\beta-\varepsilon}\prod_{i=1}^{k}h_i^{\delta(i)}b^{\gamma(i)} \quad (h_{k+1} \leqslant |J| \leqslant h_k^{2\delta(k)/\varepsilon}) \tag{5.3.43}
$$

for large k, where (5.3.31) and the fact that $\delta(k) \to 1-\beta < 1-\beta+2/\varepsilon$ are used.

Combining (5.3.39)–(5.3.43) yields

$$
M_{k+j+1}(J) \leqslant c|J|^{\beta-\varepsilon}\left(\prod_{i=1}^{k}\frac{1}{A_{Q(i)}}\right)M_{k+j+1} \tag{5.3.44}
$$

for $J \in \mathscr{D}_k$ and $|J| \leqslant h_k$.

This proves (5.3.33).

Case Ⅱ $p \geqslant 2$.

Take j_m such taht $\sigma_{j_m}(\rho_m)=\sigma^*(\rho_m)$. In this case, for any $I=[t_m(i),t_m(i)+\rho_m]\in\mathscr{F}_m$ we also have

$$\frac{\| Y(I) \|_{l^p}}{\tilde{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}} \geqslant \frac{X_{j_m}(I)}{\sigma_{j_m}(\rho_m)(2\log\rho_m^{-1})^{1/2}}.$$

Following the lines of the proof in the case I, we conclude that (5.3.15) remains true in this case as well.

Proof of Theorem 5.3.3

Since $E^*(\alpha)\subset E(\alpha)$, we only need to show that $\dim E^*(\alpha)\geqslant 1-\alpha^2$. We may suppose $P>1$. By tightening the argument used in the proof of (5.3.15), We may show that for any $\alpha\in[0,1)$, φ-mes $E(\alpha)>0$ a.s., where $\varphi(s)=s^{1-\alpha^2}(\log s^{-1})^{6P+4}$. Since

$$E(\alpha) = E^*(\alpha)\bigcup_{n=1}^{\infty}E(\alpha+1/n), \tag{5.3.45}$$

and $\dim E(\alpha+1/n)=1-(\alpha+1/n)^2$ implies φ-mes $E(\alpha+1/n)=0$, we have

$$\varphi\text{-mes } E^*(\alpha) \geqslant \varphi\text{-mes } E(\alpha) > 0 \quad \text{a.s.}$$

which implies $\dim E^*(\alpha)\geqslant 1-\alpha^2$ a.s. This proves Theorem 5.3.3.

Using the conclusions of Theorem 5.3.2 and 5.3.3, we can obtain some consequences. The first one is about the Gaussian processes with regularly varying variance functions.

Suppose that $\{X(t);t\geqslant 0\}$ is a centered Gaussian process with stationary increments $\sigma^2(h)=E(X(t+h)-X(t))^2$. Assume that $\sigma(t)$ is increasing and regularly varying at zero with index $0<\gamma<1$, given in the canonical form

$$\sigma(h) = h^{\gamma}\exp\left(\int_h^1 \frac{\varepsilon(h)}{y}\mathrm{d}y\right), \tag{5.3.46}$$

where $\varepsilon(h)\to 0$ as $h\to 0$. Theorem 2.2.5 establishes the Lévy moduli of continuity for this kind of processes. Clearly, $\{Y(t);t\geqslant 0\}$ is a fractional Brownian motion of order γ if $\varepsilon(h)\equiv 0$.

Let $p\geqslant 1$, $\{c_n;n\geqslant 1\}$ be non-negative numbers. Put

$$c(p)=\Big(\sum_{k=1}^{\infty}c_k^p\Big)^{1/p}, \tag{5.3.47}$$

$$\tilde{c}(p)=\begin{cases} c\left(\dfrac{2p}{2-p}\right) & \text{if } 1\leqslant p<2, \\ \max_{k\geqslant 1} c_k & \text{if } p\geqslant 2. \end{cases} \tag{5.3.48}$$

Let $\{Y(t);t\geqslant 0\}=\{c_kX_k(t);t\geqslant 0\}_{k=1}^{\infty}$, where $X_k(\cdot)$, $k=1,2,\cdots$ are independent copies of $X(\cdot)$. Set $\sigma_k^2(h)=c_k^2EX_k^2(h)=c_k^2\sigma^2(h)$. Let $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ be defined as before. Clearly, we have

$$\sigma(p,h) = c(p)\sigma(h), \quad \tilde{\sigma}(p,h) = \tilde{c}(p)\sigma(h). \tag{5.3.49}$$

Assume

$$0 < \sum_{k=1}^{\infty}c_k^p < \infty. \tag{5.3.50}$$

As consequences of Theorems 5.3.2 and 5.3.3, we have the following results.

Theorem 5.3.4 *Let $p\geqslant 1$, $\{Y(t);t\geqslant 0\}=\{c_kX_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be defined as above. If for any $\varepsilon>0$ there exists a constant C_ε such that $-h^{1+\varepsilon}\varepsilon'(h)\leqslant C_\varepsilon$ for all $h\in[0,1]$, then for any $\alpha\in[0,1]$, we have almost surely*

$$\dim\left\{t\in[0,1];\frac{\| Y(t+h)-Y(t) \|_{l^p}}{\tilde{c}_p\sigma(h)(2\log h^{-1})^{1/2}}\geqslant\alpha\right\}=1-\alpha^2. \tag{5.3.51}$$

If there exists a constant C such that $-h\varepsilon'(h)\leqslant C(\log(e/h))^C$ for all $h\in[0,1]$, then for any $\alpha\in[0,1]$, we have almost surely

$$\dim\left\{t\in[0,1];\frac{\| Y(t+h)-Y(t) \|_{l^p}}{\tilde{c}_p\sigma(h)(2\log h^{-1})^{1/2}}=\alpha\right\}=1-\alpha^2. \tag{5.3.52}$$

Particularly, we have the following corollary.

Corollary 5.3.1 *Let $p\geqslant 1$, $\{\xi_k(t);t\geqslant 0,k\geqslant 1\}$ be independent fractional Wiener processes of order γ, $0<\gamma<1$. Define $\{Y(t);t\geqslant 0\}=\{c_k\xi_k(t);t\geqslant 0\}_{k=1}^{\infty}$. Assume that (5.3.50) is satisfied. Then for any $\alpha\in[0,1]$, we have almost surely*

$$\dim E(\alpha)=\dim E^*(\alpha)=1-\alpha^2,$$

where

$$E(\alpha)=\left\{t\in[0,1];\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde{c}_p h^{\gamma}(2\log h^{-1})^{1/2}}\geqslant\alpha\right\},$$

$$E^*(\alpha)=\left\{t\in[0,1];\frac{\|Y(t+h)-Y(t)\|_{l^p}}{\tilde{c}_p h^{\gamma}(2\log h^{-1})^{1/2}}=\alpha\right\}.$$

Proof of Theorem 5.3.4

To show (5.3.51) and (5.3.52), we need only to verify (5.3.5) and (5.3.8) respectively. Let

$$L(x)=\exp\left(2\int_x^1\frac{\varepsilon(y)}{y}\mathrm{d}y\right).$$

Then

$$(\sigma^2(x))''=2x^{2\gamma-2}L(x)\{\gamma(2\gamma-1)+(1-4\gamma)\varepsilon(x)+2\varepsilon^2(x)-x\varepsilon'(x)\}.$$

It is easy to see that $|\varepsilon(x)|\leqslant C$ for some $C>0$ and all $x\in[0,1]$, and also for any $\delta>0$ there is a constant C_δ such that

$$L(jx)/L(x)\leqslant C_\delta j^\delta$$

for all x, $jx\in[0,1]$. Suppose that $-x^{1+\delta}\varepsilon'(x)\leqslant C_\delta$. Then

$$\frac{(\sigma^2(jx))''x^2}{\sigma^2(x)}\leqslant C_\delta j^{2\gamma-2+\delta}x^{-\delta}.$$

Note that

$$\frac{E(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)}$$

$$=\frac{\sigma^2((j+1)h)+\sigma^2((j-1)h)-2\sigma^2(jh)}{2\sigma^2(h)}=\frac{(\sigma^2(\xi))''h^2}{2\sigma^2(h)}$$

for every $1>h\geqslant 0$, $j\geqslant 6$ and some $(j-2)/h\leqslant\xi\leqslant jh$, we conclude that

$$\frac{E(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)}$$

$$\leqslant C_\delta\frac{(\sigma^2(\xi))''(\xi/j)^2}{2\sigma^2(\xi/j)}\leqslant C_\delta j^{2\gamma-2+\delta}(\xi/j)^{-\delta}\leqslant C_\delta j^{2\gamma-2+\delta}h^{-\delta},$$

which implies (5.3.5) immediately.

Suppose that $-x\varepsilon'(x)\leqslant C(\log(\mathrm{e}/x))^C$, then

$$\frac{(\sigma^2(jx))''x^2}{2\sigma^2(x)}\leqslant C_\delta j^{2\gamma-2+\delta/2}\left(\log\frac{\mathrm{e}}{jx}\right)^C\leqslant C_\delta j^{2\gamma-2+\delta}\left(\log\frac{\mathrm{e}}{x}\right)^C,$$

We also conclude that

$$\frac{E(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)}$$

$$\leqslant C_\delta j^{2\gamma-2+\delta}(\log h^{-1})^C,$$

which implies (5.3.8) immediately.

The second two consequences of Theorems 5.3.2 and 5.3.3 are on the l^p-valued fractional Ornstein-Uhlenbeck processes and the infinite series of fractional Ornstein-Uhlenbeck processes.

Let $\{Y(t);t\geqslant 0\}=\{X_k(t);t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent fractional Ornstein-Uhlenbeck processes of order β_k with coefficients γ_k and λ_k, where $0<\beta_k<1,\gamma_k\geqslant 0,\lambda_k>0$, i.e.,

$$\{X_k(t);t>0\}\text{ and }\left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{\xi_k(\mathrm{e}^{2\lambda_k t})}{\mathrm{e}^{2\beta_k\lambda_k t}};t\geqslant 0\right\}$$

have the same distribution, where $\{\xi_k(t);t\geqslant 0\}$ is a fractional Wiener process of order β_k. It is easily seen that $EX_k(t)=0$,

$$EX_k(t)X_k(s)=\frac{\gamma_k}{2\lambda_k}(\mathrm{e}^{2\beta_k\lambda_k(t-s)}+\mathrm{e}^{2\beta_k\lambda_k(s-t)}-|\mathrm{e}^{\lambda_k(t-s)}-\mathrm{e}^{\lambda_k(s-t)}|^{2\beta_k})$$

for all $t,s\geqslant 0$, and

$$\sigma_k^2(h)=E(X_k(t+h)-X_k(t))^2$$

$$=\frac{\gamma_k}{\lambda_k}\{(\mathrm{e}^{\lambda_k h}-\mathrm{e}^{-\lambda_k h})^{2\beta_k}-(\mathrm{e}^{\beta_k\lambda_k h}-\mathrm{e}^{-\beta_k\lambda_k h^2})\}.$$

Clearly, $\{X_k(t)\}$ are the usual Ornstein-Uhlenbeck processes if

$\beta_k = 1/2$ for all $k \geqslant 1$. Section 2.2.3 studied the Lévy moduli of continuity for the infinite series of Ornstein-Uhlenbeck processes (cf. Remarks 2.2.5, 2.2.6). Similar quantities for l^p-valued fractional Ornstein-Uhlenbeck processes were studied in Section 3.3 in the case that all $X_k(\cdot)$ have the same order γ, i.e., $\beta_k = \gamma$ for all $k \geqslant 1$. Here, as consequences of Theorems 5.3.2 and 5.3.3, we have the following results about the fractal nature.

Let $p \geqslant 1$. Let $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ be defined as before.

Theorem 5.3.5 *Assume that $\tilde{\sigma}(p,h)/h^A$ is quasi-increasing on $(0,A)$ for some $A>0$. Suppose that $0<\beta_k \leqslant \beta_0 < 1$ for all $k \geqslant 1$. If*

$$\sigma(p,h) = o\left(\tilde{\sigma}(p,h)\left(\log \frac{1}{h}\right)^{1/2}\right) \quad \text{as } h \to 0, \tag{5.3.53}$$

then for any $\alpha \in [0,1]$, we have almost surely

$$\dim E(\alpha) = \dim E^*(\alpha) = 1 - \alpha^2,$$

where

$$E(\alpha) = \left\{t \in [0,1]; \frac{\| Y(t+h) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}} \geqslant \alpha\right\},$$

$$E^*(\alpha) = \left\{t \in [0,1]; \frac{\| Y(t+h) - Y(t) \|_{l^p}}{\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}} = \alpha\right\}. \tag{5.3.54}$$

Proof We need only to verify (5.3.8). For any fixed $0<\gamma \leqslant \beta_0 < 1$, let

$$f(x) := f(\gamma,x) = (e^x - e^{-x})^{2\gamma} - (e^{\gamma x} - e^{-\gamma x})^2.$$

It is easy to see that

$$f'(x) = 2\gamma\{(e^x - e^{-x})^{2\gamma-1}(e^x + e^{-x}) - (e^{\gamma x} - e^{-\gamma x})(e^{\gamma x} + e^{-\gamma x})\} > 0$$

for all $x > 0$,

$$f''(x) = 2\gamma\{(2\gamma-1)(e^x - e^{-x})^{2\gamma-2}(e^x + e^{-x})^2 + (e^x - e^{-x})^{2\gamma} - 2\gamma e^{2\gamma x} - 2\gamma e^{-2\gamma x}\} < 0$$

for all $x \geqslant 0$ if $0 < \gamma \leqslant 1/2$,

$$f''(x) \leqslant 16\gamma(2\gamma-1)(e^x - e^{-x})^{2\gamma-2} \leqslant 16(e^x - e^{-x})^{2\gamma-2} =: g(\gamma,x) =: g(x) \tag{5.3.55}$$

for all $x > 0$ if $\gamma > 1/2$.

Hence in any case we have

$$f''(x) \leqslant g(x) \quad \text{for all } x > 0. \tag{5.3.56}$$

We also have

$$\begin{aligned} f'(x) &= 2\gamma\left\{(e^x - e^{-x})^{2\gamma}\frac{e^x + e^{-x}}{e^x - e^{-x}} - (e^{\gamma x} - e^{-\gamma x})^2 \frac{e^{\gamma x} + e^{-\gamma x}}{e^{\gamma x} - e^{-\gamma x}}\right\} \\ &< 2\gamma\left\{(e^x - e^{-x})^{2\gamma}\frac{e^x + e^{-x}}{e^x - e^{-x}} - (e^{\gamma x} - e^{-\gamma x})^2 \frac{e^x + e^{-x}}{e^x - e^{-x}}\right\} \\ &= 2\gamma f(x)\frac{e^x + e^{-x}}{e^x - e^{-x}}, \end{aligned}$$

which implies that $f(x)/((e^x - e^{-x})^{2\gamma})$ is decreasing. So, we have that for $x \geqslant 1$,

$$\begin{aligned} f(x) &\geqslant f(1) = e^{2\gamma}\{(1 - e^{-2})^{2\gamma} - (1 - e^{-2\gamma})^2\} \\ &\geqslant (1 - e^{-2})^{2\beta_0} - (1 - e^{-2\beta_0})^2 = c_{\beta_0} > 0; \end{aligned}$$

and for $0 < x \leqslant 1$,

$$\begin{aligned} f(x) &\geqslant \frac{f(1)}{(e - e^{-1})^{2\gamma}}(e^x - e^{-x})^{2\gamma} \\ &\geqslant \frac{f(1)}{(e - e^{-1})^{2\beta_0}}(e^x - e^{-x})^{2\gamma} = c_{\beta_0}(e^x - e^{-x})^{2\gamma}. \end{aligned}$$

Now we estimate

$$g(jx)x^2/f(x).$$

Since $(e^{jx} - e^{-jx}) \geqslant j(e^x - e^{-x})$ for all $x \geqslant 0$, we have that for $0 < x \leqslant 1$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0}\left(\frac{e^{jx} - e^{-jx}}{e^x - e^{-x}}\right)^{2\gamma-2}\left(\frac{x}{e^x - e^{-x}}\right)^2 \leqslant c_{\beta_0} j^{2\gamma-2};$$

and for $x \geqslant 1$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0} g(jx)x^2 = c_{\beta_0}\left(\frac{e^{jx} - e^{-jx}}{e^x - e^{-x}}\right)^{2\gamma-2} \frac{x^2}{(e^x - e^{-x})^{2-2\gamma}}$$

$$\leqslant c_{\beta_0} j^{2\gamma-2} \frac{x^2}{(e^x - e^{-x})^{2-2\beta_0}} \leqslant c_{\beta_0} j^{2\gamma-2}.$$

So, we conclude that for all $x>0$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0} j^{2\beta_0-2}. \tag{5.3.57}$$

Now we verify (5.3.8). Define $f_k(x)=f(\beta_k, x)$. We have that

$$\begin{aligned}
&\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{\sigma_k^2(h)} \\
&=\frac{f_k((j+1)\lambda_k h)+f_k((j-1)\lambda_k h)-2f_k(j\lambda_k h)}{2f_k(\lambda_k h)} \\
&=\frac{f''(\xi)(\lambda_k h)^2}{2f_k(\lambda_k h)}
\end{aligned}$$

for every $h>0, j\geqslant 6, k\geqslant 1$ and some $(j-1)\lambda_k h \leqslant \xi \leqslant (j+1)\lambda_k h$. Noting from (5.3.56) that

$$f''_k(\xi) \leqslant g(\beta_k, \xi) \leqslant g(\beta_k, (j-1)\lambda_k h),$$

we deduce from (5.3.57) that

$$\begin{aligned}
&\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{\sigma_k^2(h)} \\
&\leqslant \frac{1}{2} c_{\beta_0} (j-1)^{2\beta_0-2} \leqslant c_{\beta_0} j^{2\beta_0-2},
\end{aligned} \tag{5.3.58}$$

which implies (5.3.8) immediately.

Theorem 5.3.6 *Suppose that* $\{X(t)=\sum_{k=1}^{\infty} X_k(t); t\geqslant 0\}$ *is the infinite series of* $\{X_k(t)\}_{k=1}^{\infty}$ *and* $0<\beta_k\leqslant\beta_0<1$ *for all* $k\geqslant 1$. *Assume that* $\sigma(2,h)/h^{\Lambda}$ *is quasi-increasing on* $(0,\Lambda)$ *for some* $\Lambda>0$. *Then for any* $\alpha\in[0,1]$, *we have almost surely*

$$\dim E(\alpha) = \dim E^*(\alpha) = 1-\alpha^2,$$

where

$$\begin{aligned}
E(\alpha) &= \left\{ t\in[0,1]; \frac{|X(t+h)-X(t)|}{\sigma(2,h)(2\log h^{-1})^{1/2}} \geqslant \alpha \right\}, \\
E^*(\alpha) &= \left\{ t\in[0,1]; \frac{|X(t+h)-X(t)|}{\sigma(2,h)(2\log h^{-1})^{1/2}} = \alpha \right\}.
\end{aligned} \tag{5.3.59}$$

Proof From (5.3.58), we deduce that for every $j\geqslant 6, h\geqslant 0$,

$$\begin{aligned}
&E(X(h)-X(0))(X((j+1)h)-X(jh)) \\
&= \sum_{k=1}^{\infty} E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh)) \\
&\leqslant \sum_{k=1}^{\infty} \sigma_k^2(h) c_{\beta_0} j^{2\beta_0-2} = c_{\beta_0} j^{2\beta_0-2} \sigma^2(2,h),
\end{aligned}$$

which implies (5.3.8). Hence we have Theorem 5.3.6.

5.4 The Fractal Nature of Increments of the Infinite Series of Ornstein-Uhlenbeck Processes Related to the Chung Type LIL

Let $\{W(t); t\geqslant 0\}$ be a standard Wiener process. Orey and Taylor (1947) also studied the fractal nature of the set

$$B_1(\alpha):=\left\{ t\in[0,1]; \liminf_{h\to 0}\left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} \sup_{0\leqslant s\leqslant h} |W(t+s)-W(t)| \leqslant \alpha \right\} \quad (\alpha\geqslant 1).$$

They showed that for each $\alpha\geqslant 1$, $B_1(\alpha)$ is random fractal and proved:

Theorem 5.4.1 *For any* $\alpha\geqslant 1$ *we have almost surely*

$$\dim B_1(\alpha) = 1-\alpha^{-2}. \tag{5.4.1}$$

This result is corresponding to the Chung type law of the iterated logarithm and the moduli of non-differentiability. In Chapter 4, it has been shown that the Chung type law of the iterated logarithm holds for a lot of Gaussian processes. For ex-

ample, Theorems 4.3.5 and 4.5.6 gave us a Chung type law of the iterated logarithm and an exact moduli of non-differentiability for the infinite series of independent Ornstein-Uhlenbeck processes respectively. In this section, we will establish a fractal nature similar to Theorem 5.4.1 for this kind of processes.

Let $\{Y(t);-\infty<t<\infty\}=\{X_k(t);-\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k. Suppose that $\{X(t);-\infty<t<\infty\}=\{\sum_{k=1}^{\infty}X_k(t);-\infty<t<\infty\}$ is the infinite series of $\{Y(t);-\infty<t<\infty\}$. Put

$$\Gamma_0=2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}<\infty, \tag{5.4.2}$$

$$\sigma^2(h)=E(X(t+h)-X(t))^2=2\sum_{k=1}^{\infty}\frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h}), \tag{5.4.3}$$

$$\Gamma_1=2\sum_{k=1}^{\infty}\gamma_k>0. \tag{5.4.4}$$

We now define a random set similar to $B(\alpha)$, by

$$E_1(\alpha):=\left\{t\in[0,1];\liminf\left(\frac{8\log h^{-1}}{\pi^2\sigma^2(h)}\right)^{1/2}\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)|\leqslant\alpha\right\}\quad(\alpha\geqslant1). \tag{5.4.5}$$

The following theorem is due to Zhang (1998):

Theorem 5.4.2 *Assume that* $\Gamma_1=2\sum_{k=1}^{\infty}\gamma_k<\infty$. *For any* $\alpha\geqslant 1$, *we have almost surely*

$$\dim E_1(\alpha)=1-\alpha^{-2}. \tag{5.4.6}$$

Proof Noting that $\sigma^2(h)/h\to\Gamma_1$ as $h\to0$, we can replace $\sigma^2(h)$ by $\Gamma_1 h$ in the definition of $E_1(\alpha)$.

First we show that $\dim E_1(\alpha_0)\leqslant1-\alpha_0^{-2}$ for $\alpha_0\geqslant1$. It is sufficient to show that for any fixed $\alpha_0<\alpha_1<\alpha_2<\alpha$ and $\varepsilon>0$, we have

almost surely

$$s^{\beta+\varepsilon}\text{-mes } E_1(\alpha_0)<\infty \tag{5.4.7}$$

where $\beta=1-\alpha^{-2}$. For $I=[t,t+h]$ we define

$$M(I)=\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)|. \tag{5.4.8}$$

Choose $\theta>1$ near 1 enough such that $\theta^2\alpha_1<\alpha_2$. Define

$$h_n=\theta^{-n},\quad \delta_n=n^{-3},\quad n=1,2,\cdots$$

and

$$t_n(i)=i\delta_n h_n,\quad 0\leqslant i\leqslant m_0:=[(\delta_n h_n)^{-1}]-1,\quad n\geqslant1.$$

Furthermore, let for $0\leqslant i\leqslant m_n$, $n\geqslant1$

$$\mathscr{I}_{n,i}=\left\{\left(\frac{8\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}M([t_n(i),t_n(i)+h_n])<\alpha_2\right\}$$

and

$$I_{n,i}=\begin{cases}[t_n(i)-\delta_n h_n,t_n(i)+\delta_n h_n], & \text{if } \mathscr{I}_{n,i}=1,\\ \varnothing, & \text{otherwise.}\end{cases}$$

Suppose $t\in E_1(\alpha_0)$. Then there exists a sequence of positive numbers $\{v_m\}$ with $v_m\to0(m\to\infty)$ such that

$$\left(\frac{8\log v_m^{-1}}{\pi^2\Gamma_1 v_m}\right)^{1/2}M([t,t+v_m])<\alpha_1.$$

For every m large enough, there exist $n\geqslant2$ and $1\leqslant i\leqslant m_n$ such that $h_n<v_m\leqslant h_{n-1}$ and $t\in[t_n(i)-\delta_n h_n,t_n(i)+\delta_n h_n]$. It follows that

$$\begin{aligned}&\left(\frac{8\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}M([t_n(i),t_n(i)+h_n])\\ &\leqslant\left(\frac{8\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}M([t,t+h_n])\\ &\quad+2\left(\frac{8\log h_n^{-1}}{\pi^2\Gamma_1 h_n}\right)^{1/2}\sup_{0\leqslant\tau\leqslant1}\sup_{0\leqslant s\leqslant2\delta_n h_n}|X(\tau+s)-X(\tau)|\\ &=:J_n^{(1)}+J_n^{(2)},\end{aligned}$$

where

$$J_n^{(2)} \to 0 \quad \text{as } n \to \infty$$

by the moduli of continuity theorem (cf. Theorem 2.2.5), and

$$J_n^{(1)} < \left(\frac{8\log h_n^{-1}}{\pi^2 \Gamma_1 h_n}\right)^{1/2} \left(\frac{8\log h_{n-1}^{-1}}{\pi^2 \Gamma_1 h_{n-1}}\right)^{-1/2} \alpha_1 \leqslant \theta\alpha_1.$$

Hence for n large enough, we have

$$\left(\frac{8\log h_n^{-1}}{\pi^2 \Gamma_1 h_n}\right)^{1/2} M([t_n(i), t_n(i) + h_n]) < \alpha_2,$$

which implies $t \in I_{n,i}$. Hence we conclude that

$$E_1(\alpha_0) \subset \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{i=1}^{m_n} I_{n,i} \quad \text{a.s.} \tag{5.4.9}$$

By the definition, we have

$$s^{\beta+\epsilon}\text{-mes}\Big(\bigcup_{n=k}^{\infty} \bigcup_{i=1}^{m_n} I_{n,i}\Big) \leqslant \sum_{n=k}^{\infty} \sum_{i=1}^{m_n} (2\delta_n h_n)^{\beta+\epsilon} \mathscr{I}_{n,i}. \tag{5.4.10}$$

Let $E_n = E\mathscr{I}_{n,i}$. For any $\delta > 0$ satisfying $(1+\delta)\alpha_2^2 < \alpha^2$, by Lemma 4.3.5 we have that for n large enough,

$$E_n = P\left\{\left(\frac{8\log h_n^{-1}}{\pi^2 \Gamma_1 h_n}\right)^{1/2} \sup_{0 \leqslant s \leqslant h_n} |X(s) - X(0)| < \alpha_2\right\}$$

$$\leqslant \frac{4}{\pi} \exp\left(-\frac{\log h_n^{-1}}{(1+\delta)\alpha_2^2}\right) = \frac{4}{\pi} h_n^{\alpha^{-2}}.$$

It follows that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{m_n} (2\delta_n h_n)^{\beta+\epsilon} E_n \leqslant C \sum_{n=1}^{\infty} n^{3(1-\beta-\epsilon)} h_n^{\epsilon} < \infty,$$

which together with (5.4.9) and (5.4.10) implies (5.4.7).

We now turn to the proof of the opposite inequality. We may assume that $\alpha_0 > 1$, otherwise there is nothing to prove. It is sufficient to show that for any fixed $1 < \alpha < \alpha_0$ and $\epsilon > 0$, we have almost surely

$$\dim E_1(\alpha_0) \geqslant \beta - 2\epsilon, \tag{5.4.11}$$

where $\beta = 1 - \alpha^{-2}$.

We will apply Lemma 5.3.1 with K chosen as a suitable subset of $E_1(\alpha_0)$ and $c = \beta - 2\epsilon$.

Let $\mathscr{T}$ denote the collection of intervals $[u, u+h] \subset [0,1]$ such that

$$\left(\frac{8\log h^{-1}}{\pi^2 \Gamma_1 h}\right)^{1/2} M([u, u + h]) < \alpha_0. \tag{5.4.12}$$

Let $\alpha < \alpha_2 < \alpha_1 < \alpha_0$. If

$$\left(\frac{8\log h^{-1}}{\pi^2 \Gamma_1 h}\right)^{1/2} M([u, u + h]) < \alpha_1, \tag{5.4.13}$$

then for $0 < b < 1$ and any $t \in [u, u+bh]$, we have

$$M([t, t + (1 - b)h]) < \alpha_1 \left(\frac{\pi^2 \Gamma_1 h}{8\log h^{-1}}\right)^{1/2}.$$

Hence there exists $b > 0$ depending only on α_0 and α_1 such that, for every sufficiently small $I = [u, u+h] \subset [0,1]$, (5.4.13) implies that $[t, t+(1-b)h] \in \mathscr{T}$ for all $t \in I(b) = [u, u+bh]$. As in Section 5.3, we assume that b is the reciprocal of an integer, and that ρ_m is the reciprocal of an integer, $\rho_{m+1} < b\rho_m$ and $b\rho_m/\rho_{m+1}$ is an integer for $m = 1, 2, \cdots$. Let $A_m = [9(\log \rho_m^{-1})^4]$ and

$$\mathscr{T}_m = \{[2iA_m\rho_m, 2iA_m\rho_m + \rho_m]; i = 0, 1, \cdots, [(\rho_m^{-1} - 1)/(2A_m)]\}. \tag{5.4.14}$$

Let $\{W_j(t); t \geqslant 0\}_{j=1}^{\infty}$ be a sequence of independent standard Wiener processes. We can write

$$X(t) = \sum_{j=1}^{\infty} \left(\frac{\gamma_j}{\lambda_j}\right)^{1/2} \frac{W_j(e^{2\lambda_j t})}{e^{\lambda_j t}}, \quad t \geqslant 0. \tag{5.4.15}$$

For any $I = [2iA_m\rho_m, 2iA_m\rho_m + \rho_m] \in \mathscr{T}_m$ we define

$$\xi_{m,I}(s) = \sum_{j=1}^{\infty} \left(\frac{\gamma_j}{\lambda_j}\right)^{1/2} \left\{\frac{W_j(e^{2\lambda_j(2iA_m\rho_m + s)}) - W_j(e^{2\lambda_j(2i-1)A_m\rho_m})}{e^{\lambda_j(2iA_m\rho_m + s)}} - \frac{W_j(e^{2\lambda_j(2iA_m\rho_m)}) - W_j(e^{2\lambda_j(2i-1)A_m\rho_m})}{e^{\lambda_j(2iA_m\rho_m)}}\right\}, \tag{5.4.16}$$

$$\eta_{m,I}(s) = \sum_{j=1}^{\infty}\left(\frac{\gamma_j}{\lambda_j}\right)^{1/2}(1-\mathrm{e}^{-\lambda_j s})\frac{W_j(\mathrm{e}^{2\lambda_j(2i-1)A_m\rho_m})}{\mathrm{e}^{\lambda_j(2iA_m\rho_m)}},$$
$$(0\leqslant s\leqslant\rho_m, 0\leqslant[(\rho_m^{-1}-1)/(2A_m)]).$$

Then for each m, $\{\xi_{m,I}(s); 0\leqslant s\leqslant\rho_m, I\in\mathscr{F}_m\}$ are independent, the distribution of $\{\xi_{m,I}(s); 0\leqslant s\leqslant\rho_m\}$ does not depend on $I\in\mathscr{F}_m$, and for $I\in\mathscr{F}_m$ we have

$$\begin{aligned}&\sup_{0\leqslant s\leqslant\rho_m}|\xi_{m,I}(s)|-\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|\\ &\leqslant M(I)\leqslant\sup_{0\leqslant s\leqslant\rho_m}|\xi_{m,I}(s)|+\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|.\end{aligned}\tag{5.4.17}$$

We define

$$\begin{aligned}&\mathscr{F}_m^+=\left\{I\in\mathscr{F}_m;\sup_{0\leqslant s\leqslant\rho_m}|\xi_{m,I}(s)|<\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\},\\ &\mathscr{F}_m^+(b)=\{I(b)=[u,u+b(v-u)];I=[u,v]\in\mathscr{F}_m^+\},\\ &N_m(J)=\mathrm{Card}\{I\in\mathscr{F}_m^+;I\subset J\},\quad N_m=N_m([0,1]),\\ &l_m(J)=\mathrm{Card}\{I\in\mathscr{F}_m;I\subset J\},\\ &l_m=l_m([0,1])=[(\rho_m^{-1}-1)/(2A_m)]+1,\\ &\rho_m^{1-\beta(m)}=P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\xi_{m,I}(s)|<\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}.\end{aligned}\tag{5.4.18}$$

Next we show that $\beta(m)\to\beta$ as $m\to\infty$ and

$$\lim_{m\to\infty}\max_{J\in\mathscr{F}_m}\left(\frac{8\log\rho_m^{-1}}{\pi^2\Gamma_1\rho_m}\right)^{1/2}\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,J}(s)|=0\quad\text{a.s.}\tag{5.4.19}$$

Note that for any $I\in\mathscr{F}_m$ and $0\leqslant s\leqslant s+h\leqslant\rho_m$ we have

$$\begin{aligned}E(\eta_{m,I}(s+h)-\eta_{m,I}(s))^2&\leqslant\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda_j}\mathrm{e}^{-2\lambda_jA_m\rho_m}(1-\mathrm{e}^{-\lambda_jh})^2\\ &\leqslant\sum_{j:\lambda_j\rho_m\leqslant(\log A_m)/A_m}\mathrm{e}^{-2\lambda_jA_m\rho_m}\frac{\gamma_j}{\lambda_j}(1-\mathrm{e}^{-\lambda_jh})\lambda_j\rho_m\\ &\quad+\sum_{j:\lambda_j\rho_m\geqslant(\log A_m)/A_m}\mathrm{e}^{-2\lambda_jA_m\rho_m}\frac{\gamma_j}{\lambda_j}(1-\mathrm{e}^{-\lambda_jh})\\ &\leqslant\frac{1}{2}\left(\frac{\log A_m}{A_m}+\frac{1}{A_m}\right)\sigma^2(h)\leqslant\frac{\log A_m}{A_m}\Gamma_1h.\end{aligned}$$

Hence by Theorem 1.1.3 we have

$$P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|\geqslant x(1+\sqrt{8\pi})\left(\frac{\log A_m}{A_m}\Gamma_1\rho_m\right)^{1/2}\right\}\leqslant A\mathrm{e}^{-x^2/2}$$

for any $x\geqslant0$. Then for any $\delta>0$, we have for m large enough

$$\begin{aligned}&P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|\geqslant\delta\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\leqslant\exp(-(\log\rho_m^{-1})^2),\end{aligned}\tag{5.4.20}$$

which implies

$$\begin{aligned}&\sum_{m=1}^{\infty}P\left\{\sup_{J\in\mathscr{F}_m}\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,J}(s)|\geqslant\delta\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\leqslant\sum_{m=1}^{\infty}\rho_m^{-1}\exp(-(\log\rho_m^{-1})^2)<\infty.\end{aligned}\tag{5.4.21}$$

It follows that (5.4.19) holds true.

Also for any $\delta>0$, by (5.4.20) and Lemma 4.3.4 we have for m large enough that

$$\begin{aligned}&\frac{2}{\pi}\exp\left(-\frac{\log\rho_m^{-1}}{(1-\delta)^2\alpha^2}\right)-\exp(-(\log\rho_m^{-1})^2)\\ &\leqslant P\left\{\sup_{0\leqslant s\leqslant\rho_m}|X(s)-X(0)|\leqslant(1-\delta)\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\quad-P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|\geqslant\delta\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\leqslant P_m:=P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\xi_{m,I}(s)|<\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\leqslant P\left\{\sup_{0\leqslant s\leqslant\rho_m}|X(s)-X(0)|\leqslant(1+\delta)\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\quad+P\left\{\sup_{0\leqslant s\leqslant\rho_m}|\eta_{m,I}(s)|\geqslant\delta\alpha\left(\frac{\pi^2\Gamma_1\rho_m}{8\log\rho_m^{-1}}\right)^{1/2}\right\}\\ &\leqslant\frac{2}{\pi}\exp\left(-\frac{\log\rho_m^{-1}}{(1+\delta)^2\alpha^2}\right)+\exp(-(\log\rho_m^{-1})^2).\end{aligned}$$

It follows that

$$\frac{1}{(1+\delta)^2\alpha^2} \leqslant \liminf_{m\to\infty} \frac{\log P_m}{\log \rho_m} \leqslant \limsup_{m\to\infty} \frac{\log P_m}{\log \rho_m} \leqslant \frac{1}{(1-\delta)^2\alpha^2},$$

which implies that $\lim_{m\to\infty}(\log P_m)/\log\rho_m = \alpha^{-2}$ by letting $\delta\to 0$. Hence $\beta(m)\to\beta$ as $m\to\infty$. Without loss of generality, we can assume that $\beta(m)>0$ for all $m\geqslant 1$.

Now by (5.4.17) and (5.4.19) we have that for m large enough, $I=[u,u+h]\in\mathscr{F}_m^+$ implies that (5.4.13) holds. And then we conclude that $[t,t+(1-b)h]\in\mathscr{T}$ for all $t\in I(b)\in\mathscr{F}_m^+(b)$.

Clearly, $N_m(J)$ has a binomial distribution with parameters $p=\rho_m^{1-\beta(m)}$ and $n=l_m(J)$. So with similar proofs to those of Lemmas 5.3.5, 5.3.6 we have the following two lemmas.

Lemma 5.4.1 *Given $\varepsilon>0$ and $\delta>0$, with probability 1 there exists an integer $m_0=m_0(\varepsilon,\delta)$ such that*

$$|N_m(J)-EN_m(J)| < \varepsilon EN_m(J)$$

for all $J\subset[0,1]$ with $|J|\geqslant\delta$, and all $m\geqslant m_0(\varepsilon,\delta)$.

Lemma 5.4.2 *Given $\beta'<\beta=1-\alpha^{-2}$, there exists an absolute constant c such that with probability 1 there exists $m_1=m_1(\beta')$ such that*

$$N_m(J)\leqslant c|J|^{\beta'}N_m([0,1])$$

for all $J\subset[0,1]$ with $|J|\geqslant\rho_m$, $m\geqslant m_1$.

With Lemmas 5.4.1 and 5.4.2 instead of Lemmas 5.3.5 and 5.3.6, the remainder of the proof (5.4.11) is similar to that of (5.3.15). This proves (5.4.6).

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Subject Index

Remark' § n. 0 denotes the introduction of Chapter n.